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BOLTZMANN APPROACH TO CONDUCTIVITY CALCULATIONS

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The basic set of equations is derived for the calculation of the time-dependent electrical conductivity of an initially unionized gas generated by a propagating relativistic electron beam. In contrast with previous work, which used thermal averages, conductivity is determined from the complete, non-equilibrium electron energy distribution. The evolution of the distribution function, resulting from interaction with gas molecules, ions, beam induced electric field, and beam current, is described by a Boltzmann equation.

E. B.

1. INTRODUCTION

This report is concerned with the calculation of conductivity generated by a propagating relativistic electron beam in a gaseous medium. In this calculational scheme, conductivity is determined from the detailed electron energy distribution, which results from the interaction of the plasma with the electromagnetic fields and with the gaseous medium at the atomic and molecular level. The evolution of the plasma is described by a Boltzmann equation. While the general theoretical framework is well known,^{1,2} the purpose of this report is to derive the basic formulae as they apply to the relativistic electron beam (REB) problem. Survey of existing atomic data and numerical results are not within the scope of this work, but will be the subject of future reports.

2. BOLTZMANN EQUATION

In the Boltzmann description of a plasma, the basic quantity of interest is a distribution f which is a function of seven variables (the position $\vec{r}$, velocity $\vec{v}$, and time t). $f(\vec{r}, \vec{v}, t) d^3r d^3v$ is the number of plasma electrons in the six-dimensional phase space $d^3r d^3v$. In general, the distribution function f satisfies a Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{e}{m} \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \cdot \vec{\nabla}_v f = L(\vec{r}, \vec{v}, t), \quad (1)$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, and e and m are the electron charge and mass respectively; c is the speed of light. The operator $\vec{\nabla}_v$ is defined by

$$\vec{A} \cdot \vec{\nabla}_v f \equiv A_x \frac{\partial f}{\partial v_x} + A_y \frac{\partial f}{\partial v_y} + A_z \frac{\partial f}{\partial v_z}$$

where $\vec{A}$ is any vector.

$L(\bar{r}, \bar{v}, t)$ describes all atomic processes which affect the distribution function, and will be treated in more detail later.

The plasma density, current and temperature are expressible as different weighted averages of the distribution function. The plasma density is given by

$$n_e(\bar{r}, t) = \int f(\bar{r}, \bar{v}, t) d^3v \quad (2)$$

while the plasma current density is

$$\bar{J}_e(\bar{r}, t) = e \int \bar{v} f(\bar{r}, \bar{v}, t) d^3v \quad (3)$$

The plasma temperature is defined to be 2/3 of the average electron energy $\bar{\epsilon}(\bar{r}, t)$, where

$$n_e \bar{\epsilon} = \int \frac{mv^2}{2} f(\bar{r}, \bar{v}, t) d^3v \quad (4)$$

3. ATOMIC PROCESSES

To complete the Boltzmann equation, an explicit form must be written down for $L(\bar{r}, \bar{v}, t)$, the atomic collisions operator. In a gaseous medium such as air, many different processes can contribute to $L(\bar{r}, \bar{v}, t)$. The more important ones are listed below.

1. Direct ionization of the gas by beam particles. This process constitutes one of the major sources of electrons.
2. Recombination of electrons with ions. This is a key channel for electron loss. Three modes of recombination are possible and must all be taken into account.

a) Dissociative recombination

where XY^+ is a molecular ion.

- b) Three-body recombination--neutral third body

where M is a neutral atom or molecule

- c) Three-body recombination--electron as third body

3. Attachment of electrons to neutrals. This is another electron loss mechanism. The dissociative and three-body attachment to O_2 are examples of this process.

4. Discrete excitations. While discrete excitations do not result in a net change in electron number, energy is lost, leading to a rearrangement of the distribution function. The important modes of excitation are as follows:

- a) Electronic excitation
- b) Dissociative excitation
- c) Vibrations
- d) Rotations

5. Elastic collisions. While elastic collisions result in little energy loss, momentum is transferred. This again results in rearrangement of the distribution function.

6. Ionization by plasma electrons. In this process a plasma electron loses energy in the collision, and at the same time creates an additional electron. This process is both a mechanism for energy loss as well as an electron source.

7. Ion-neutral collisions. To determine the recombination rates described in 2, the population of ions must first be known. Thus, the history of ion-neutral collisions must be followed.

The above list is not meant to be exhaustive. Additional processes of importance will be included as we become aware of them.

The atomic collisions operator $L(\vec{r}, \vec{v}, t)$ represents the aggregate effect of all these processes.

$$L(\vec{r}, \vec{v}, t) = \sum_i L^{(i)}(\vec{r}, \vec{v}, t) \quad (5)$$

where the sum is performed over all atomic processes and molecular species.

While the number of terms in the sum may be many, they can be grouped into three different categories.

A) The source term, described in 1 above has the form

$$L^{\text{source}} = (J_b/e)n_g \frac{d^3 \sigma^{\text{ion}}}{dv^3} (T, \vec{v}) \quad (6)$$

where J_b is the beam current density, and n_g is the gas density. The total ionization cross section for a relativistic electron with kinetic energy T is

$$\sigma^{\text{ion}}(T) = \int \frac{d^3 \sigma^{\text{ion}}}{dv^3} d^3 v \quad (7)$$

B) Recombination and attachment, described in 2 and 3 above are electron loss channels. Each loss term has the form

$$L^{\text{loss}} = - f(\bar{r}, \bar{v}, t) n_i v_i \quad (8)$$

n_i represents the density of the ion in question in the case of recombination, and represents the density of O_2 in the case of O_2 attachment. v_i is the collision rate per ion/molecule for recombination/attachment. In two-body processes (dissociative recombination and dissociative attachment),

$$v_i = v\sigma \quad (9)$$

where v is the speed of the electron and σ is the cross section for the process. In three-body processes, v_i is proportional to the density of the third body.

$$v_i = v^{\text{three-body}} n_M \quad (10)$$

$v^{\text{three-body}}$ is a rate coefficient for the three-body process and has the dimensions of $\text{cm}^6 - \text{sec}^{-1}$.

