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IN A PLASMA

BY

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On the Energy of Waves in a Plasma

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A formula is given for the energy of an arbitrary linear wave in a plasma which may be magnetized, collisional, homogeneous or weakly inhomogeneous, in terms of the electric and magnetic fields of the wave and derivatives of the wave frequency with respect to the plasma parameters. This formula makes it possible to determine the energy of a wave from its dispersion relation (and the magnitude of the fields). The derivation rests on a simple invariance property - essentially dimensional analysis - of the Fokker-Planck equation together with well-known results from the theory of dispersive media.

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I. INTRODUCTION

A plasma is one example of a dispersive medium, i.e., a medium whose dielectric function $\epsilon(\vec{k}, \omega)$ depends on frequency. In such a medium the energy required to build up a specified electric field $\vec{E}(\vec{x}, t)$ is not necessarily related in any simple way to $\vec{E}$. This, of course, is because building up the field from zero amplitude is a time-dependent process; thus, the field is a superposition of fields of various frequencies, each of which has its own value of ϵ . Suppose, however, that the medium is only weakly dissipative, and that the field is electrostatic and of the form $\vec{E}(\vec{x}, t) = \vec{E}(t) \exp(-i\omega t + i\vec{k} \cdot \vec{x}) + \text{complex conjugate}$, where ω and $\vec{k}$ are real, and $\vec{E}(t)$ varies very slowly compared to the exponential. In this situation, as is well known,¹ W , the energy per unit volume required to set up the field, is

$$W = \frac{\partial}{\partial \omega} (\omega \epsilon') \frac{\langle \vec{E}(\vec{x}, t)^2 \rangle}{8\pi}, \quad (1)$$

where $\epsilon = \epsilon' + i\epsilon''$, with ϵ' and ϵ'' real, and the angular brackets denote a time average over the rapid variation of the exponential. (Note that $\langle \vec{E} \rangle^2 = 2|\vec{E}|^2$). If the electric field $\vec{E}$ is due to an electrostatic wave with no externally imposed electric fields driving it, then $\epsilon'' = 0$, in which case $\partial(\omega \epsilon') / \partial \omega = \omega \partial \epsilon' / \partial \omega$. The energy of an electrostatic wave is, therefore,

$$W = \omega \frac{\partial \epsilon'}{\partial \omega} \frac{\langle \vec{E}(\vec{x}, t)^2 \rangle}{8\pi}. \quad (2)$$

The object of this paper is to derive a rather general formula for the quantity $\omega \partial \epsilon / \partial \omega$ in terms of ω , k and the dependence of the frequency ω on various parameters of the plasma, such as collision frequency ν , and applied magnetic field $\vec{B}_0$. If the plasma is weakly inhomogeneous, the dependence of ϵ on L , the scale length of the inhomogeneity, must also be taken into account. For an electrostatic wave, the relation is

$$\frac{\omega}{\partial \epsilon} = \frac{2(1-\epsilon)}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k}) - (\vec{B}_0/\omega) \cdot (d\omega/d\vec{B}_0) - (\nu/\omega) (d\omega/d\nu) + (L/\omega) (d\omega/dL)} \quad (3)$$

The derivatives in Eq. (3) are taken with ϵ fixed; all of the parameters are also fixed, except the parameter being varied. For example, $d\omega/d\vec{B}_0$ is the derivative of the wave frequency, at fixed $\vec{k}$, ν , L . This formula gives, via Eq. (2), the energy of an electrostatic wave, provided that three conditions are met: (a) $\epsilon = 0$, i.e., the wave is a normal mode; (b) the wave is weakly damped, so that the derivatives in the denominator are essentially real; (c) $\partial \epsilon / \partial \omega \ll \partial \epsilon / \partial \omega$. These are also the conditions for Eq. (2) to represent the energy of a wave, so that Eq. (3) represents the wave energy whenever the wave energy is defined.

This relation, and its generalization to electromagnetic waves, are useful for several reasons. They allow formulation of a concise criterion for when a wave is a negative energy wave. They also make evaluation of the wave energy possible if the dielectric function has not been explicitly evaluated. Several general conclusions follow from the form of the denominator of Eq. (3); for example, in an unmagnetized, collisionless, homogeneous plasma, if a wave is

spatially non-dispersive (ω proportional to k) the electric field associated with a wave of finite energy must vanish. Finally, there are several situations where integrals of $1/v$ are required, e.g., in the fluctuation-dissipation theorem² and in sum rules involving $\text{Im} 1/\epsilon$.³ Equation (3) gives a concise expression for the residue of the poles at $\omega = 0$.

The organization of this paper is as follows: in Sec. II, Eq. (3) is derived by using a simple invariance property of the Fokker-Planck equation which is little more than dimensional analysis, together with the assumption that the plasma response is linear. Section III gives a rather novel derivation of the dielectric function in a magnetized collisionless plasma, in order to illustrate the somewhat abstract derivation in Sec. II. Section IV derives the generalization of Eq. (3) to electromagnetic waves. Finally, Sec. V discusses various applications of this work.

