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SYNCHRONOUS PARTICLE AND BUCKET DYNAMICS

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ALTERNATING GRADIENT SYNCHROTRON DEPARTMENT

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ABSTRACT

In accelerator theory of particle longitudinal motion, a classical definition of a synchronous particle (synchronous energy, synchronous phase, synchronous orbit) assumes that there is a one-to-one relationship between the strength of the guiding magnetic field and the radio frequency of the accelerating electrical field. In practice, that relationship may not be sustained because of errors in the magnetic field, radio frequency, or because sometimes we do not want to keep that relationship for some reason.

In this report we formulate the concept of adiabatic equilibrium between particle and accelerator, and we introduce a definition of synchronous particle when the magnetic field and radio frequency are independent functions of time.

The result is that size and shape of the bucket (separatrix) depends not only on the field rate of change $\dot{B}$, but on the frequency rate $\dot{f}$ as well. This means, for example, we can have a stationary bucket even at the rising field $\dot{B} > 0$.

Having a frequency f in addition to field B and voltage V as parameters controlling the bucket dynamics, we will show how to decrease particle losses during injection and capture.

Introduction

A full description of a particle dynamical state in an accelerator requires six-dimensional phase space composed of three space coordinates and three corresponding momenta. In this phase space, particle motion would be represented by a six-dimensional curve where each point corresponds to each moment of time. The phase curve is obtained by solving dynamical equations which include detailed information about all accelerator magnets and electrical cavities composing the accelerator, as well as the particles' initial conditions right after injection.

Much input information on accelerator structure, a large number of differential equations (three for each particle), and a lot of difficulties in solving them--all of these things are the price paid for the full description of particle dynamical state.

A partial description of a particle's state can be achieved in many different ways. One of these is by the separation of the six-dimensional phase space into two independent transverse (four-dimensional) and longitudinal (two-dimensional) spaces. The main reason for such a separation is that the period of betatron oscillations in transverse space is much less than the period of synchrotron oscillations in longitudinal space. Depending on the problem under consideration, we can then work in either of these spaces regardless of the other one.

In this report we will concentrate on one particular problem of accelerator operational physics: how to optimize frequency and voltage control for electrical cavities in order to reduce particle losses during injection and capture.

To approach this problem, we will restrict ourselves in longitudinal phase space. We will see that solution of the problem depends largely on bucket dynamics. There is a difference between particle dynamics and bucket dynamics. While particle motion is governed by dynamical equations, the bucket size and shape are adiabatic invariants¹ of these equations determined by magnetic field, B , radio frequency, f , and cavity voltage V . That is why we can separate the longitudinal description into two parts: bucket evolution and particle dynamics.

If all three $B=B(t)$, $f=f(t)$, $V=V(t)$ are slowly changing functions compared to synchrotron period then the bucket would be truly adiabatic. This means the particles within the bucket will not leave the bucket as well as particles outside the bucket will not enter the bucket.

However, if any of B , f , V are not slow enough then adiabaticity will be lost. As a result some of the particles could lose their stability and leave the bucket. On the other hand some of the particles from outside could enter the bucket and become stable. We call that last case "non-adiabatical capture".

We will show in this report (Section V) how to organize non-adiabatical capture without losing stable particles from the bucket.

In Section I we recapitulate a classical definition of a synchronous particle which requires it to stay on the central (design) orbit. Later we will remove that requirement.

In Section II we describe adiabatic variables and how their equilibrium equations are derived. The main equilibrium equations were introduced earlier without derivation by C. Bovet et al.² Many other equilibrium relations were derived either with constant magnetic field or with constant rf frequency, or with constant orbit radius.^{3,4} In this report we consider adiabatic variables such as field B , frequency f , momentum p , and orbit radius R as a full set of parameters which describe adiabatic equilibrium between two systems--accelerator (B, f) and (synchronous) particle (p, R). Such an approach reflects the fact that in real accelerators our main goal is to control particle energy and orbit using the variables which are at our disposal, that is B and f (and voltage V).

We use these control parameters B and f in Section III to determine synchronous radius, energy, and phase in the case when the synchronous particle is out of central orbit.

In Section IV, we show how the same parameters determine the size and the shape of the bucket.

All developed theory is then applied in the last section, Section V, to demonstrate with the use of computer modeling how frequency/voltage manipulation can reduce particle losses during injection and capture.

This report is divided into sections numbered by Roman numerals from I to V. Some sections consist of several subsections numbered by the usual integers. From programming languages we adopted "the principle of localization" for purposes of formula references. For example, Reference (3) means Formula (3) within the same subsection where this reference occurs; (2.1) means Formula (1) of Subsection 2 within the same section where this reference occurs; (II.2.B3) means Formula (B3) of Subsection (2) of Section II.

Notations:

c	velocity of light
e	particle charge
f	accelerating frequency
B	magnetic flux density
ω	particle revolution frequency
E_r	particle rest energy
h	harmonic number
V	peak accelerating voltage per turn
R_o	radius of central (design) orbit
E_o	total energy of the particle staying on central orbit (R_o -orbit)
p_o	momentum for the same particle
ω_o	revolution frequency for the same particle

We omitted charge e in all formulae.

If necessary e can be recovered easily by means of dimensionality.

I. Synchronous Particle on the Central Orbit

The beauty of each accelerator comes from its two main parts--the guiding-focusing magnet system and the accelerating electrical system. Those are two servants catering to one master--the synchronous particle.