C) The excitation processes, elastic collisions, and ionization by plasma electrons, described in 4 to 6 above, all result in rearrangement of the distribution function. The rearrangement terms have the form

$$L^{\text{rearr}} = -f(\bar{v}) n_g v \sigma(v) + n_g \int v' \frac{d^3\sigma}{dv^3}(\bar{v}', \bar{v}) f(\bar{v}') d^3v' \quad (11)$$

$\frac{d^3\sigma}{dv^3}(\bar{v}', \bar{v})$ is the differential cross section for a collision process with an initial electron of velocity $\bar{v}'$ and a final electron of velocity $\bar{v}$. For excitation and elastic collisions, the speed v is related to v' by energy conservation. Thus $\frac{d^3\sigma}{dv^3}(\bar{v}', \bar{v})$ contains some implicit energy conserving delta functions. The usual total cross sections for excitation and elastic collisions are retrieved by integrating over all final velocities.

$$\sigma(v') = \int \frac{d^3\sigma}{dv^3}(\bar{v}', \bar{v}) d^3v \quad (12)$$

It is clear that for excitation and elastic collisions

$$\int L^{\text{rearr}} d^3v = 0$$

i.e., these processes produce no net change in electron number.

In ionizing collisions, the final electron with velocity v could be either the primary electron or the secondary electron. Therefore for ionization, we have

$$2\sigma(v') = \int \frac{d^3\sigma}{dv^3}(\bar{v}', \bar{v}) d^3v \quad (13)$$

The factor of 2 is a consequence of the fact that each ionization collision results in two electrons. The net change in electron number due to ionization is nonzero.

Finally, to derive a close set of equations, we need rate equations for the production and loss of ions. These equations have the structure

$$\begin{aligned} \frac{dn_i}{dt} = & \sum_j \alpha_{ij} n_j - n_i \sum_j \alpha_{ji} - n_i \sum_k \beta_{ik} n_k^- + \left(\frac{j_b}{e} \right) n_g \sigma_i(T) \\ & + n_g \int f(\vec{v}) v_i^{(ion)} d^3v - n_i \int f(\vec{v}) v_i^{(recom)} d^3v \end{aligned} \quad (14)$$

The rate equation for the i th positive ion (with density n_i) consists of five types of contributions. The first two terms on the right hand side describe ion-neutral interactions. α_{ij} is the rate coefficient for the conversion of the j^{th} ion species to the i^{th} species. In general α_{ij} is proportional to the density of the neutral reactant. In the case of three-body interactions, α_{ij} is in addition, proportional to the density of the third body. The third term describes positive ion--negative ion recombination, with n_k^- the density of the k^{th} negative ion and β_{ik} , the rate coefficient for ion-ion recombination. The fourth and the fifth terms correspond to the production of the ion by direct ionization by beam electrons and ionization by plasma electrons respectively. The final term is the loss from recombination.

A similar rate equation could be written down for the negative ions. The last three terms in Eq. (14), describing electron-positive ion relations are replaced by the rate for the creation of negative ions by attachment

$$n_g \int f(\vec{v}) v_i^{(att)} d^3v .$$

4. APPROXIMATIONS FOR REB

Up to this point, to the extent that the atomic collisions operator could be determined, the Boltzmann equation Eq. (1) is exact. However, Eq. (1) is a seven-dimensional equation which is much too difficult to handle numerically. For the relativistic electron beam problems of interest, three approximations can generally be made:

1. Negligible spatial diffusion. The second term on the LHS of Eq. (1), $\vec{v} \cdot \vec{\nabla} f$, describes the effect of spatial diffusion. If the problems of interest take place in a very short time (e.g. during the passage of a beam with short pulse time τ_p), the plasma electrons do not have sufficient time to diffuse out; therefore, the diffusion term can be ignored.

2. Small magnetic force. The third term on the LHS of Eq. (1) describes the effect of diffusion in velocity space under the influence of an external force. The force has an electric and magnetic component. For the problems of interest, the magnetic term is generally small compared to the electric term. One reason for the smallness of the magnetic term is that we are looking at low energy electrons, for which $\vec{v}/c$ is small. Combining approximations 1 and 2, we can rewrite the Boltzmann equation as

$$\frac{\partial f}{\partial t} + \frac{e}{m} \vec{E} \cdot \vec{\nabla}_v f = L(\vec{r}, \vec{v}, t) \quad (15)$$

In this simplified version of the Boltzmann equation, the position $\vec{r}$ enters only as an external parameter. Equation (15) is therefore essentially four-dimensional.

3. Near isotropy, or weak field approximation. The assumption here is that the distribution function is nearly isotropic in velocity space. The distribution function cannot be completely isotropic

since this would preclude a nonzero plasma current. Near isotropy is equivalent to the statement that the drift velocity is small compared to the thermal velocity.

In the REB problem, the electron source is basically isotropic. Thus, the anisotropy comes from the driving electric field. A small drift velocity implies that the electric field cannot be too strong. Thus, the near isotropy approximation is also a weak field approximation.

The near isotropy assumption is basic to the Legendre expansion which is to follow. While the first two assumptions could be "patched up" in situations where these approximations are not valid, the third assumption of near isotropy is fundamental to the present approach. When it fails, all the analyses that will follow become invalid.

However, insofar as the above three approximations hold, our analysis is general. Thus the conductivity which will be derived can be coupled to either one-dimensional or three-dimensional codes for fields and currents, and is valid both on and off beam axis.

5. EXPANSION IN LEGENDRE FUNCTIONS

By dropping the spatial diffusion term and the magnetic term, the Boltzmann Equation is somewhat simplified. However, the resultant four-dimensional equation is still rather complicated. Our next attempt is to reduce the four-dimensional equation to two-dimensional, with time t and the electron energy ϵ as the two independent variables. To provide the ansatz for making the near isotropy approximation, it is necessary to first recast Eq. (15) into a different form. The approach is to perform a Legendre function decomposition for the distribution function.

Let (v, θ, ϕ) be the spherical polar coordinates of the vector $\vec{v}$. The polar axis is defined to lie along the electric field E . To a good approximation, the distribution function is azimuthally symmetric about this axis. The beam current J_b which creates plasma electrons by direct ionization could in principle break the azimuthal symmetry if J_b is not parallel to E . However, the ejection of slow secondary electrons from the ionization process is by and large isotropic. The direction of the beam current therefore has little effect on the distribution function. By assuming ϕ independence, we can write the distribution as a sum of Legendre functions. This is possible because the Legendre functions are a complete set of functions. Thus, we have

$$f(\vec{v}) = \frac{1}{4\pi} \sum_{\ell=0} f_{\ell}(v) P_{\ell}(\cos\theta) . \quad (16)$$

The Legendre coefficients f_{ℓ} are independent of angle and depend only on v , the magnitude of the velocity vector.