II. ELECTROSTATIC CASE

For the case of a collisionless, unmagnetized, homogeneous plasma, results akin to Eq. (3) are implicit in the literature.^{4,5} Probably the simplest way to derive the result is to note that the dielectric function⁶

$$\epsilon(\vec{k}, \omega) = 1 - \frac{\omega_p^2}{k^2} \int d^3v \frac{1}{\vec{k} \cdot \vec{v} - \omega} \quad (4)$$

is of the form⁷

$$\epsilon(\vec{k}, \omega) = 1 + \frac{1}{k^2} \epsilon\left(\frac{\vec{k}}{\omega}\right) . \quad (5)$$

It follows from this form that

$$\omega \frac{\partial \epsilon}{\partial \omega} + \vec{k} \cdot \frac{\partial \epsilon}{\partial \vec{k}} = 2(1-\epsilon) . \quad (6)$$

Now

$$\left. \frac{d\omega}{d\vec{k}} \right|_L = - \frac{(\partial \epsilon / \partial \vec{k})}{(\partial \epsilon / \partial \omega)} , \quad (7)$$

which immediately yields

$$\omega \frac{\partial \epsilon}{\partial \omega} = \frac{2(1-\epsilon)}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k})} . \quad (8)$$

This proof is direct but somewhat difficult to generalize to more complex situations. The first step in this section will be to rederive Eq. (8) by an alternate procedure which is readily generalized; the procedure exploits an almost trivial invariance of the linear Vlasov equation.

Consider a plasma which is excited by an externally applied electric field $\vec{E}_e$. The total electric field is $\vec{E}_t = \vec{E}_e + \vec{E}_p$, where $\vec{E}_p$ is the electric field whose source is the plasma particles. Both $\vec{E}_e$ and $\vec{E}_p$ are assumed to be electrostatic fields. If all fields have space-time dependence $\exp(-i\omega t + i\vec{k} \cdot \vec{x})$ the dielectric function $\epsilon(\vec{k}, \omega)$ is defined by

$$\vec{E}_e = \epsilon(\vec{k}, \omega) \vec{E}_t . \quad (9)$$

Note that $\epsilon(\vec{k}, \omega)$ is a scalar; this is because both $\vec{E}_e$ and $\vec{E}_T$ are electrostatic fields, and thus proportional to $\vec{k}$. The case where $\epsilon(\vec{k}, \omega)$ is a tensor is discussed in Sec. IV.

The equations to be solved are the linearized Vlasov equation and Poisson's equation:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 + \frac{e}{m} \vec{E}_e \cdot \frac{\partial f_0}{\partial \vec{v}} = 0, \quad (10)$$

$$\nabla \cdot \vec{E}_e = 4\pi n_0 e \int d^3v f_1(\vec{x}, \vec{v}, t). \quad (11)$$

Here n_0 , f_0 are the number density and distribution function of the unperturbed plasma; f_1 is the perturbation of the distribution function. Only one species of particle is explicitly shown, but the generalization to several species is clear.

These equations are invariant under the scale transformation $\vec{x} \rightarrow \lambda \vec{x}$, $t \rightarrow \lambda t$, $\vec{E} \rightarrow \vec{E}/\lambda$, $n_0 \rightarrow n_0/\lambda^2$. More formally, define $\vec{x}^* = \lambda \vec{x}$, $t^* = \lambda t$, $\vec{v}^* = \vec{v}$, $n_0^* = n_0/\lambda^2$, and

$$f_1^*(\vec{x}^*, \vec{v}^*, t) = f_1(\vec{x}, \vec{v}, t), \quad (12)$$

$$\vec{E}^*(\vec{x}^*, \vec{v}^*, t) = \frac{1}{\lambda} \vec{E}(\vec{x}, \vec{v}, t). \quad (13)$$

It is easily verified that if f_1 and $\vec{E}_1$ satisfy Eqs. (10) and (11), f_1^* and $\vec{E}_1^*$ also satisfy these equations, provided that n_0 is replaced by n_0^* . The scaling of n_0 as $1/\lambda^2$ seems surprising at first. This is explained by the scaling $t \rightarrow \lambda t$, which implies that frequencies scale as $\omega \rightarrow \omega/\lambda$. Since n_0 is proportional to

plasma frequency squared ($\omega_p^2 = 4\pi n_0 e^2/m$), n_0 must scale as $1/\lambda^2$.

Both the Vlasov equation and Poisson's equation are linear; therefore, there must exist a scalar Green's function $G(\vec{x}, t; n_0)$, which obeys

$$\tilde{E}_p(\vec{x}, t) = \int d^3x_1 dt_1 G(\vec{x}_1, t_1; n_0) \tilde{E}_t(\vec{x} - \vec{x}_1, t - t_1) \quad (14)$$