Firstly, the synchronous particle is one whose energy E_0 and momentum p_0 correspond to a given magnetic field B_0 to provide the given constant trajectory curvature ρ_0 :

$$p_0 = \rho_0 B_0, \quad (1)$$

$$E_0 = (c^2 p_0^2 + E_r^2)^{1/2}. \quad (2)$$

The constant curvature ρ_0 together with the length of the straight section determines constant so-called "the radius R_0 of the machine", which we will refer to as the radius of the central (design) orbit. If the accelerator is composed of m straight sections, each of length L , and m circular arches, each of radius ρ_0 , then the machine radius R_0 (Fig. 1) is determined by:

Fig. 1. The solid line is the machine central orbit. The dashed line is the circle of the same length.

$$2\pi R_0 = mL + 2\pi\rho_0. \quad (3)$$

Secondly, the synchronous particle is one whose angular frequency ω_0 is synchronized with rf frequency f_0 :

$$h\omega_0 = 2\pi f_0. \quad (4)$$

Staying on the central orbit R_0 , synchronous particle will have the velocity $v = \omega_0 R_0$ and energy ($\beta_0 = v/c$):

$$E_0 = \gamma_0 E_r = (1 - \beta_0^2)^{-1/2} E_r. \quad (5)$$

The first and the second requirement can be unified by equating (2) and (5). With the use of (4), we can write:

$$f_o = \frac{ch}{2\pi R_o} \left[\left(\frac{E_r}{c\rho_o B_o} \right)^2 + 1 \right]^{-1/2}. \quad (6)$$

Thus, the synchronous particle will stay on the central orbit R_o if and only if two servants--magnetic field B_o and radio frequency f_o --are tied to each other by frequency-field relation (6).

II. Adiabatic Variables and Equilibrium Equations

In a theory of particle longitudinal motion, there are four values B , f , p , R , which we call adiabatic variables:

B is magnetic field,

f is supplied rf frequency,

p is particle momentum, and

R is radius of synchronous orbit for that particle.

The set of those four variables is called adiabatic quartet or the main quartet. In this section, we will derive relations connecting any three adiabatic variables of the main quartet.

1. Adiabatic Variables

Suppose the state of an accelerator can be described uniquely by two variables B and f , while the state of the particle is described by p and R . Some of the particles, namely the synchronous one can be at the state of equilibrium with the accelerator. Such an equilibrium can be expressed by the unique functions:

$$p = p(B, f), \quad R = R(B, f), \quad (1)$$

which can be inverted again as a unique functions

$$B = B(p, R), \quad f = f(p, R). \quad (2)$$

If relations such as (1) and (2) exist, we will call them adiabatic relations or equilibrium equations. For example, equations (I.1) and (I.6) are equilibrium equations for central orbit $R = R_o$ ($B = B_o$, $f = f_o$).

The concept of equilibrium state allows us to simplify a full description of a particle dynamical state:

- Instead of three particle coordinates changing along the particle trajectory, we consider now synchronous orbit characterized by a single adiabatic variable--orbit radius R . That orbit is not only the result of averaging over betatron oscillations around some closed off-momentum orbit, but also the result of the averaging of the closed orbit itself.*
- Instead of the three-dimensional vector-momentum associated with three-dimensional particle coordinates, we have one scalar p which is longitudinal momentum associated not with the point, but with the whole orbit of radius R .
- Instead of many rf cavities allocated over the accelerator ring, we have one variable f associated with the ring as a whole and uniquely connected with the synchronous particle revolution frequency by $h\omega = 2\pi f$.
- Similarly, we consider the mean field $\bar{B}$ as a local field $B(s)$ averaged along the synchronous orbit

$$\bar{B} = \frac{1}{2\pi R} \oint B(s) ds. \quad (3)$$

If $\bar{B}$ would be the dipole field allocated homogeneously along a circle orbit of radius R , then to stay on that orbit the particle should have a momentum

$$p = R\bar{B}. \quad (4)$$

---

*More precisely, let p_0 be a momentum of a particle staying on the central orbit of radius R_0 (on-momentum particle). Let p be a momentum of a particle in question oscillating around closed off-momentum orbit of radius $r(s) = R_0 + \frac{p - p_0}{p_0} D(s)$. Here s is the path length along the closed orbit and $D(s)$ is the dispersion function determined by the betatron motion theory. Then the radius of the synchronous orbit is defined as the average $R = \bar{r}(s) = R_0 + \frac{p - p_0}{p_0} \bar{D}$.

While this is a true adiabatic equation, there is no practical way to use it. Indeed, according to its definition, $\bar{B}$ depends not only on the radius R , but also on the actual path of the orbit chosen for averaging. It would be very inconvenient to characterize the magnetic field by the contour integral depending on a contour that we do not know well. This is why to approximate real accelerator magnets by a simple model we consider the so-called isomagnetic field, in which the central orbit is composed of a curved section of radius ρ_0 and by straight sections, which may contain quadrupoles. The advantage of this approximation comes from two facts:

- The average field $\bar{B}_0$ over the central orbit R_0 is proportional to the local dipole field B_0 over the curved section

$$\bar{B}_0 = \text{const.} \oint B_0(s) ds = \text{const.} 2\pi R_0 B_0 \quad (5)$$

- To the first order in $\delta = (p - p_0)/p_0$ average field $\bar{B}$ over any other orbit R can be expressed by B_0 and by the compaction factor $\alpha = p dR/R dp$:

$$\bar{B} = \bar{B}(B_0, \alpha). \quad (6)$$

Spatial averaging is one reason which led to the concept and to the name of adiabatic variables. Another reason is time averaging.

While each argument in (1) and (2) can be dependent on time, none of those functions contain time explicitly. It can be sustained only if adiabatic variables change with time slowly enough. In other words, adiabatic variables are the result of time averaging over a large enough period T . For those particles which are close to the bucket center T is exactly a synchrotron period, but for the particles close to the separatrix, synchrotron period can increase infinitely. However, those are the particles whose stability is most sensitive to varying parameters. To say in a more practical way, T should be large enough to keep stable particles from losing their stability (to keep particles within the bucket). For example, if the frequency $f = f(t)$ increased abruptly during acceleration and as a result some particles left the bucket, then we will say that f was increased non-adiabatically. Thus, period T should be defined not so much from the synchrotron period but rather from the physical or computational experiment.