The Legendre functions are an infinite set of polynomials, the first few of which are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2} (3x^2 - 1) . \end{aligned} \quad (17)$$

The Legendre functions are orthogonal (but not orthonormal) satisfying

$$\int P_{\ell}(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell\ell'} . \quad (18)$$

Using Eqs. (2), (16), (17) and (18), we can express the plasma particle density in terms of the zeroth Legendre coefficient only.

$$\begin{aligned} n_e &= \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \left[\int f_{\ell}(v) v^2 dv \right] \left[\int P_{\ell}(\cos \theta) d\Omega \right] \\ &= \frac{1}{2} \sum_{\ell=0}^{\infty} \left[\int f_{\ell}(v) v^2 dv \right] \left[\int P_{\ell}(\cos \theta) P_0(\cos \theta) d\cos \theta \right] = \int f_0(v) v^2 dv \end{aligned} \quad (19)$$

Similarly, the average electron energy is given in terms of f_0 while the plasma current is related to f_1 .

$$\begin{aligned} n_e \bar{\epsilon} &= \frac{m}{2} \int f_0(v) v^4 dv \\ J_e &= (J_e)_z = \frac{e}{3} \int f_1(v) v^3 dv \end{aligned} \quad (20)$$

The next step is to make a Legendre function decomposition for each term in the Boltzmann Equation of Eq. (15). Since in atomic collisions, the electron energy $\epsilon = \frac{mv^2}{2}$ is more conventional as the independent variable than v , the Legendre function representation of the Boltzmann Equation is more conveniently expressed in terms of ϵ and corresponding scaled Legendre coefficients

$$\bar{f}_{\ell}(\epsilon) \equiv f_{\ell}(v) v^2 \frac{dv}{d\epsilon} = f_{\ell}(v) \left(\frac{v}{m} \right). \quad (22)$$

The first term on the LHS of Eq. (15) is easiest to decompose since that is just

$$\frac{\partial f}{\partial t} = \sum_{\ell=0}^{\infty} \left[\frac{1}{4\pi} \left(\frac{m}{v} \right) \frac{\partial \bar{f}_{\ell}}{\partial t} \right] P_{\ell}(\cos \theta). \quad (23)$$

The other terms require more work.

The Electric Term

We wish now to decompose the electric term into an infinite series of Legendre functions

$$\frac{e}{m} \bar{E} \cdot \bar{\nabla}_v f = \frac{e}{m} E \frac{\partial f}{\partial v_z} = \sum_{\ell=0}^{\infty} g_{\ell} P_{\ell}(\cos \theta). \quad (24)$$

The ∇_v operator is first transformed from Cartesian to polar coordinates

$$\frac{\partial f}{\partial v_z} = \cos \theta \frac{\partial f}{\partial v} + \frac{\sin^2 \theta}{v} \frac{\partial f}{\partial \cos \theta} \quad (25)$$

Reduction of the angular components could be performed by means of two identities involving Legendre functions.

$$\cos \theta P_\ell(\cos \theta) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(\cos \theta) + \frac{\ell}{2\ell+1} P_{\ell-1}(\cos \theta) \quad (26)$$

$$\sin^2 \theta \frac{dP_\ell(\cos \theta)}{d \cos \theta} = \frac{\ell(\ell+1)}{(2\ell+1)} \left[P_{\ell-1}(\cos \theta) - P_{\ell+1}(\cos \theta) \right] \quad (27)$$

After some algebra, we find that the coefficient of the electric term is

$$g_\ell = \frac{eEm}{4\pi} \left\{ \left(\frac{\ell}{2\ell-1} \right) \epsilon^{\ell/2} \frac{d}{d\epsilon} \left(\frac{\bar{F}_{\ell-1}}{\epsilon^{\ell/2}} \right) + \left(\frac{\ell+1}{2\ell+3} \right) \frac{1}{\epsilon^{(\ell+1)/2}} \frac{d}{d\epsilon} \left(\epsilon^{(\ell+1)/2} \bar{F}_{\ell+1} \right) \right\} \quad (28)$$

The Atomic Collisions

Legendre function expansions of the various atomic collision terms must next be performed

$$L(i) = \sum_{\ell=0}^{\infty} L_\ell(i) \quad (29)$$

For the source term of Eq. (6), we will first rewrite the ionization cross section in terms of more conventional variables. The differential cross section in energy and solid angle is

$$\frac{d\sigma_{\text{ion}}}{d\epsilon d\Omega} d\epsilon d\Omega = \frac{d^3\sigma_{\text{ion}}}{dv^3} dv^3 \quad (30)$$

The source term is then

$$L^{\text{source}} = (J_b/e) n_g \left(\frac{m}{v} \right) \frac{d\sigma_{\text{ion}}}{d\epsilon d\Omega}$$

Making the approximation that the secondary electron spectrum is isotropic, the zeroth coefficient is the only nonvanishing term

$$L_{\ell}^{\text{source}} = \frac{1}{4\pi} \left(\frac{m}{v} \right) (j_b/e) n_g \frac{d\sigma^{\text{ion}}}{d\epsilon} \delta_{\ell 0} \quad (31)$$

The decomposition of the loss terms of Eq. (8) is simple because the only angular dependence comes in through the distribution function itself.

$$L_{\ell}^{\text{loss}} = - \frac{1}{4\pi} \left(\frac{m}{v} \right) \bar{F}_{\ell} n_i v_i \quad (32)$$

The rearrangement terms of Eq. (11) have two components, one representing loss, and the other representing gain. The loss component is identical in form to the recombination terms, and its decomposition is given by Eq. (32). The gain component is somewhat more complicated. We will next decompose this component

$$L^{\text{gain}} = n_g \int v' \frac{d^3 v'}{dv'^3} (\bar{v}' \cdot \bar{v}) f(\bar{v}') d^3 v' \quad (33)$$

into Legendre functions.

The gain term consists of an integral which is a function of the two vectors $\bar{v}$ (with polar coordinates (v, θ, ϕ)) and $\bar{v}'$ (with polar coordinates (v', θ', ϕ')) as shown in Figure 1. χ is the angle between the two vectors $\cos \chi = \bar{v} \cdot \bar{v}' / vv'$. The

differential cross section could in general be represented by Legendre functions of $\cos \chi$

$$\frac{d^3\sigma}{dv^3}(v', v) = \frac{m}{v} \frac{d\sigma}{d\epsilon} \frac{d\Omega}{d\Omega} = \frac{m}{v} \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{4\pi} \right) \frac{d\sigma_{\ell}}{d\epsilon} P_{\ell}(\cos \chi) \quad (34)$$

where

$$\frac{d\sigma_{\ell}}{d\epsilon}(\epsilon', \epsilon) \equiv \int \frac{d\sigma}{d\epsilon} \frac{d\Omega}{d\Omega} P_{\ell}(\cos \chi) d\Omega \quad (35)$$