(Causality requires that G vanish for $t_1 < 0$, but this fact is not required here.) Note that $\tilde{E}_p$ appears on the left, while $\tilde{E}_t$ is on the right. This is because the plasma particles, which produce $\tilde{E}_p$, respond to the total field $\tilde{E}_t$. Now, $\tilde{E}^*$ and f_1^* obey the same equations as do $\tilde{E}$ and f_1 , provided n_0^* replaces n_0 . Thus, it must also be true that

$$\tilde{E}_p^*(\vec{x}^*, t^*) = \int d^3x_1^* dt_1^* G(\vec{x}_1^*, t_1^*; n_0^*) \tilde{E}_t^*(\vec{x}^* - \vec{x}_1^*, t^* - t_1^*) \quad (15)$$

where $G(\vec{x}^*, t^*; n_0^*)$ is the same function of its arguments as $G(\vec{x}, t; n_0)$. Using the scaling for $\tilde{E}^*$, Eq. (13), and $d^3x^* dt^* = \lambda^4 d^3x dt$, it follows that

$$G(\lambda\vec{x}, \lambda t; n_0^*) = \frac{1}{\lambda^4} G(\vec{x}, t; n_0) \quad (16)$$

This equation relates the Green's functions of two physically distinct plasmas, namely, a plasma with unperturbed density n_0 and one with density n_0^* . However, as follows from Poisson's equation, for a given f_1 [i.e., for a given total field $E_t(t)$] E_p is proportional to n_0 ; hence, G must also be proportional to n_0 . Thus, Eq. (16) implies that

$$G(\lambda \vec{x}, \lambda t, n_0) = \frac{1}{\lambda^2} G(\vec{x}, t, n_0) \quad (17)$$

(The parametric dependence of G on n_0 will be suppressed henceforth.) If the electric fields vary as $\vec{E}(\vec{x}, t) = \vec{E}(\vec{k}, \omega) \exp(-i\omega t + i\vec{k} \cdot \vec{x})$, Eq. (14) yields

$$\vec{E}_p(\vec{k}, \omega) = G(\vec{k}, \omega) \vec{E}_t(\vec{k}, \omega) \quad (18)$$

where

$$G(\vec{k}, \omega) = \int d^3x dt \exp(i\omega t - i\vec{k} \cdot \vec{x}) G(\vec{x}, t) \quad (19)$$

The scaling law Eq. (17) implies the central result

$$G\left(\frac{\vec{k}}{\lambda}, \frac{\omega}{\lambda}\right) = \lambda^2 G(\vec{k}, \omega) \quad (20)$$

Differentiating this with respect to λ , and then setting $\lambda = 1$, yields

$$\vec{k} \cdot \frac{\partial G}{\partial \vec{k}} + \omega \frac{\partial G}{\partial \omega} = -2G(\vec{k}, \omega) \quad (21)$$

From Eq. (9), it follows that $\epsilon(\vec{k}, \omega) = 1 - G(\vec{k}, \omega)$, which implies Eqs. (6) and (7).

The obvious virtue of this argument is that the generalization to more complicated systems is trivial. As an example, consider electrostatic waves in a homogeneous, magnetized, collisional plasma. The equations of motion are now Poisson's equation, as before, and the Fokker-Planck equation

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 + \frac{e}{m} \vec{E}_t \cdot \frac{\partial f_0}{\partial \vec{v}} + \frac{e}{mc} \vec{v} \times \vec{B}_0 \cdot \frac{\partial f_1}{\partial \vec{v}} = \nu C f_1 \quad (22)$$

where $\vec{B}_0$ is the applied magnetic field; νC is the collision operator, where ν is a parameter with dimensions of frequency, and C is a dimensionless operator which acts on velocity variables only. These equations are invariant under the scaling $\vec{x} \rightarrow \lambda \vec{x}$, $t \rightarrow \lambda t$, $\vec{E} \rightarrow \vec{E}/\lambda$, $\vec{B}_0 \rightarrow \vec{B}_0/\lambda$, $\nu \rightarrow \nu/\lambda$, $n_0 \rightarrow n_0/\lambda^2$. Following the same argument as before, the Green's function G , which depends parametrically on n_0 , $\vec{B}_0$ and ν , obeys the scaling equation

$$\vec{k} \cdot \frac{\partial G}{\partial \vec{k}} + \omega \frac{\partial G}{\partial \omega} + \nu \frac{\partial G}{\partial \nu} + \vec{B}_0 \cdot \frac{\partial G}{\partial \vec{B}_0} = -2G \quad (23)$$

Again, $\epsilon = 1 - G$, and so

$$\omega \frac{\partial \epsilon}{\partial \omega} = \frac{2(1-\epsilon)}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k}) - (\vec{B}_0/\omega) \cdot (d\omega/d\vec{B}_0) - (\nu/\omega) (d\omega/d\nu)} \quad (24)$$

As another example, consider the effect of making the plasma weakly inhomogeneous in the z direction, with the inhomogeneity characterized by a scale length L . Suppose that, at a given z , there is a local Green's function in the x - y plane, which depends parametrically on L , i.e., that

$$\vec{E}_p(x, y, t) = \int dx_1 dy_1 dt_1 G(x_1, y_1, t_1; n_0; L) \vec{E}_t(x-x_1, y-y_1, t-t_1) \quad (25)$$

The parameter L must be scaled $L \rightarrow \lambda L$, of course. This produces a term $-L \partial G / \partial L$ on the left side of Eq. (23), which gives Eq. (3).