Non-adiabatical behavior cannot be described by equilibrium equations, which contain no time explicitly. It can be done, however, with the use of dynamical equations for longitudinal phase space.

2. Equilibrium Equations

We will need several times following mathematical definition. If a function $F(x)$ has a derivative $F'(x)$, then the logarithmic differential and logarithmic derivative are defined as

$$d \ln F(x) = \frac{F'(x)}{F(x)} dx = \frac{dF}{F}, \quad [\ln F(x)]' = \frac{F'(x)}{F(x)}.$$

The most useful feature is that logarithmic differential (derivative) of the two function product is the sum of their logarithmic differentials (derivatives): $\frac{d(FG)}{FG} = \frac{dF}{F} + \frac{dG}{G}$.

Now, we will begin derivations of the equilibrium equations, starting with one which is the only exact. All the others will be approximate.

If a particle has a revolution frequency $\omega = 2\pi f/h$ along the orbit of radius R , then the particle velocity is $v = \beta c$ or

$$\beta c = \omega R. \quad (1)$$

Replacing β by p from $\beta = cp/E$, E by p from $E^2 = c^2 p^2 + E_r^2$, and using f instead of ω , we will have

$$\boxed{f = \frac{h}{2\pi R} \frac{c^2 p}{(c^2 p^2 + E_r^2)^{1/2}}} = \frac{ch}{2\pi R} \frac{(E^2 - E_r^2)^{1/2}}{E}. \quad (A1)$$

This is the first equilibrium equation, written in closed form. Some times it is easier to use (A1) in differential form, which we will get faster from (1) than from (A1). Indeed, by taking the logarithmic differential of (1), we come up with $d\beta/\beta = dR/R + d\omega/\omega$. Then using the relativistic $\gamma^2 d\beta/\beta = dp/p$ and f instead of ω we have

$$\boxed{\frac{dp}{p} = \gamma^2 \frac{dR}{R} + \gamma^2 \frac{df}{f}} = \frac{dE}{\beta^2 E}. \quad (B1)$$

This is the first equilibrium equation in differential form. If it is considered as a differential equation (with $\gamma = \gamma(p)$), then (A1) is the solution.

Now let us consider orbit radius R as a function of the particle momentum p and magnetic field B :

$$R = R(p, B). \quad (2)$$

For the central orbit R_0 , equilibrium equation (2) reads as (see (I.1), (I.3))

$$p_0 = \rho_0 B_0. \quad (3)$$

For any other orbit, it will be the same Lorentz equation

$$p = \rho B. \quad (4)$$

The difference between (3) and (4) is that the first has orbit radius R_0 known a priori, while in the second equation, R is unknown. Let us find it.

If the magnetic field and particle momentum deviate from (p, B) by a small amount (dp, dB) , then the radius differential $dR = R(p + dp, B + dB) - R(p, B)$ can be Taylor expanded to the first order as

$$dR = \frac{\partial R}{\partial p} dp - \frac{\partial R}{\partial B} dB. \quad (5)$$

The minus sign is here because the radius should decrease when the field increases. With use of the compaction factor $\alpha = p dR / R dp$, the first coefficient in (5) is

$$\frac{\partial R}{\partial p} = \alpha \frac{R}{p}. \quad (6)$$

Taking this and (4) into account, the second coefficient can be written as

$$\frac{\partial R}{\partial B} = \frac{\partial R}{\partial p} \cdot \frac{\partial p}{\partial B} = \alpha \frac{R}{p} \cdot \rho = \alpha \frac{R}{p} \cdot \frac{p}{B} = \alpha \frac{R}{B}. \quad (7)$$

Substituting (6), (7) into (5), we have

$$\boxed{\frac{dR}{R} = \alpha \left(\frac{dp}{p} - \frac{dB}{B} \right)} \quad (B2)$$

This is the second adiabatic relation in differential form. If we consider (B2) as a differential equation with initial condition (3), then the solution is

$$R = R_0 \left(\frac{p}{p_0} \right)^\alpha \quad (8)$$

This is the second equilibrium equation written in a closed form. Sometimes in literature² equilibrium equation (8) is expressed as so-called fundamental equation

$$\boxed{p = p_0 \left(R/R_0 \right)^{1/\alpha} B} \quad (A2)$$

which is equivalent to (8).

The following differential relations can be obtained by linear combination. Namely, multiplying (B1) by α and adding to (B2)

$$\boxed{\frac{dR}{R} = \frac{1}{\gamma_{tr}^2 - \gamma^2} \left(\gamma^2 \frac{df}{f} - \frac{dB}{B} \right)} \quad (B3)$$

Equating this to (B2), we find

$$\boxed{\frac{dp}{p} = \frac{1}{-\eta} \left(\frac{df}{f} - \alpha \frac{dB}{B} \right)} = \frac{dE}{\beta^2 E} \quad (B4)$$

where $\eta = \gamma_{tr}^{-2} - \gamma^{-2}$.

These two adiabatic relations allow us to determine the state of the particle (p , R) knowing the state of the accelerator (B , f). Actual calculations could be performed by solving equilibrium equations (A1) and (A2) simultaneously.