$\frac{d\sigma}{d\epsilon}$ is a function of the incident energy ϵ' and the final energy ϵ , and is independent of the angle. In particular,

$$\frac{d\sigma_0}{d\epsilon} = \frac{d\sigma}{d\epsilon} \quad (36)$$

In order to perform the $\bar{v}'$ integral, we need to convert $P_{\ell}(\cos \chi)$ in Eq. (34) into functions that depend on θ' and ϕ' . This can be done by means of the Addition Theorem

$$P_{\ell}(\cos \chi) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (37)$$

where the $Y_{\ell m}$'s are spherical harmonics. Next, the distribution function $f(\bar{v}')$ in Eq. (33) is again cast into a sum over $P_{\ell}(\cos \theta')$. The angular part of the v' integral can now be performed by noting that

$$\begin{aligned} \int Y_{\ell m}^*(\theta', \phi') P_{\ell}(\cos \theta') d\Omega' &= \sqrt{\frac{4\pi}{2\ell+1}} \int Y_{\ell m}^*(\theta', \phi') Y_{\ell 0}(\theta', \phi') d\Omega' \\ &= \sqrt{\frac{4\pi}{2\ell+1}} \delta_{\ell\ell'} \delta_{m0} \end{aligned} \quad (38)$$

The net result of the angular integration is

$$\int v' \frac{d^3 \sigma}{dv^3} f(v') d^3 v' = \frac{1}{4\pi} \frac{m}{v} \sum_{\ell=0}^{\infty} \int \frac{v' d\sigma_{\ell}}{d\epsilon} (\epsilon' \epsilon) \bar{f}_{\ell}(\epsilon') d\epsilon' P_{\ell}(\cos \theta) \quad (39)$$

Thus the Legendre function decomposition of the rearrangement terms give

$$L_{\ell}^{\text{rearr}} = \frac{1}{4\pi} \left(\frac{m}{v} \right) \left(-\bar{f}_{\ell}(\epsilon) n_g v \sigma(\epsilon) + n_g \int \frac{v' d\sigma_{\ell}}{d\epsilon} (\epsilon' \epsilon) \bar{f}_{\ell}(\epsilon') d\epsilon' \right) \quad (40)$$

where $\frac{d\sigma_{\ell}}{d\epsilon}$ is defined in Eq. (35).

Equation (40) gives the general form of a rearrangement term. We will now consider the elastic collisions, excitations, and ionizations separately.

Elastic collisions result in no energy change. Therefore, we have

$$\frac{d\sigma_{\ell}}{d\epsilon} (\epsilon' \epsilon) = \sigma_{\ell}(\epsilon') \delta(\epsilon' - \epsilon) \quad (41)$$

where

$$\sigma_{\ell}(\epsilon) \equiv \int \frac{d\sigma(\epsilon)}{d\Omega} P_{\ell}(\cos \theta) d\Omega \quad (42)$$

Integrating over ϵ' , we get for the elastic collisions

$$L_{\ell}^{\text{el}}(\epsilon) = \frac{1}{4\pi} \left(\frac{m}{v} \right) \bar{f}_{\ell}(\epsilon) n_g v \left(\sigma_{\ell}^{\text{el}}(\epsilon) - \sigma(\epsilon) \right) \quad (43)$$

From Eq. (42), it is evident that $\sigma_0 = \sigma$. Therefore $L_0^{\text{el}} = 0$. This result reflects the fact that plasma electrons do not lose energy by elastic collisions.

For the excitation processes, we have

$$\frac{d\sigma_{\ell}}{d\epsilon} (\epsilon' \epsilon) = \sigma_{\ell}(\epsilon') \delta(\epsilon' - \epsilon - \epsilon_x) \quad (44)$$

where ϵ_x is the excitation energy. Integrating over energy ϵ' results in

$$L_{\ell}^{\text{ex}}(\epsilon) = \frac{1}{4\pi} \left(\frac{m}{v} \right) n_g \left(\bar{f}_{\ell}(\epsilon + \epsilon_x) \sqrt{\frac{2(\epsilon + \epsilon_x)}{m}} \sigma_{\ell}^{\text{ex}}(\epsilon + \epsilon_x) - \bar{f}_{\ell}(\epsilon) \sqrt{\frac{2\epsilon}{m}} \sigma_{\ell}^{\text{ex}}(\epsilon) \right) \quad (45)$$

Ionization processes do not involve discrete energy jumps, and therefore there are no delta functions in the differential cross sections. However, we need to take into consideration the fact that two electrons result from each ionization process. The ionization term is

$$L_{\ell}^{\text{ion}}(\epsilon) = \frac{1}{4\pi} \left(\frac{m}{v} \right) n_g \left\{ -\bar{f}_{\ell} v \sigma_{\ell}^{\text{ion}}(\epsilon) + \int v' \left[\frac{d\sigma_{\ell}^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon) + \frac{d\sigma_{\ell}^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon - \epsilon_I) \right] \bar{f}_{\ell}(\epsilon') d\epsilon' \right\} \quad (46)$$

where ϵ_I is the ionization threshold energy.

6. CONDUCTIVITY AND BOLTZMAN EQUATION IN ENERGY VARIABLE

The Legendre function expansion of the Boltzmann Equation of Eq. (15) is now complete. We may rewrite Eq. (15) in the form

$$\sum_{\ell=0}^{\infty} I_{\ell} P_{\ell}(\cos \theta) = 0 \quad (47)$$

using Eqs. (23), (24), (28), (31), (32), (43), (45), and (46). Since the Legendre polynomials are orthogonal, Eq. (47) implies that $I_{\ell} = 0$ for each ℓ . Writing this equation out explicitly, we have

$$\begin{aligned}
 \frac{\partial \bar{f}_\ell}{\partial t} + eEv \left\{ \left(\frac{\ell}{2\ell+1} \right) \epsilon^{\ell/2} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_{\ell-1}}{\epsilon^{\ell/2}} \right) + \left(\frac{\ell+1}{2\ell+3} \right) \frac{1}{\epsilon^{(\ell+1)/2}} \frac{\partial}{\partial \epsilon} \right. \\
 \left. \left(\epsilon^{(\ell+1)/2} \bar{f}_{\ell+1} \right) \right\} = \sum \left\{ n_g (J_b/e) \frac{d\sigma_{\ell}^{\text{ion}}}{d\epsilon} (T, \epsilon) \delta_{\ell 0} \right. \\
 - n_i \bar{f}_\ell v_i^{\text{loss}} + n_g \bar{f}_\ell v (\sigma_{\ell}^{e1} - \sigma^{e1}) \\
 + n_g \left(\bar{f}_\ell (\epsilon + \epsilon_x) \sqrt{\frac{2(\epsilon + \epsilon_x)}{m}} \sigma_{\ell}^{\text{ex}} (\epsilon + \epsilon_x) - \bar{f}_\ell (\epsilon) \sqrt{\frac{2\epsilon}{m}} \sigma^{\text{ex}} (\epsilon) \right) \\
 + n_g \left(\int d\epsilon' \bar{f}_\ell (\epsilon') v' \left[\frac{d\sigma_{\ell}^{\text{ion}}}{d\epsilon} (\epsilon' \epsilon) + \frac{d\sigma_{\ell}^{\text{ion}}}{d\epsilon} (\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] \right. \\
 \left. - \bar{f}_\ell (\epsilon) v \sigma_{\ell}^{\text{ion}} (\epsilon) \right\} \quad \ell = 0, 1, 2, \dots \quad (48)
 \end{aligned}$$

The five terms on the right hand side represent direct ionization by the beam, electron loss processes, elastic, excitation, and electron ionization processes respectively. The sum is performed over all relevant processes and all atomic species.