III. MAGNETIZED COLLISIONLESS PLASMA

As an illustration of the results given in Sec. II, a novel derivation of the dielectric constant for electrostatic waves in a homogeneous, collisionless, magnetized plasma will now be presented. The method is a generalization of an approach due to Kaufman;⁸ it is also related to the work of Bakshi and Bellow.⁹ Only one species will be considered explicitly; adding other species is straightforward.

The perturbed electron distribution function f_1 satisfies Eq. (22), with $\nu = 0$. It will prove to be convenient to take $f_1(\vec{x}, \vec{v}, t) = g(\vec{v}, t) \exp(-i\omega t + i\vec{k} \cdot \vec{x})$, where $g(\vec{v}, t)$ varies slowly compared to $\exp(-i\omega t)$. Similarly, $\vec{E}(\vec{x}, t) = \vec{E}(t) \exp(-i\omega t + i\vec{k} \cdot \vec{x})$. Equation (22) then implies

$$\frac{\partial g}{\partial t} - i(\omega - \vec{k} \cdot \vec{v})g + (\vec{v} \times \vec{\Omega}_O) \cdot \frac{\partial g}{\partial \vec{v}} = -\frac{e}{m} \vec{E}_t \cdot \frac{\partial f_O}{\partial \vec{v}}, \quad (26)$$

where $\vec{\Omega}_O = e\vec{B}_O/mc$. Define an operator O by $O = -i(\omega - \vec{k} \cdot \vec{v}) + (\vec{v} \times \vec{\Omega}_O) \cdot \partial / \partial \vec{v}$. Equation (26) is solved by noting that

$$e^{-O t} \frac{\partial}{\partial t} (g e^{O t}) = \frac{\partial g}{\partial t} - i(\omega - \vec{k} \cdot \vec{v})g + \vec{v} \times \vec{\Omega}_O \cdot \frac{\partial g}{\partial \vec{v}}. \quad (27)$$

Hence, assuming that $g(0) = 0$, i.e., that at $t = 0$ the plasma is unperturbed,

$$g(t) = -\frac{e}{m} \int_0^t dt' e^{-O t'} \vec{E}_{t'} \cdot \frac{\partial f_O}{\partial \vec{v}}. \quad (28)$$

The field $\vec{E}_p(t)$ may be found by solving Poisson's equation, using Eq. (28) to get ρ_1 . The result is $\vec{E}_p = \vec{k} E_p$, where $\vec{k} = \vec{k}/k$, and E_p satisfies

$$E_p(t) = \int_0^t dt K(\vec{k}, \omega, \vec{\Omega}; t) E_t(t-t) \quad , \quad (29)$$

$$K(\vec{k}, \omega, \vec{\Omega}; t) = \frac{4\pi e^2}{k} \int d^3v e^{-i\omega t} \vec{k} \cdot \frac{\partial f_0}{\partial \vec{v}} \quad (30)$$

This formal expression for K is evaluated in Appendix A, in order to demonstrate that it in fact yields the correct dielectric function. The actual value of K is not really of interest here; rather, the important question is how K changes when the scale transformations are performed. Clearly, when $\vec{k} \rightarrow \vec{k}/\lambda$, $\omega \rightarrow \omega/\lambda$, $\vec{\Omega} \rightarrow \vec{\Omega}/\lambda$, $t \rightarrow \lambda t$, the change in K is $K \rightarrow \lambda K$. (Scaling n_0 is superfluous, since K is manifestly linear in n_0 .) Thus, differentiating with respect to λ , then setting $\lambda = 1$,

$$-\left\{ \vec{k} \cdot \frac{\partial K}{\partial \vec{k}} + \omega \frac{\partial K}{\partial \omega} + \vec{\Omega} \cdot \frac{\partial K}{\partial \vec{\Omega}} \right\} + t \frac{\partial K}{\partial t} = K \quad . \quad (31)$$

The dielectric constant is

$$\epsilon(\vec{k}, \omega, \vec{\Omega}) = 1 - \int_0^\infty dt K(\vec{k}, \omega, \vec{\Omega}; t) \quad . \quad (32)$$

This may be seen by letting $t \rightarrow \infty$ in Eq. (29), assuming that $E_p(t)$ and $E_t(t)$ are essentially constant for a very long time, i.e., E_p and E_t vary essentially as $\exp(-i\omega t)$. Inserting Eq. (31) into Eq. (32) and integrating by parts, one again derives Eq. (24) (with $v = 0$).

As mentioned previously, the dielectric function given by Eq. (32) is identical to the dielectric function calculated in a conventional way.² The value of this derivation is that it produces a compact representation of $\epsilon(\vec{k}, \omega, \vec{B})$ which obviously has the proper scaling behavior. A subsidiary benefit is that it yields a direct construction of the Fourier transform of the Green's function required in Eq. (14) :

$$G(\vec{k}, \omega) = \int_0^{\infty} dt K(\vec{k}, \omega, t) . \quad (33)$$

IV. ELECTROMAGNETIC WAVES

The goal of this section is to present a result, analogous to Eq. (3), which is valid for electromagnetic waves. The only complication necessitating special treatment for this case is that the dielectric function is a tensor, which makes the manipulations more involved.