III. Synchronous Particles Out of Central Orbit

1. Synchronous (Orbit) Radius R_s

As we have seen in Section I, the necessary and sufficient condition for existence of synchronous particles on R_o -orbit is a frequency-field relation (I.6) which can be written as

$$f_o = f_o(B_o, R_o). \quad (1)$$

We can write (1) as an orbital equation just by interchanging f_o with R_o in (I.6):

$$R_o = R_o(B_o, f_o). \quad (2)$$

In a real accelerator, magnetic field B and radio frequency f can be controlled independently. Usually the frequency-field relation (1) is kept during the whole acceleration cycle. However, sometimes there are errors in magnetic field ΔB and/or in frequency Δf which make (1) and (2) invalid. Moreover, we will see later in Section V that sometimes during injection and capture, it is worthwhile to deviate from (2) by frequency manipulation.

If we supply the field $B = B_o + \Delta B$ and frequency $f = f_o + \Delta f$ then it will change the synchronous radius by ΔR . Replacing the differential relation (II.2.B3) by the difference equivalent we have

$$\frac{\Delta R}{R_o} = \frac{1}{\gamma_{tr}^2 - \gamma_o^2} \left(\gamma_o^2 \frac{\Delta f}{f_o} - \frac{\Delta B}{B_o} \right), \quad (3)$$

where γ_o is the synchronous energy (in units of the rest energy) of the particle provided it stay on the non-distorted central orbit (2):

$$\gamma_o^2 = \left(\frac{c \rho_o B_o}{E_r} \right)^2 + 1 = \left[1 - \left(\frac{2 \pi R_o f_o}{ch} \right)^2 \right]^{-1}. \quad (4)$$

Thus, the new synchronous orbit will have a synchronous radius

$$R_s = R_o + \Delta R. \quad (5)$$

2. Synchronous Energy E_s

After radius R_s has been determined, the synchronous energy can be found by either of two ways. Either from (II.2.A2)

$$P_s = \rho_0 (R_s/R_0)^{1/\alpha} B, \quad (1)$$

$$E_s = E_B(B, R_s) = (c^2 P_s + E_r^2)^{1/2}, \quad (2)$$

or

$$E_s = E_f(f, R_s) = \gamma_s E_r = \left[1 - \left(\frac{2\pi R_s f}{ch}\right)^2\right]^{-1/2} E_r. \quad (3)$$

Since (2) and (3) provide approximate results (due to approximate R_s), it is a good idea to check the accuracy

$$|E_B - E_f| < \epsilon. \quad (4)$$

If for a given small $\epsilon > 0$ inequality (4) is true, then both approximations R_s and E_s are good enough.

Perhaps the most important relation is for the rate of change of the synchronous energy. The simplest way to get it is dividing differential relation (II.2.B4) by the time differential dt :

$$\dot{E}_s = \frac{\beta_s^2 E_s}{-\eta_s} \left(\frac{\dot{f}}{f} - \alpha \frac{\dot{B}}{B} \right), \quad (5)$$

where β_s , η_s should be calculated by using E_s , which in turn is determined as it is stated above. In particular, if $f = f_0$, $B = B_0$, $R = R_0 = \text{const.}$, then (II.2.B3) can be written as $\gamma^2 \dot{f}/f = \dot{B}/B$ and (5) become

$$\dot{E}_s = \frac{\beta_s^2 E_s}{-\eta_s} \left(\frac{1}{\gamma_s^2} - \frac{1}{\gamma_{tr}^2} \right) \frac{\dot{B}}{B} = \beta_s^2 E_s \frac{\dot{B}}{B},$$

which is equivalent to the Lorentz equation (II.2.4).

It is easy to see that synchronous energy will be constant $E_s = \text{const.}$ not only when $f = \text{const.}$, $B = \text{const.}$, but more generally when

$$\frac{\dot{f}}{f} = \alpha \frac{\dot{B}}{B} \quad (6)$$

$$f = f_0 \left(\frac{B}{B_0} \right)^\alpha. \quad (7)$$

Having condition (6) and according to (II.2.B3), the synchronous radius will change as

$$R_s = R_0 \left(\frac{B}{B_0} \right)^\alpha. \quad (8)$$

Let us estimate, for example, how long a particle with such synchronous radius will survive before it hits the aperture limits during the injection period in the Brookhaven AGS.

Example 1. During injection in the AGS we have the following parameters and functions: $R_0 = 128.5 \text{ m}$; $B(t) = B_0 + \dot{B}t$, $B_0 = 250 \text{ Gs}$, $\dot{B} = 4.9 \text{ Gs/ms}$; $\alpha = 0.014$.

Suppose there is an aperture limit $R_u < R_s < R_\ell$ and R_s starts moving at $t = 0$ from the upper limit $R_u = R_0 + 2 \text{ cm}$ until the lower limit $R_\ell = R_0 - 2 \text{ cm}$ when time is $t = t_\ell$. If time t_ℓ is short enough, then the linear expansion in (8) will be a good approximation for R_s :

$$R_s \approx R_0 \left(1 - \alpha \frac{\dot{B}}{B_0} t \right). \quad (9)$$

With this approximation, the difference $R_u - R_\ell$ between initial $t = 0$ and the final $t = t_\ell$ position can be expressed by the equation

$$4 \text{ cm} = R_0 + 2 - (R_0 + 2) \left(1 - \alpha \frac{\dot{B}}{B_0} t_\ell \right) = (R_0 + 2) \alpha \frac{\dot{B}}{B_0} t_\ell,$$

from which the time of survival can be found as

$$t_\ell = \frac{4}{12852 \cdot 0.014 \cdot 4.9/250} = 1.1 \text{ ms.}$$

3. Synchronous Phase ϕ_s

If a particle stays on the central orbit R_o , then the particle's synchronous phase is determined by the energy gain in the electrical system

$$\dot{E}_s = \frac{\omega_s}{2\pi} V \sin \phi_s, \quad \omega_s = \frac{2\pi f_o}{h} \quad (1)$$