Equation (48) is equivalent in content to the Boltzmann equation (15). What we have done is to replace a single 4-D equation (in velocity and time) by an infinite hierarchy of coupled 2-D equations (in energy and time). Equation (48), in its exact form is certainly no simpler than Eq. (15). However, Eq. (48) provides us with an ansatz to make the near isotropy approximation to which we now turn.

We observe the following two features about Eq. (48). First, if the electric field were identically zero, there would be no coupling among different $\bar{f}_\ell$'s. Secondly, the source term is isotropic, and contributes

only to $\bar{f}_0$. It follows that in the $E = 0$ case, if the initial distribution function is isotropic, then the distribution function is isotropic at all times (i.e. $\bar{f}_\ell = 0$ for $\ell \neq 0$). If the electric field is slowly turned on, successive $\bar{f}_\ell$'s become important, with $\bar{f}_1$ being linearly proportional to E , and $\bar{f}_\ell$ proportional to E^ℓ in general. In a weak-field approximation, we assume that $\bar{f}_0$ and $\bar{f}_1$ are the only nonvanishing functions. This approximation truncates the infinite hierarchy to two equations with two unknowns. The $\ell = 0$ equation is

$$\begin{aligned} \frac{\partial \bar{f}_0}{\partial t} + \frac{eE}{3} \sqrt{\frac{2}{m}} \frac{\partial}{\partial \epsilon} (\epsilon^{1/2} \bar{f}_1) = & \sum \left\{ n_g \left(\frac{j_b}{e} \right) \frac{d\sigma^{\text{ion}}}{d\epsilon} (\epsilon, \epsilon) - n_i \bar{f}_0(\epsilon) v_i^{\text{loss}} \right. \\ & + n_g \left(\bar{f}_0(\epsilon + \epsilon_X) \sqrt{\frac{2(\epsilon + \epsilon_X)}{m}} \sigma^{\text{ex}}(\epsilon + \epsilon_X) - \bar{f}_0(\epsilon) \sqrt{\frac{2\epsilon}{m}} \sigma^{\text{ex}}(\epsilon) \right) \\ & + n_g \left(\int d\epsilon' \bar{f}_0(\epsilon') v' \left[\frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon) + \frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] \right. \\ & \left. \left. - \bar{f}_0(\epsilon) v \sigma^{\text{ion}}(\epsilon) \right) \right\} \quad (49) \end{aligned}$$

The $\ell = 1$ equation could be written as

$$\frac{\partial \bar{f}_1}{\partial t} + eE \sqrt{\frac{2}{m}} \epsilon \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) = - n_g \bar{f}_1 v_m + R \quad (50)$$

where

$$v_m = v \int (1 - \cos \theta) \frac{d\sigma^{\text{tot}}}{d\Omega} d\Omega + \sum_i \frac{n_i}{n_g} v_i^{\text{loss}} \quad (51)$$

The total cross section in Eq. (51) refers to the sum over all elastic, excitation and ionization processes. v_m is the usual rate for momentum transfer plus additional contributions from loss processes. It is clear

that loss processes do result in momentum transfer, and caution must be exercised to ascertain the magnitude of these effects. Fortunately, for the beam parameters of interest, these effects are small.

In Eq. (50), R is a remainder term given by

$$\begin{aligned}
 R = n_g \sum \left\{ \bar{f}_1(\epsilon + \epsilon_x) \sqrt{\frac{2(\epsilon + \epsilon_x)}{m}} \sigma_1^{\text{ex}}(\epsilon + \epsilon_x) - \bar{f}_1(\epsilon) \sqrt{\frac{2\epsilon}{m}} \sigma_1^{\text{ex}}(\epsilon) \right. \\
 \left. + \int d\epsilon' v' \bar{f}_1(\epsilon') \left[\frac{d\sigma_1^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon) + \frac{d\sigma_1^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] \right. \\
 \left. - \bar{f}_1(\epsilon) v \sigma_1^{\text{ion}}(\epsilon) \right\} \quad (52)
 \end{aligned}$$

The remainder term is related to the anisotropy of inelastic collisions, and is generally small. We set R equal to zero.

From Eq. (21), we see that $\bar{f}_1$ is related to the plasma current.

The variation of the plasma current takes place with time scales which are of the order of the beam current rise time. This variation is slow compared to rates for atomic collisions. Thus, we may also set $\frac{\partial \bar{f}_1}{\partial t} = 0$.

With these approximations, Eq. (50) reduces to

$$\bar{f}_1(\epsilon) = - \frac{eE}{mn_g v_m} \sqrt{2m} \epsilon \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) \quad (53)$$

By taking the appropriately weighted integral of Eq. (53), using Eqs. (21) and (22), we get the plasma current and an expression for conductivity.

$$j_e = \frac{e}{3} \int v \bar{f}_1(\epsilon) d\epsilon = \sigma E \quad (54)$$

$$\sigma = - \frac{2e^2}{3mn_g} \int \frac{\epsilon^{3/2}}{v_m(\epsilon)} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) d\epsilon \quad (55)$$

Substituting $\bar{f}_1(\epsilon)$ derived in Eq. (53) into Eq. (49), a Boltzmann equation for $\bar{f}_0$ is obtained.