The first requirement is a generalization of Eq. (9), the definition of the dielectric function, to include electromagnetic waves. The definition to be adopted here is that which produces the simplest scaling properties. This turns out to be defining a dielectric tensor $\epsilon_{ij}(\vec{k}, \omega)$ by

$$E_{e,i} = \epsilon_{ij}(\vec{k}, \omega) E_{t,j} \quad (34)$$

for fields with space-time dependence $\exp(-i\omega t + i\vec{k} \cdot \vec{x})$; the dielectric tensor will be denoted $\vec{\epsilon}$ henceforth. In terms of the conductivity tensor $\vec{\sigma}$, defined by $\vec{J}_p = \vec{\sigma} \cdot \vec{E}_p$, ($\vec{J}_p$ is the plasma current),

$$\vec{\epsilon} = \vec{1} + \frac{4\pi i}{\omega} \vec{P} + \frac{4\pi i}{\omega} \frac{c^2 k^2}{\omega^2 - c^2 k^2} \vec{P} \cdot \vec{P} \quad (35)$$

where $\vec{1}$ is the identity tensor and $\vec{P} = \vec{1} - \hat{k}\hat{k}$ is the transverse projection operator. (The product $\vec{A} \cdot \vec{B}$ means, in terms of components, $(\vec{A} \cdot \vec{B})_{ij} = A_{ir} B_{rj}$.)

The energy of a wave is most simply expressed in terms of the dispersion tensor¹⁰ $\vec{D}$:

$$\vec{D} = \vec{1} - \frac{c^2 k^2}{\omega^2} \vec{P} + \frac{4\pi i}{\omega} \vec{P} \cdot \vec{P} \quad (36)$$

For a wave $\vec{E} = \vec{E} \exp(-i\omega t + \vec{k} \cdot \vec{x})$ plus complex conjugate, the wave energy is¹⁰

$$W = \frac{\omega}{4\pi} \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial \omega} \cdot \vec{E} \quad (37)$$

provided that the anti-hermitean part of $\vec{D}$ is small, and that $\vec{D} \cdot \vec{E} = 0$, i.e., the electric field $\vec{E}$ is a normal mode. This expression for W includes the electric and magnetic field energy as well as the particle kinetic energy. The relation between $\vec{D}$ and $\vec{\epsilon}$ will be required; it is

$$\vec{D} = \left(\vec{1} - \frac{c^2 k^2}{\omega^2} \vec{P} \right) \cdot \vec{\epsilon} \quad (38)$$

The determinant of $(\vec{1} - c^2 k^2 \vec{P} / \omega^2)$ is $(1 - c^2 k^2 / \omega^2)^2$, which vanishes only for a light wave in a vacuum. Thus, $\vec{D} \cdot \vec{E} = 0$ implies $\vec{\epsilon} \cdot \vec{E} = 0$, and vice versa; i.e., the normal modes may be calculated either from $\det \vec{D} = 0$ or $\det \vec{\epsilon} = 0$.

The derivation of a theorem analogous to Eq. (6) is a simple generalization of the derivation given in Sec. II and will be omitted. The result is that a relation similar to Eq. (6) is true for each component of $\vec{D}$; e.g., for a homogeneous, collisionless, unmagnetized plasma,

$$\left(\omega \frac{\partial}{\partial \omega} + \vec{k} \cdot \frac{\partial}{\partial \vec{k}} \right) D_{ij} = 2 (D_{ij} - \delta_{ij} D) \quad (39)$$

Using Eq. (38), the analogous result for $\vec{D}$ is

$$\left(\omega \frac{\partial}{\partial \omega} + \vec{k} \cdot \frac{\partial}{\partial \vec{k}} \right) \vec{D} = 2 \left(\vec{D} - \frac{c^2 k^2}{\omega^2} \vec{P} \cdot \vec{D} \right) \quad (40)$$

To extract some information from this concerning wave energy, note that a normal mode $\vec{E}(\vec{k}, \omega)$ satisfies $\vec{D}(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega) = 0$. (Dependence on the parameters $\vec{B}_0, \nu, L$, etc., may easily be included; for brevity this will be omitted.) Now consider varying ω and $\vec{k}$ (and the parameters $\vec{B}_0, \nu, L$, etc.,) in such a way that this equation is maintained, i.e.,

$$\left(\omega \frac{\partial \vec{D}}{\partial \omega} + \vec{k} \cdot \frac{\partial \vec{D}}{\partial \vec{k}} \right) \cdot \vec{E} + \vec{D} \cdot \delta \vec{E} = 0 \quad (41)$$

Assuming that the anti-hermitean part of $\vec{D}$ is small, multiplying this tensor equation on the left by $\vec{E}^*$ gives

$$\delta \omega \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial \omega} \cdot \vec{E} + \delta k_i \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial k_i} \cdot \vec{E} = 0 \quad (42)$$

Thus

$$\frac{d\omega}{dk_i} = - \frac{\vec{E}^* \cdot (\partial \vec{D} / \partial k_i) \cdot \vec{E}}{\vec{E}^* \cdot (\partial \vec{D} / \partial \omega) \cdot \vec{E}} \quad (43)$$

where the derivative $d\omega/dk_1$ is taken at fixed $\vec{B}_0, \nu, L$, etc., and with $\text{det} \vec{D} = 0$. Using this result in Eq. (40) one finds, provided $\vec{D} \cdot \vec{E} = 0$.