and by the corresponding field gain in the magnetic system

$$\dot{E} = \frac{dE}{dp} \dot{p} = v \dot{p} = \omega_s R_o \rho_o \dot{B}_o, \quad (2)$$

which results in

$$\sin \phi_s = \frac{2\pi R_o \rho_o}{V} \dot{B}_o. \quad (3)$$

However, if the synchronous orbit is not the central one, then the energy gain (1) should be combined not with field-gain alone but with frequency-field gain (2.5), which results in

$$\sin \phi_s = \frac{h^2}{-aVf} \left(\frac{\dot{f}}{f} - \alpha \frac{\dot{B}}{B} \right), \quad (4)$$

where

$$a = \frac{h\eta_s}{\beta_s^2 E_s}. \quad (5)$$

Example 2. During injection in the AGS we have conditions described in Example 1. However, at the beginning of the injection the frequency rate of change does not follow its nominal* value of $\dot{f} = 33$ kHz/ms. Instead, it is chosen to be $\dot{f} = 22$ kHz/ms. What difference will it make in the synchronous phase? With $\rho_o = 85$ m, $V = 150$ kV, $\dot{B} = 4.9$ Gs/ms, $f = 2520$ kHz, $\gamma = 1.2$ calculations according to (4) will result in $\phi_s = 0.14$, while according to (3) $\phi_s = 0.23$. Thus, 33% of deviation in $\dot{f}$ from its nominal value lead to 39% of deviation in synchronous phase.

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*We consider frequency to be nominal if it is obtained from frequency-field relation (1.6) with nominal field $B_o = 250 + 4.9 t$, $[B_o] =$ Gs, $[t] =$ ms.

IV. Asynchronous Particle Motion

Since we established the basic formulae for synchronous particle (synchronous orbit R_s , energy E_s , and phase ϕ_s) we have a reference frame to describe a motion of all the other asynchronous particles. There is nothing new except for bucket height and length.

Let $E = E_s + \Delta E$ and $-\pi \leq \phi \leq \pi$ be an energy and phase of an arbitrary particle. The differential equations for particle motion are known as

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_s} \right) = \frac{V}{2\pi} (\sin \phi - \sin \phi_s), \quad (1)$$

$$\frac{d\phi}{dt} = a\omega_s \Delta E, \quad (2)$$

$$a = \frac{h\eta_s}{\beta_s^2 E_s}, \quad b = a\omega_s^2, \quad \omega_s = \frac{2\pi f}{h}, \quad (3)$$

where $a < 0$, $b < 0$ if $\eta_s < 0$.

Equations (1) and (2) can be derived from the Hamiltonian

$$H = b \frac{W^2}{2} + U, \quad (4)$$

where

$$W = \frac{\Delta E}{\omega_s}, \quad U = \frac{V}{2\pi} [\cos \phi + (\phi - \phi_s) \sin \phi_s - \cos \phi_s]. \quad (5)$$

Hamiltonian is the first integral of the system (1), (2), and formula (4) determines one-parameter (H) family of trajectories in phase space $(\Delta E, \phi)$.

Two singular or fixed points determined by

$$\partial H / \partial W = 0, \quad \partial H / \partial \phi = 0 \quad (6)$$

are $(\Delta E = 0, \phi = \phi_s)$ and $(\Delta E = 0, \phi = \pi - \phi_s)$.

The separatrix is the curve determined by the equation

$$H(\Delta E, \phi) = H(0, \pi - \phi_s) \quad (7)$$

or

$$(\Delta E)^2 = \frac{V}{a\pi} [(\pi - \phi_s - \phi) \sin \phi_s - \cos \phi_s - \cos \phi]. \quad (8)$$

The closed part of this curve is called the "bucket" (Figure 2). The synchronous particle sits at the center of the bucket, which is a stable, fixed point $(\Delta E = 0, \phi = \phi_s)$. The height of the bucket can be found from (8) at $\phi = \phi_s$ as

$$E_h = 2 \sqrt{\frac{2V}{a\pi} \left[\left(\frac{\pi}{2} - \phi_s \right) \sin \phi_s - \cos \phi_s \right]}. \quad (9)$$

Figure 2. Particle trajectories and bucket in phase space.

If ϕ_s is close enough to zero to approximate $\cos \phi_s$ by 1 and $\phi_s \sin \phi_s$ by 0, then the height can be approximated by

$$E_h = 2 \sqrt{\frac{2V}{|a|\pi} \left(1 - \frac{\pi}{2} \sin \phi_s \right)} \quad (10)$$

or with $\sin \phi_s$ taken from (III.3.4)

$$E_h = 2 \sqrt{\frac{2V}{|a|\pi} \left[1 + \frac{\pi h^2}{2aVf} \left(\frac{\dot{f}}{f} - \alpha \frac{\dot{B}}{B} \right) \right]}. \quad (11)$$

In particular, if $\frac{\dot{f}}{f} \gg \alpha \frac{\dot{B}}{B}$ which is the case at the injection in the AGS, then

$$E_h = 2\sqrt{\frac{2V}{|a|\pi} - \frac{h^2}{a^2} \frac{\dot{f}}{f^2}}. \quad (12)$$

Example 3. If the frequency rate $\dot{f}$ deviates from its nominal value as it was described in Example 2 of Section III, then the bucket height will deviate from its nominal value. Indeed, with the nominal $\phi_s = \phi_{sn} = 0.23$ and with the deviation $\phi_s = \phi_{sd} = 0.14$, we will have from (9) the ratio of corresponding heights as $E_{hd}/E_{hn} = 1.09$.