$$\begin{aligned} \frac{\partial \bar{f}_0}{\partial t} - \left(\frac{2e^2}{3m} \right) \epsilon^2 \frac{\partial}{\partial \epsilon} \left[\frac{\epsilon^{3/2}}{n_g v_m} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) \right] &= \sum \left(n_g \left(\frac{J_b}{e} \right) \frac{d\sigma^{ion}}{d\epsilon} (T, \epsilon) \right. \\ &- n_i \bar{f}_0(\epsilon) v_i^{loss} + n_g \left(\bar{f}_0(\epsilon + \epsilon_x) \sqrt{\frac{2(\epsilon + \epsilon_x)}{m}} \sigma^{ex}(\epsilon + \epsilon_x) \right. \\ &- \bar{f}_0(\epsilon) \sqrt{\frac{2\epsilon}{m}} \sigma^{ex}(\epsilon) \Big) \\ &+ n_g \left(\int_0^{\epsilon_{max}} d\epsilon' \bar{f}_0(\epsilon') v' \left[\frac{d\sigma^{ion}}{d\epsilon}(\epsilon'\epsilon) + \frac{d\sigma^{ion}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] \right. \\ &- \bar{f}_0(\epsilon) v \sigma^{ion}(\epsilon) \Big) \Big\} \quad (56) \end{aligned}$$

Equation (56) is an integro-differential equation involving the independent variables, ϵ and t . If the electric field and beam current density are externally specified, the Boltzmann distribution $\bar{f}_0(\epsilon, t)$ could be derived. This in turn gives rise to the conductivity via Eq. (55). More generally, Eqs. (56) and (55) are coupled with circuit equations for E and J_b . The conductivity, fields and currents are then derived self-consistently.

Particle and energy conservation laws could be retrieved by taking appropriate moments of the Boltzmann Equation. Integrating Eq. (56) over energy, we obtain a particle density conservation equation

$$\begin{aligned} \frac{dn_e}{dt} = \sum \left\{ n_g \left(\frac{j_b}{e} \right) \sigma^{ion}(T) - n_i \int d\epsilon \bar{f}_0(\epsilon) v_i^{loss}(\epsilon) \right. \\ \left. + n_g \int d\epsilon \bar{f}_0(\epsilon) v_i^{ion}(\epsilon) \right\} \end{aligned} \quad (57)$$

Similarly, by taking the first ϵ moment of Eq. (56), the energy density conservation equation is retrieved

$$\begin{aligned} \frac{d}{dt} (n_e \tilde{\epsilon}) = \sigma E^2 + \sum \left\{ n_g \left(\frac{j_b}{e} \right) \int d\epsilon \epsilon \frac{d\sigma^{ion}}{d\epsilon} (T, \epsilon) \right. \\ - n_i \int d\epsilon \epsilon \bar{f}_0(\epsilon) v_i^{loss}(\epsilon) - n_g \epsilon_x \int d\epsilon \bar{f}_0(\epsilon) v^{ex}(\epsilon) \\ \left. - n_g \epsilon_T \int d\epsilon \bar{f}_0(\epsilon) v^{ion}(\epsilon) \right\}. \end{aligned} \quad (58)$$

The conductivity formula of Eq. (55) may seem unconventional. In a special limit, namely, when v_m is independent of ϵ , we can perform a partial integration on Eq. (55) to retrieve the more conventional formula

$$\sigma = \frac{e^2 n_e}{m n_g v_m} \quad v_m \text{ constant} \quad (59)$$

Intuition based on familiarity with Eq. (59) might tempt one to make the generalization for arbitrary Boltzmann distribution by making the replacement

$$\frac{n_e}{v_m} \rightarrow \int \frac{\bar{f}_0(\epsilon)}{v_m(\epsilon)} d\epsilon.$$

However, this straightforward generalization could be in error by as much as a factor of two or three. For example, if v_m is proportional to ϵ^n , then partial integration of Eq. (55) leads to

$$\sigma = \left(1 - \frac{2n}{3}\right) \left(\frac{e^2}{m n_g}\right) \int_0^\infty \frac{\bar{f}_0(\epsilon)}{v_m(\epsilon)} d\epsilon \quad v_m \propto \epsilon^n \quad (60)$$

For N_2 , $n \sim 1$ for low energies ($\epsilon < .1$ eV), whereas for higher energies, $n \sim 1/2$.

THE ELECTRON SHOWER

Conventionally, the effect of direct ionization by the beam is treated as an instantaneous shower of electrons which include the slow secondaries resulting from the primary collision as well as tertiaries and quaternaries from repeated ionizations of fast secondaries. The shower concept could be incorporated into the present framework, and would result in a much better calculational scheme.

Equation (56) is an equation of two variables ϵ and t . As it stands, ϵ covers a broad range from 0 to the maximum energy $\epsilon_{\max}$ which is approximately one half of the initial beam energy (in the MeV region). To numerically integrate Eq. (56) over this broad range is a rather cumbersome task. This task could be significantly eased by making some simplifying approximations which hold at high energies. We start by choosing a cutoff energy ϵ_c , typically like 30 eV to 50 eV. For $\epsilon > \epsilon_c$, contributions from electron loss processes as well as some excitation processes (e.g., vibrations and rotations) become negligible. The change in energy induced by Joule heating is a small fraction of the total electron energy. The effect of this small change can also be

neglected. Finally, if the change in beam current density is slow compared to the ionization and excitation times, then a steady-state approximation $\frac{\partial \bar{f}_0}{\partial t} = 0$ is reasonable. With these approximations, Eq. (56) becomes, after some rearrangement of terms, the following equation

$$\bar{f}_0(\epsilon) \equiv f'(\epsilon) \left(\frac{J_b}{e} \right) \quad (61)$$

$$f'(\epsilon) = \frac{1}{\sum_{ex} \nu^{ex}(\epsilon) + \nu^{ion}(\epsilon)} \left[\frac{d\sigma}{d\epsilon}(T, \epsilon) + \sum_{ex} f'(\epsilon + \epsilon_x) \nu^{ex}(\epsilon + \epsilon_x) \right. \\ \left. + \int_{\epsilon + \epsilon_I}^{\epsilon_{max}} \left[\frac{d\sigma}{d\epsilon'}(\epsilon', \epsilon) + \frac{dG}{d\epsilon'}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] \nu' f'(\epsilon') d\epsilon' \right] \\ \text{for } \epsilon > \epsilon_c \quad (62)$$

Thus, in the high energy regime, Eq. (56) which is a second-order differentio-integral equation of two variables reduces to Eq. (62), a simple integral equation of one variable. Equation (62) can be solved by iteration. The iterative procedure will converge very fast.