$$\omega \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial \omega} \cdot \vec{E} = \frac{2 \vec{E}^* \cdot [\vec{D} - (c^2 k^2 / \omega^2) \vec{P}] \cdot \vec{E}}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k})} \quad (44)$$

[It is possible to give a generalization of Eq. (44), analogous to Eq. (3), to the case $\vec{D} \cdot \vec{E} \neq 0$. This does not seem very useful, however, and will be omitted.] For magnetized, collisional, weakly inhomogeneous plasmas, the denominator here would be replaced by the denominator in Eq. (3). For a transverse wave ($\vec{P} \cdot \vec{E} = \vec{E}$) the result is simplified:

$$\omega \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial \omega} \cdot \vec{E} = \frac{2[1 - (c^2 k^2 / \omega^2)] |\vec{E}|^2}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k})} \quad (45)$$

In general, using Faraday's law, Eq. (44) may be rewritten

$$\omega \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial \omega} \cdot \vec{E} = \frac{2(|\vec{E}|^2 - |\vec{B}|^2)}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k})} \quad (46)$$

The final result, valid for an electromagnetic wave in a magnetized, collisional, weakly inhomogeneous plasma is that the energy of a wave is

$$W = \frac{1}{4\pi} \frac{\langle \vec{E}^2 \rangle - \langle \vec{B}^2 \rangle}{1 - (\vec{k}/\omega) \cdot (d\omega/d\vec{k}) - (\vec{B}_0/\omega) \cdot (d\omega/d\vec{B}_0) - (\nu/\omega) (d\omega/d\nu) + (L/\omega) (d\omega/dL)} \quad (47)$$

In this equation, $\vec{B}_0$ is the applied magnetic field, ν is the collision frequency [see Eq. (22) for a more precise definition], and L is a length scale characterizing the inhomogeneity. $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields of the wave; the space-time dependence

is $\vec{E} = \vec{E}(\vec{x}, t) \exp(-i\omega t + i\vec{k} \cdot \vec{x})$ plus complex conjugate, where $\vec{E}$ is slowly varying, with a similar form for $\vec{B}$. The angular bracket means a time average over the rapid variation of the exponential phase factor. For electrostatic waves $\vec{B} = 0$; for transverse waves $\vec{B}^2 = (c^2 k^2 / \omega^2) \cdot \vec{E}^2$.

V. APPLICATIONS

This section presents a more or less random sample of applications of Eqs. (3) and (47), as well as a warning concerning the use of these results when supposedly small terms have been neglected in the derivation of dispersion relations.

As a first example, consider the transverse electromagnetic wave in a cold homogeneous unmagnetized plasma, whose dispersion relation is $\omega^2 = \omega_p^2 + k^2 c^2$. Equation (45) applies, because the wave is transverse. Now $(\vec{k}/\omega) \cdot d\omega/d\vec{k} = c^2 k^2 / \omega^2$; thus $W = 2 \langle \vec{E}^2 \rangle / 8\pi$, which is indeed correct. Obviously, half the energy of this wave is in the electric field; the remainder is in the particles and the magnetic field.

Next, consider electron Bernstein waves in a low density plasma ($\omega_{pe}^2 \ll \Omega_e^2$). The dispersion relation for these electrostatic waves is $\omega_n^2 = n^2 \Omega_e^2 + f_n(k\rho)$, where n is a positive integer, ρ is the electron gyroradius, and $f_n(z) = 2n^2 (\omega_{pe}^2 / z^2) I_n(z^2) \exp(-z^2)$. Using Eq. (3), these waves have energy

$$W_n = \frac{\langle \vec{E}_n^2 \rangle}{4\pi} \frac{\omega_n^2}{\omega_n^2 - n^2 \Omega_e^2} \quad (48)$$

These waves all have positive energy, since $\epsilon_n > 0$. For small values of (k_y) , the denominator in Eq. (48) scales as $(k_y)^{2(n-1)}$. Thus, if the W_n are all of comparable magnitude, the associated electric fields scale as $E_n \sim (k_y)^{n-1}$.

The electrostatic drift wave is an example where the neglect of apparently small terms may cause problems. Consider a plasma slab, with cold ions and warm fluid electrons, with a density gradient in the X direction and an applied magnetic field in the Z direction. For drift waves with $k_z \ll k_y$ (but $\omega/k_z < v_t$, the electron thermal velocity),

$$\omega = - \frac{\omega_{pi}^2 \lambda_D^2 k_y}{1 + k_y^2 (\rho^2 + \lambda_D^2)} \frac{r}{\Omega_i} \quad (49)$$

where ω_{pi} is the ion-plasma frequency, λ_D the electron Debye length, Ω_i the ion-cyclotron frequency, $\rho = v_t/\omega$, and $r = 0 \ln n/\partial x$ is the inverse of the density gradient scale length.¹² Normally, $\rho^2 \ll 1$, and the λ_D^2 term in the denominator is usually omitted. This corresponds to neglecting the vacuum contribution to the electron dielectric function $\epsilon_e = 1 + 1/(k_y \lambda_D)^2$, i.e., neglecting the 1. Evaluating the energy of this wave using Eq. (3), one finds

$$W = \left\{ 1 + \frac{1}{(k_y \lambda_D)^2} + \frac{\omega_{pi}^2}{\Omega_i^2} \right\} \frac{\langle \tilde{E}^2 \rangle}{8\pi} \quad (50)$$