The bucket width $\Delta\phi$ is a distance between two buckets extreme points, which is an unstable fixed point $\phi_u = \pi - \phi_s$ and the turning point ϕ_e (Figure 2):

$$\Delta\phi = \phi_u - \phi_e = \pi - \phi_s - \phi_e. \quad (13)$$

Because the Hamiltonian $H(\Delta E, \phi)$ has a constant value in any point of the separatrix, the extreme point ϕ_e can be found from the equation

$$H(0, \phi) = H(0, \phi_u)$$

or

$$\cos\phi + \phi \sin\phi_s = (\pi - \phi_s) \sin\phi_s - \cos\phi_s \quad (14)$$

The numerical solution ($\phi_e = \phi_e(\phi_s)$) of this equation was tabulated⁵ for different values of ϕ_s . An analytical approximation can be obtained with some precautions. If ϕ_e is an exact solution of (14), then by differentiating it with respect to ϕ_s we will find

$$\frac{d\phi_e}{d\phi_s} (\sin\phi_s - \sin\phi_e) = (\pi - \phi_e - \phi_s) \cos\phi_s. \quad (15)$$

Knowing that $\phi_s = 0$ corresponds to a stationary bucket with $\phi_e = -\pi$, we will find from (15) that

$$\lim_{\phi_s \rightarrow 0} \frac{d\phi_e}{d\phi_s} = \infty, \quad (16)$$

which means that ϕ_e cannot be approximate in the vicinity of $\phi_s = 0$ by any linear or polynominal function. A quick and rough approximation can be constructed as follows. Expanding trigonometrical functions in (14) in a power series around $\phi = -\pi$ and $\phi_s = 0$, we will retain terms with ϕ up to the second order and with ϕ_s to the first order:

$$\phi^2 + 2(\pi + \phi_s)\phi + \pi^2 - 2\pi\phi_s = 0. \quad (17)$$

The solution $\phi_e \geq -\pi$ of this equation is

$$\phi_e = -(\pi + \phi_s) + \sqrt{4\pi\phi_s + \phi_s^2}. \quad (18)$$

Retaining here the terms with the lowest order in ϕ_s we come to

$$\phi_e = -\pi + 2\sqrt{\pi\phi_s}. \quad (19)$$

Satisfying to (16) this approximation is supposed to illustrate the analytical behavior of ϕ_e rather than to serve as a tool for computations although the relative error in (19) is less than 10% for $0 \leq \phi_s \leq 20^\circ$.

Differentiation of (19) with respect to time yields

$$\dot{\phi}_e = \dot{\phi}_s \sqrt{\frac{\pi}{\phi_s}}. \quad (20)$$

For a stationary bucket $\phi_s = 0$ and (20) shows that transition from stationary to accelerating bucket should be smooth and fast enough in order to preserve the particles from leaving the left corner of the stationary bucket. For instance, if the stable, fixed point ϕ_s starts to move as $\phi_s = t^2$ during transition from stationary to accelerating bucket, then the turning point ϕ_e will move from $\phi_e = -\pi$ with speed of $\dot{\phi}_e = 2\sqrt{\pi}$.

However, if ϕ_s is not fast enough, lets say $\phi_s = t^{2-\epsilon}$, then

$$\dot{\phi}_e = (2 - \epsilon) \sqrt{\frac{\pi}{\epsilon}} \cdot \frac{1}{t} . \quad (21)$$

This means that the turning point initially ($t = 0$) will move so fast that particles which were near that point will inevitably be lost from the bucket.

We have seen above that all bucket parameters - the center (or stable fixed point), the size (height and width), and the shape - are determined by the state of the accelerator, i.e. magnetic field $B(t)$, radio frequency $f(t)$, and* cavity voltage $V(t)$.

This means that the existence of the bucket as a certain area in phase space and within the machine does not depend on the presence of actual physical particles in an accelerator. Such an independence of the bucket makes it a unique reference frame moving in an accelerator according to prescribed control functions $B(t)$, $f(t)$, $V(t)$.

Now, if there are actual particles in the accelerator, then they have two choices:

First. If the triplet (B, f, V) represents three constants or three functions slowly changing with time, then the Hamiltonian surface $H(\Delta E, \phi, t)$ similarly will be constant or almost constant. The lines of level $H(\Delta E, \phi) = \text{const.}$ will be called streamlines. The importance of this first case is the fact that in phase space, the actual particle trajectories coincide with the streamlines of the Hamiltonian surface. The important consequence is that the particles within the bucket will remain there while particles outside the bucket will remain there.

Second. If at least one of B, f, V is not adiabatic (changes rapidly), then the particle's trajectories do not coincide with the streamlines. Moreover, trajectories can crossover and they can cross the separatrix.

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*Earlier we defined the state of the accelerator as a pair (B, f) which was enough for synchronous particles (p_s, R_s) only. For bucket description and for asynchronous particle description, we will extend the concept of accelerator state to the triplet (B, f, V) .

Turning back to the adiabatic quartet (B, f, p, R) and to the equilibrium equations such as, for instance

$$p = p(B, f), \quad R = R(B, f)$$

we can ask a question: What is the meaning of the synchronous particle (p, R) , if the state of the accelerator $B(t), f(t)$ is not adiabatic?

The answer is: The synchronous particle (p, R) does not represent a physical particle, but a reference frame which will hold an actual particle in adiabatic conditions and will otherwise provide synchronous momentum (energy) and synchronous (orbit) radius as a reference frame for dynamical equations, describing asynchronous particle motion.

V. Computer Simulations of Injection and Capture

A computer program was developed to solve dynamical equations (IV.1), (IV.2), by the usual method of difference approximation. The definition of a synchronous particle was adopted as it was stated above in Section III, namely:

- the synchronous radius R_s calculated according to (III.1.2-5),
- the synchronous energy E_s calculated from (III.2.1-4),
- the synchronous phase ϕ_s calculated from (III.3.4).