Once the high-energy solution is derived, we can return to Eq. (56) and write for the low-energy regime:

$$\begin{aligned}
 \frac{\partial \bar{f}_0}{\partial t} - \left(\frac{2e^2}{3m} \right) \epsilon^2 \frac{\partial}{\partial \epsilon} \left[\frac{\epsilon^{3/2}}{n_g v_m} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) \right] &= S(\epsilon) \\
 - \sum_i n_i \bar{f}_0(\epsilon) v_i^{\text{loss}} - n_g \bar{f}_0(\epsilon) v^{\text{inel}}(\epsilon) & \\
 + \sum' n_g \bar{f}_0(\epsilon + \epsilon_X) v^{\text{ex}}(\epsilon + \epsilon_X) + n_g \int_{\epsilon + \epsilon_I}^{\epsilon} \left[\frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon) \right. & \\
 \left. + \frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] v' \bar{f}_0(\epsilon') d\epsilon' & \\
 \epsilon < \epsilon_c &
 \end{aligned} \tag{63}$$

where the total rate for inelastic collisions is

$$v^{\text{inel}}(\epsilon) = v \left[\sum \sigma^{\text{ex}}(\epsilon) + \sigma^{\text{ion}}(\epsilon) \right] \tag{64}$$

and $\sum'$ implies summation over all excited states which satisfy

$\epsilon + \epsilon_X < \epsilon_c$. The shower is now described by the function

$$S(\epsilon) \equiv n_g \left(\frac{j_b}{e} \right) S'(\epsilon) \tag{65}$$

$$\begin{aligned}
 S'(\epsilon) &= \frac{d\sigma^{\text{ion}}}{d\epsilon}(T, \epsilon) + \sum'' f'(\epsilon + \epsilon_X) v^{\text{ex}}(\epsilon + \epsilon_X) \\
 &+ \int_{\epsilon_c}^{\epsilon_{\text{max}}} \left[\frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon) + \frac{d\sigma^{\text{ion}}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] v' f'(\epsilon') d\epsilon'
 \end{aligned} \tag{66}$$

In Eq. (66), $\sum''$ implies summation over all states such that

$\epsilon + \epsilon_X > \epsilon_c$. $f'(\epsilon)$ is given in Eq. (62).

The shower function $S'(\epsilon)$ is independent of the beam current or pressure, and is a function only of the atomic cross sections. For a given beam energy T and cutoff energy ϵ_c , $S'(\epsilon)$ needs to be calculated

only once. The resultant function can then be input into Eq. (63).

Since Eq. (63) needs to be solved only in the range $0 < \epsilon < \epsilon_c$, it represents a significant improvement over Eq. (56).

The shower calculation could be further simplified by making a change in the variable of integration in Eqs. (62) and (66). By introducing a new variable

$$y = \frac{R}{\epsilon} \quad (67)$$

where R is some constant, the integral in Eq. (62) becomes

$$\int_{R/\epsilon_{\max}}^{R/(\epsilon + \epsilon_I)} R \frac{dy}{y^2} \left(\frac{2R}{my} \right)^{1/2} f' \left(\frac{R}{y} \right) \left[\frac{d\sigma}{d\epsilon} \left(\frac{R}{y} \epsilon \right) + \frac{d\sigma}{d\epsilon} \left(\frac{R}{y} \frac{R}{y} - \epsilon - \epsilon_I \right) \right]$$

The range of the integral is small, and the function is reasonably well-behaved as long as the incident beam energy does not get too high. It is therefore a much more tractable integral. When the beam energy gets very high, $y' = \ln \frac{\epsilon}{R}$ would be a more useful variable of integration to use.

7. SUMMARY

The derivation of the Boltzmann equation for conductivity is now complete. In this section, the results are summarized. The notation in this section is self-contained, but may not be identical to those in earlier sections, if the latter is deemed too cumbersome for future reference. The equations will be made more explicit where possible.

We again stress a point made earlier that the list of atomic processes included in this report is by no means final. Direct heating by beam electrons, electron-electron scattering, energy loss due to

elastic scattering, finite gas temperature effects, and other atomic processes could be added where these effects become important. The object of the present report is to provide the general theoretical framework.

In summary, the program for calculating conductivity as derived in this report has five components. The Boltzmann equation for low energy electrons is the major component of the calculational scheme. The shower equations supply the source function for the Boltzmann equation, and also calculates the high energy end of the distribution function. The ion rate equations specify the population of various ionic species, so that the effects of electron-ion recombination may be determined. In order to obtain a close set of equations, circuit equations for electric field and beam current are generally coupled to the Boltzmann equation. The final component consists of the relation of conductivity and other plasma parameters to the electron distribution function

Boltzmann Equation

$$\begin{aligned}
 \frac{\partial f(\bar{x}, \epsilon, t)}{\partial t} - \left(\frac{2e^2}{3m} \right) E^2(\bar{x}, t) \frac{\partial}{\partial \epsilon} \left[\frac{\epsilon^{3/2}}{\sum n_g v_m(\epsilon)} \frac{\partial}{\partial \epsilon} \left(\frac{f(\bar{x}, \epsilon, t)}{\epsilon^{1/2}} \right) \right] \\
 = S(\bar{x}, \epsilon, t) \\
 - f(\bar{x}, \epsilon, t) \left\{ \sum n_g v_{att}(\epsilon) + \sum n_i^+ v_{i(\epsilon)}^{recom} + \sum n_g v_{inel}(\epsilon) \right\} \\
 + \sum' f(\bar{x}, \epsilon + \epsilon_x, t) n_g v_{ex}(\epsilon + \epsilon_x) \\
 + \sum \int_{\epsilon + \epsilon_I}^{\epsilon_C} f(\bar{x}, \epsilon', t) n_g \sqrt{\frac{2\epsilon}{m}} \left[\frac{d\sigma^{ion}}{d\epsilon}(\epsilon', \epsilon) \right. \\
 \left. + \frac{d\sigma^{ion}}{d\epsilon}(\epsilon', \epsilon' - \epsilon - \epsilon_I) \right] d\epsilon' \quad 0 \leq \epsilon \leq \epsilon_C \quad (S1)
 \end{aligned}$$

where $f(\bar{x}, \epsilon, t)$ - electron distribution function

- $\bar{x}$ - position
- t - time
- ϵ - energy of electron
- e - electronic charge
- m - mass of electron
- n_g - gas density
- n_i^+ - positive ion density, as calculated from ion rate equation
- v_m - momentum transfer rate
- v_{att} - attachment rate
- v_i^{recom} - recombination rate

- ν^{inel} - total inelastic collision rate (ionization plus all excitations)
- ν^{ex} - excitation rate
- $\frac{d\sigma^{ion}}{d\epsilon}(\epsilon', \epsilon)$ - differential cross section for ionization with incident electron energy ϵ' and secondary electron energy ϵ
- ϵ_x - excitation energy
- ϵ_I - ionization energy
- ϵ_c - cutoff energy, externally specified
- $E(\bar{x}, t)$ - electric field, from circuit equations
- $S(\bar{x}, \epsilon, t)$ - shower function. From shower equations.
- $\sum$ implies summation over all relevant gas (or ion) species, processes and levels. $\sum'$ implies summation over excitation levels which satisfy $\epsilon + \epsilon_x < \epsilon_c$.