If the $(k_y \lambda_D)^2$ term in the denominator of Eq. (49) were neglected, W would apparently be infinite! The interpretation of the three terms in Eq. (50) is straightforward: the ω_{pi}^2/Ω_i^2 term is the kinetic energy of the ions due to their $\hat{E} \times \hat{B}$ drift, the $1/(k_y \lambda_D)^2$ term is the electron kinetic energy due to motion parallel to the magnetic field,¹³

and the remaining term is the electric field contribution. Alternatively, the first two terms may be lumped into $\epsilon_e \langle \vec{E}^2 \rangle / 8\pi$, which is the energy required to polarize the electron fluid, considered as a nondispersive (ϵ_e independent of ω) medium.

The vanishing of the denominator of Eq. (3) for the drift wave example when $k_{\perp} v_D$ is neglected raises an interesting issue: how should Eq. (3) be interpreted when the denominator is very small or zero? Taking the attitude that the energy W must be finite (for example, of order Boltzmann's constant times temperature), if the denominator is small or zero, then the electric field $\vec{E}$ associated with such a wave must also be small or zero. One example is the long-wavelength ion-acoustic wave, for which $(k/\omega) d\omega/dk \approx 1$. Another example is the electron Bernstein wave mentioned previously, when $kp \ll 1$.

The results of this paper may be used to give a simple test for whether a wave is a negative energy wave. In view of Eq. (47), the criterion will depend on whether the wave is "electric" ($\langle \vec{E}^2 \rangle > \langle \vec{B}^2 \rangle$) or "magnetic" ($\langle \vec{B}^2 \rangle > \langle \vec{E}^2 \rangle$). For an electric wave, the wave has negative energy if

$$\omega < \vec{k} \cdot \frac{d\omega}{d\vec{k}} + \vec{B}_0 \cdot \frac{d\omega}{d\vec{B}_0} + \frac{v}{\omega} \frac{d\omega}{dv} - \frac{L}{\omega} \frac{d\omega}{dL} ; \quad (51)$$

for a magnetic wave the inequality must be reversed. For electrostatic waves in a homogeneous, unmagnetized, collisionless plasma, this implies that a wave has negative energy if the group velocity is greater than the phase velocity.

One final example, of some conceptual interest but dubious practical value, is the application of these results to sum rules based on the Kramers-Kronig relations.³ For sufficiently high frequency, the dielectric constant of any plasma must behave as $\epsilon \sim 1 - \omega_p^2/\omega^2$.¹⁴ For a collisionless, unmagnetized homogeneous plasma, Eq. (6) makes it possible to extend this as an asymptotic series in powers of $1/\omega$. Specifically, suppose that ϵ has the asymptotic expansion

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{a_3(k)}{\omega^3} - \frac{a_4(k)}{\omega^4} \dots \quad (52)$$

Equation (6) implies that $a_n(k) = \alpha_n k^{n-2}$. The coefficients α_n are not known a priori. However, comparing Eq. (52) with Eq. (4), when $f_0(\vec{v}) = f_0(-\vec{v})$ one finds for example that $\alpha_3 = 0$, $\alpha_4 = 3\langle v^2 \rangle$. If there are no unstable modes, $1/\epsilon(\omega)$ is analytic in the upper-half plane.¹⁵ Thus, a Kramers-Kronig relation exists for $1/\epsilon$:

$$\text{Re} \left\{ \frac{1}{\epsilon(\omega)} - 1 \right\} = \frac{1}{\pi} P \int_{-\infty}^{+\infty} d\omega' \frac{\omega'}{\omega'^2 - \omega^2} \text{Im} \frac{1}{\epsilon(\omega')} \quad (53)$$

Expanding both sides in an asymptotic series in powers of $1/\omega$, and comparing equal powers on either side, a series of sum rules emerge. The lowest sum rule is well known:³

$$\omega_p^2 = - \frac{2}{\pi} \int_0^\infty d\omega \omega \text{Im} \frac{1}{\epsilon(k, \omega)} \quad (54)$$

The next sum rule is

$$\omega_p^2 (\omega_p^2 + 3k^2 v^2) = -\frac{2}{\pi} \int_0^{+\infty} d\omega \, \omega^3 \operatorname{Im} \epsilon(k, \omega) \quad (55)$$