In the computer model, we also introduced aperture limits. Those are distances $\Delta R = \pm 2.5$ cm from the central orbit of radius $R_0 = 128.5$ m. During the tracking, each particle is subject to counting provided the particle orbit radius R satisfies $R_0 - \Delta R < R < R_0 + \Delta R$. If this condition is not satisfied, then the particle is lost (hits aperture) and is removed from further calculations. The results of the computer simulations are presented in Figures 3-7.

In this section we will discuss some features of injection and capture, we will interpret the current AGS controlling program using a computer model, and finally we will propose one particular program which can reduce particle losses.

At Brookhaven National Laboratory (BNL) during 0.3 ms of the injection, protons at 200 MeV come from the Linac to the AGS for subsequent capture into 12 buckets and for further acceleration. The injection particles are not monoenergetic. The energy spread is ± 0.4 MeV.

Suppose for simplicity that injected particles are homogeneously distributed in phase space within a rectangle $200 - 0.4 \leq \Delta E \leq 200 + 0.4$ and $-\pi \leq \phi \leq \pi$. This rectangle can be represented by "injected ribbon" as can be seen in Figure 3.

In stationary conditions, it would be possible to adjust the frequency f to the field B in such a way that the injected ribbon would be placed exactly at the middle of the bucket with a minimum of lost particles. The price for such high efficiency capture is very dense particle population at the center of the bucket which leads to increasing unwanted space charge effects during later acceleration and transition. For this reason and for other reasons (to avoid a rippling field, for example) injection into the AGS is made into the moving bucket. Obviously, the width of the moving bucket is less than that of the stationary one and the shorter part of the injected ribbon will be captured. In addition, because the injected ribbon holds constant energy while the bucket increases its energy, the first and the last portion of the beam will not be placed into the bucket's widest part. Rather, these ribbons will be separated (Figure 4), which will result in further

particle losses. Even with the maximum available voltage $V = 0.3$ MV and with frequency adjusted to keep the synchronous particles on the central orbit ($\dot{f} = 33$ kHz/ms), the losses are almost one-third of the injected number of particles.

Fig. 3. Injection in stationary bucket: $B=251$ Gs, $f=2522.8$ kHz, $V=0.3$ MV.

Fig. 4. Injection into the moving bucket: $V=0.3$ MV, $B=250.6+4.9T$, $\dot{f}=2517.8+33t$; $[B]=\text{Gs}$, $[f]=\text{kHz}$, $[t]=\text{ms}$.

Thus, the necessity to inject into the moving bucket is the first obstacle that decreases the capture efficiency. The second obstacle comes from injection technology.⁶ During injection, the AGS central orbit is locally distorted forming an orbit bump located at the stripping foil, where injected H^- ions are stripped from electrons and "new-born" protons continue their way along the orbit. After injection the local bump, controlled by two magnets, collapses adiabatically onto the central orbit.

Significant dilution of the horizontal and vertical linac emittances during injection require reduction in the bucket height and consequently in the bucket width. To do this, the cavity voltage $V(t)$ should be reduced to 150 kV during injection versus $V = 300$ kV afterwards.

Thus, the second factor limiting capture efficiency is reduced voltage, resulting in reduced bucket size. If such reduced voltage would stay constant during the whole injection and capture period, then the losses would reach up to 40% as we can see in Figure 5.

To avoid such big losses, a control program was implemented at the AGS. This program combines frequency and voltage control as follows.⁶

When injection starts, the voltage is 150 kV and the frequency rate is $\dot{f} = 22$ kHz/ms. Then, the voltage ramps up in 300 μ s to the final 300 kV and frequency switches to $\dot{f} = 33$ kHz/ms. The frequency transition from $\dot{f} = 22$ to 33 is not smooth. During the voltage ramp up, " $\dot{f}$ dips to near zero briefly (≈ 100 μ s) before attaining the nominal synchronous value"⁶ of 33 kHz/ms.

The results of computer simulations of the program are presented in Figure 6.

Fig. 5. Injection into reduced bucket: $V=.15$ MV, B and f are the same as in Fig. 4.

Fig. 6. Computer modeling representing AGS control program.

There are three graph sections in Figure 6. The upper graph $E(t)$ has three curves. The middle curve $E_s(t)$ represents an evolution of synchronous energy with the time. Roughly speaking, it characterizes the energy of the bucket as a whole. After each 0.1 ms, the curve E_s is crossed by vertical lines representing the height of the bucket at that time. Two other curves below and above E_s are the aperture limits in energy units. If the bucket reaches one of those limits, then the part of the bucket particles hit the aperture and are lost. Thus, the upper and lower aperture curves compose the aperture corridor where the bucket is moving. In this corridor, the solid horizontal line ending at the vertical time line represents constant injection energy. The time line crossing all the graphs represents the duration of the injection which starts at $T = 0$ and finishes at $T = 0.3$ ms.

The middle graph shows the frequency program $f(t)$. It starts with constant $f = 22$ kHz/ms and during 100 μ s between $T = 0.2$ ms and $T = 0.3$ ms it stays with $f = 0$ and then goes with $f = 33$ kHz/ms.

The lower graph shows the voltage program, which is linear between $T = 0.2$ ms and $T = 0.5$ ms and which is constant elsewhere. We can also see an intensity curve $I(t)$, which is the total number of particles at each instant or total charge vs. time.

Five pictures in Figure 6 represent particle position in phase space $(\Delta E, \phi)$. Each dot is a particle. The first picture shows the bucket at the beginning of injection, injection ribbon, and the first 39 particles after the first turn (that is why the particles are shifted from the ribbon) when the time of 4.7 μ s shown after rounding as 0 ms.