Shower Equations

For $\epsilon > \epsilon_c$, the electron distribution function is

$$f(\bar{x}, \epsilon, t) = f'(\epsilon) \frac{j_b(\bar{x}, t)}{e} \quad (S2)$$

where

$$f'(\epsilon) = \frac{1}{\nu^{inel}(\epsilon)} \left\{ \frac{d\sigma^{ion}}{d\epsilon}(\epsilon, \epsilon) + \sum f'(\epsilon + \epsilon_x) \nu^{ex}(\epsilon + \epsilon_x) + \int_{R/\epsilon_{max}}^{R/(\epsilon + \epsilon_I)} \frac{dy}{y^2} \sqrt{\frac{2R}{my}} f' \left(\frac{R}{y} \right) \left[\frac{d\sigma^{ion}}{d\epsilon} \left(\frac{R}{y}, \epsilon \right) + \frac{d\sigma^{ion}}{d\epsilon} \left(\frac{R}{y}, \frac{R}{y} - \epsilon - \epsilon_I \right) \right] \right\} \quad (S3)$$

- J_b - beam current density
 R - constant, may be set equal to ϵ_I
 T - kinetic energy of incident beam electron
 $\epsilon_{\max} = \frac{T - \epsilon_I}{2}$ - maximum energy of secondary electron

The shower function is

$$S(\bar{x}, \epsilon, t) = S'(\epsilon) \left(\frac{J_b(\bar{x}, t)}{e} \right) n_g \quad (S4)$$

$$\begin{aligned}
 S'(\epsilon) = & \frac{d\sigma^{\text{ion}}}{d\epsilon} (T, \epsilon) + \sum f'(\epsilon + \epsilon_x) v^{\text{ex}}(\epsilon + \epsilon_x) \\
 & + \int_{R/\epsilon_{\max}}^{R/\epsilon_c} \frac{R dy}{y^2} \sqrt{\frac{2R}{my}} f' \left(\frac{R}{y} \right) \left[\frac{d\sigma^{\text{ion}}}{d\epsilon} \left(\frac{R}{y}, \epsilon \right) + \frac{d\sigma^{\text{ion}}}{d\epsilon} \left(\frac{R}{y}, \frac{R}{y} - \epsilon - \epsilon_I \right) \right]
 \end{aligned} \quad (S5)$$

where $\sum$ implies summation over all excited levels satisfying $\epsilon +$

$\epsilon_x > \epsilon_c$. Equations (S4) and (S5) are defined only for the range

$$0 \leq \epsilon \leq \epsilon_c$$

Ion Rate Equations

Positive ions:

$$\begin{aligned}
 \frac{\partial n_i^+(\bar{x}, t)}{\partial t} = & \sum_j \alpha_{ij} n_j^+(\bar{x}, t) - n_i^+(\bar{x}, t) \sum_j \alpha_{ji} \\
 & - n_i^+(\bar{x}, t) \sum_k \beta_{ik} n_k^-(\bar{x}, t) \\
 & + \left(\frac{J_b(\bar{x}, t)}{e} \right) n_g \sigma_i^{\text{ion}}(T) + n_g \int_0^{\epsilon_{\max}} f(\bar{x}, \epsilon, t) v_i^{\text{ion}}(\epsilon) d\epsilon \\
 & - n_i^+(\bar{x}, t) \int_0^{\epsilon_{\max}} f(\bar{x}, \epsilon, t) v_i^{\text{recom}}(\epsilon) d\epsilon \quad (S6)
 \end{aligned}$$

- $n_i^+(\bar{x}, t)$ - density of i^{th} positive ion
 α_{ij} - rate for j^{th} ion to turn into i^{th} ion via ion-neutral reactions. α_{ij} is the sum of all such reactions. For two-body ion-neutral reactions, the rate is proportional to the density of the neutral; for three-body reactions, the rate is in addition proportional to the density of third body
 $n_k^-(\bar{x}, t)$ - density of the k^{th} negative ion
 β_{ik} - recombination rate for i^{th} positive ion to the k^{th} negative ion
 $\sigma_i^{\text{ion}}, \nu_i^{\text{ion-}}$ - cross section and rate for the creation of i^{th} positive ion by ionization.

The two energy integrals in Eq. (S6) are extended from 0 to ϵ_{max} . The low energy end of the electron distribution function ($\epsilon \leq \epsilon_c$) is obtained from Eq. (S1) while the high energy end results from Eqs. (S2) and (S3). In practice, the contributions to recombination from high energy electrons are negligible.

Negative Ions

$$\begin{aligned}
 \frac{\partial n_i^-(\bar{x}, t)}{\partial t} = & \sum_j \gamma_{ij} n_j^-(\bar{x}, t) - n_i^-(\bar{x}, t) \sum_j \gamma_{ji} \\
 & - n_i^-(\bar{x}, t) \sum_k \beta_{ki} n_k^+(\bar{x}, t) + n_g \int_0^{\epsilon_{\text{max}}} f(\bar{x}, \epsilon, t) \nu_i^{\text{att}}(\epsilon) d\epsilon
 \end{aligned}
 \tag{S7}$$

γ_{ij} - rate for j^{th} negative ion to turn into i^{th} negative ion via ion-neutral reactions.

Circuit Equations

The electric field $E(\bar{x}, t)$ and the current $J_b(\bar{x}, t)$ are determined by circuit equations. Description of these EM aspects of the problem is outside of the scope of this report.

Conductivity and Plasma Parameters

Conductivity:

$$\sigma(\bar{x}, t) = -\frac{2e^2}{3m} \int_0^{\epsilon_{\max}} \frac{\epsilon^{3/2}}{\sum n_g v_m(\epsilon)} \frac{\partial}{\partial \epsilon} \left(\frac{f(\bar{x}, \epsilon, t)}{\epsilon^{1/2}} \right) d\epsilon \quad (S8)$$

Plasma density:

$$n_e(\bar{x}, t) = \int_0^{\epsilon_{\max}} f(\bar{x}, \epsilon, t) d\epsilon \quad (S9)$$

Plasma temperature:

$$n_e(\bar{x}, t) T_e(\bar{x}, t) = \frac{2}{3} \int_0^{\epsilon_{\max}} \epsilon f(\bar{x}, \epsilon, t) d\epsilon \quad (S10)$$

In the above three equations, the integral in ϵ extends over the entire range $0 < \epsilon < \epsilon_{\max}$. One expects, however, the important contributions to come only from $\epsilon < \epsilon_c$.

(S1) to (S10) constitute the basic set of equations for the REB conductivity calculation.

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