If there is a weakly damped mode at $\omega = \omega_0(k) - i\gamma$, Eq. (8) may be used to show that the contribution of this mode to the first sum rule is $\omega_0^2 [1 - (k/\omega_0) (d\omega_0/dk)]$, and to the second sum rule is $\omega_0^4 [1 - (k/\omega_0) (d\omega_0/dk)]$. The Langmuir wave at $\omega_0^2 = \omega_p^2 + 3k^2 v^2$ therefore saturates these sum rules for small k , i.e., the entire integral comes from this mode. (The integrand is negative definite, so there are no cancellations). Indeed, if it were known a priori that a single wave saturated these sum rules, the dispersion relation $\omega_0^2 = \omega_p^2 + 3k^2 v^2 + \dots$ could be deduced from these sum rules. This is clearly not the most efficacious means to derive the Langmuir wave dispersion relation. However, it does demonstrate that the analytic properties of $\epsilon(k, \omega)$ imply the existence of relations between the high frequency ($\omega \gg \omega_p$) properties of the plasma and the low frequency ($\omega \sim \omega_p$) normal modes. One final remark in this connection is in order: the ion-acoustic waves do not contribute significantly to these sum rules, since they obey $(k/v) (d\omega/dk) \ll 1$.

In conclusion, this paper has demonstrated a simple relation between the energy of a wave and its dispersion relation. This relation is a consequence of a simple invariance property - essentially dimensional analysis - of the linear Fokker-Planck equation, together with the linear dependence on density of the plasma generated field.

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APPENDIX A

The purpose of this appendix is to evaluate the expression Eq. (30) for the response function $K(\vec{k}, \omega, \vec{\Omega}, \tau)$. The task is to evaluate the operator expression $\exp(-O\tau) = \exp i(\omega - k_{\parallel} v_{\parallel}) \exp(A+B)$, where $B = -ik_{\perp} \cdot \vec{v}_{\perp} \tau$ and $A = -\tau \vec{v} \times \vec{\Omega}_O \cdot \partial / \partial \vec{v}$, and the subscripts $\parallel$ and $\perp$ mean parallel and perpendicular to $\vec{\Omega}_O$. [Note that $(\omega - k_{\parallel} v_{\parallel})$ commutes with A and B, so it may be factored out as given, but A does not commute with B]. This will be accomplished using Feynman's theorem¹⁶

$$e^{A+B} = T_Z^{-1} \left(\exp \int_0^1 dz e^{zA} B e^{-zA} \right) e^A, \quad (A1)$$

where T_Z^{-1} stands for the operation of ordering the terms in the power series expansion of the exponential so that terms with smaller values of Z are to the left of terms with larger values of Z . Fortunately, this ordering turns out to be unnecessary here. In order to evaluate the integral in the exponential the following relation is required:¹⁷

$$e^{zA} B e^{-zA} = B + z[A, B] + \frac{z^2}{2!} [A, [A, B]] + \dots \quad (A2)$$

where $[A, B] \equiv AB - BA$. With A and B as defined previously

$$e^{zA} B e^{-zA} = -i\tau [\cos(\theta z) k_{\perp} \cdot v_{\perp} - \sin(\theta z) \vec{k}_{\perp} \cdot (\vec{v} \times \hat{\Omega}_O)] \quad (A3)$$

where $\theta = \Omega_O \tau$ and $\hat{\Omega}_O = \vec{\Omega}_O / \Omega_O$. Note that the two terms on the right side of Eq. (A3) contain no derivatives, and hence they commute with each other. This makes the Z -ordering operation in

Eq. (A1) unnecessary. Performing the z integration in Eq. (A1) one finds

$$e^{A+B} = \exp \left\{ -\frac{i}{\Omega_0} [\sin \theta \mathbf{k}_\perp \cdot \mathbf{v}_\perp - (1 - \cos \theta) \hat{\mathbf{k}}_\perp \cdot (\vec{v} \times \hat{\mathbf{n}}_0)] \right\} e^A. \quad (A4)$$

Finally, using Eq. (A2) again, it may be shown that $\exp(A)$ is a rotation operator, which rotates vectors by angle θ counter-clockwise about $\hat{\mathbf{n}}_0$:

$$e^A f(v_\parallel, \vec{v}_\perp) = f(v_\parallel, \cos \theta \vec{v}_\perp - \sin \theta \vec{v}_\perp \times \hat{\mathbf{n}}_0). \quad (A5)$$

Using these relations, $K(\vec{k}, \omega, \hat{\mathbf{n}}_0; \tau)$ may be evaluated. For example, if f_0 is a Maxwellian with equal perpendicular and parallel temperature T , then

$$K(\vec{k}, \omega, \hat{\mathbf{n}}_0; \tau) = \frac{1}{k^2 \lambda_D^2} e^{i\omega\tau} \frac{d}{d\tau} \left\{ \exp - \frac{k_\parallel^2 v_t^2 \tau^2}{2} - k_\perp^2 \rho^2 (1 - \cos \theta) \right\} \quad (A6)$$

where $v_t^2 = T/m$, $\rho^2 = v_t^2 / \Omega_0^2$, $\theta = \Omega_0 \tau$. Using Eq. (32), the dielectric function is

$$\epsilon(\vec{k}, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} \left[1 + i\omega \int_0^\infty d\tau \exp \left(i\omega\tau - \frac{k_\parallel^2 v_t^2 \tau^2}{2} - k_\perp^2 \rho^2 (1 - \cos \theta) \right) \right] \quad (A7)$$

which agrees with the Gordeyev integral representation¹⁸ of $\epsilon(\vec{k}, \omega)$.

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