Please note the two pairs of pointers at the frame of each picture. The two pointers above the bucket represent the current position of the outer (high) aperture. The two pointers below the bucket represent the level of the inner (low) aperture.

The third picture ($T = 0.3$ ms) shows two injection ribbons--two adjacent horizontal rectangles. The upper ribbon represents the first portion of particles (as in the first picture), the lower ribbon represents the last portion of 39 injected particles. At the bottom of the two last pictures, we can see mountain ranges. The mountain range is a series of curves representing particle intensity distribution versus phase angle from $-\pi$ to $+\pi$ and versus revolution turns, in this case after each 20 turns.

Each of the five pictures shows bucket instant contour. The last picture shows three buckets: at the beginning of injection ($T = 0$), at the end of injection ($T = 0.3$), and at the later time ($T = 1.6$ ms) when all unstable particles have been lost.

To explain the main features of the AGS controlling program, we should consider its two elements:

1. For the fixed duration of injection into the moving bucket, the distance between the first and the last injected ribbon depends on the "speed of the bucket acceleration", that is $\dot{E}_s$, which in turn depends on $\dot{f}$ (see III.3.1, III.3.4). The higher the $\dot{f}$, the bigger the distance between ribbons, which means less number of captured particles. That is why in the AGS injection starts not with nominal $\dot{f} = 33$ kHz/ms but $\dot{f} = 22$ kHz/ms. However a reduction in $\dot{f}$ will lead to a reduction of synchronous radius R_s (II.2.B3): the bucket will move closer and closer to the inner aperture. The smaller $\dot{f}$ is, the smaller the bucket's time of survival will be (see Example 1 in Section III.).

Thus, the actual initial value of $\dot{f}$ is the AGS experimental optimum between capture efficiency and bucket survival.

2. At the time of $T = 0.2$ ms when the frequency program switches from $\dot{f} = 22$ kHz/ms to $\dot{f} = 0$, it creates a bucket which covers almost all particles. If the transition from $\dot{f} = 0$ to the final $\dot{f} = 33$ kHz/ms would be slow and smooth enough, then the corresponding evolving bucket could retain all the particles. However, $\dot{f}$ changes abruptly at $T = 0.3$ ms from 0 to 33. As a result, the bucket jumps from a stationary position to an accelerating one losing 16% of the particles as we can see in the picture. Even the rising voltage cannot help to recapture those particles and they will be ultimately lost.

Now we will consider another controlling program, enhancing some features of an existing AGS program. The following program consists of two parts. First, we will block all the injected particles from going below the bucket during injection. Second, we will capture the particles out of the bucket into it by appropriately raising the voltage. Figure 7 represents computer simulations according to such a program.

Fig. 7. Proposed controlling program for injection and capture.

To block the particles from moving below the bucket, the later should be stationary and injected beam should be placed into the upper half of the bucket as it is shown in the first picture $T = 0$ ms. Because the magnetic field increases as $B(t) = 249.3 + 4.9 t$ ($[B] = \text{Gs}$, $[t] = \text{ms}$), the frequency rate, calculated from (III.2.6) should be $\dot{f} = 0.7 \text{ kHz/ms}$. Such a rate can produce a stationary bucket and keep it for 300 μs of injection. During that time, the bucket will move toward the inner aperture as we can see on the picture with graphs $E(t) = [\text{MeV}]$. After 0.2 ms of injection, the voltage starts to ramp up from 50 kV linearly with $\dot{V} = 650 \text{ kV/ms}$. The rising voltage widens the bucket allowing capture, from the outside, of those particles which are close to the bucket. At the end of injection ($T = 0.3 \text{ ms}$) the frequency rate changes from $\dot{f} = 0.7$ to $\dot{f} = 10 \text{ kHz/ms}$ and stays at that level for 0.2 ms. This intermediate period serves as a transition from stationary to fully accelerating mode, preventing particle spill below the bucket due to the particle inertia.

The voltage slope $\dot{V} = 650 \text{ kV/ms}$ is a compromise between two requirements. One is the capture requirement: the steeper slope provides more efficient (non-adiabatic) capture. Another is the acceleration requirement. After capture completion (see picture with $T = 0.41 \text{ ms}$) too many particles stay dangerously close to the bucket boundary. Those particles could spill out due to a small deviation in frequency. However, increasing voltage widens the bucket, rounds off trajectories and introduces a protective layer free of particles. The longer the voltage is ramping, the fewer particles are found in the inner vicinity of the bucket boundary. The last picture ($T = 1.7 \text{ ms}$) shows three buckets. The stationary one, the intermediate one (at the end of the injection, $T = 0.3 \text{ ms}$) and the final bucket, containing almost all injected particles.

References

1. L.D. Landau and E.M. Lifshits, Mechanics, Third Edition, Pergamon Press.
2. C. Bovet, R. Gouiran, I. Bumowski, K.H. Reich, A Selection of Formulae and Data Useful for the Design of A.G. Synchrotrons, CERN/MPS-SI/Int. DL/70/4.
3. E.D. Courant and H.S. Snyder, Theory of the Alternating-Gradient Synchrotron, Annals of Physics, 3, 1-48 (1958).
4. H. Bruck, Accélérateurs Circulaires de Particules, Paris, Presses Universitaires de France (1966).
5. I. Gumowski, CPS RF Bucket Width, Height and Area, MPS/Int. RF67-1, 20.3.1967.
6. E. Raka, L. Ahrens, W. Frey, E. Gill, J.W. Glenn, R. Sanders, W. Weng, Increased Intensity Performance of the Brookhaven AGS, IEEE Trans. on Nucl. Sci., Oct. 1985, Vol. NS-32, No. 5, Part II, 3110-3112.