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ANALYSIS OF A LATTICE WESS-ZUMINO SCHEME FOR CHIRAL FERMIONS[†]

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We show that the four-dimensional version of the chiral lattice fermion scheme proposed by Aoki and Funakubo and Kashiwa leads, in the continuum limit, to a transverse, but quadratically divergent, contribution to the effective vacuum polarization. Hence, the scheme does not make sense without the addition of counterterms beyond those that are required in the continuum theory.

1. INTRODUCTION

In recent years, there has been much research devoted to the problem of formulating a gauge invariant chiral fermion theory on a spacetime lattice. The problem is interesting from both a theoretical and a practical standpoint: a solution would shed light on the properties of a quantum field theory containing fermions and would enable one to perform numerical simulations of the standard electroweak model.

The minimal requirements for a satisfactory lattice chiral fermion theory are that it contain a single chiral fermion species for each flavor in the Lagrangian and that it be gauge invariant under the transformations of the full chiral gauge group. The property of gauge invariance is necessary to ensure the renormalizability and the unitarity of the theory. One sees immediately that the choice of the lattice fermion derivative is the major stumbling block in formulating a lattice theory of chiral fermions. Since one cannot choose any lattice derivative that results in fermion species doubling, the simplest candidate is the Wilson derivative.¹ In the Wilson case, S_{ferm} , the fermionic part of the lattice action in d dimensions, consists of a sum of two terms:

$$\begin{aligned}
 S_{ferm} &= S_{Dirac}(\bar{\psi}, \psi) + S_{Wilson}(\bar{\psi}, \psi) \\
 &= -a^d \sum_{n,\mu} \bar{\psi}(n) \gamma_\mu \frac{1}{2a} [\psi(n + a\hat{\mu}) - \psi(n - a\hat{\mu})] \\
 &\quad + a^d \sum_{n,\mu} \bar{\psi}(n) \frac{1}{2a} [\psi(n + a\hat{\mu}) + \psi(n - a\hat{\mu}) \\
 &\quad \quad - 2\psi(n)],
 \end{aligned}
 \tag{1.1}$$

where $\psi(n)$ denotes the fermion field located at the

site n , a denotes the lattice spacing, $\hat{\mu}$ denotes a unit vector in the μ direction, and the γ_μ denote the Euclidean γ matrices which satisfy the anti-commutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. We define γ_5 such that, $\{\gamma_5, \gamma_\mu\} = 0$ and $\gamma_5^2 = 1$. The first term in (1.1) is the usual lattice transcription of the Dirac (naive) derivative. The second term is Wilson's momentum-dependent "mass" term which lifts the species degeneracy by giving the extra species an infinite mass in the continuum limit. Unfortunately, the Wilson term breaks the axial part of the chiral gauge invariance because it couples left-handed and right-handed fields which, in general, have independent gauge transformations. Hence, it appears that the Wilson formulation does not lead to a satisfactory theory.

2. THE AOKI, FUNAKUBO AND KASHIWA (AFK) STRATEGY

In an attempt to circumvent the difficulty of the gauge-noninvariance of the Wilson chiral theory, Aoki² and Funakubo and Kashiwa,³ have suggested a lattice chiral fermion formulation based on the lattice analog of the well-known Wess-Zumino⁴ construction.⁵ Their construction involves the introduction into the Wilson term of scalar fields $g^L(n)$ and $g^R(n)$, whose gauge transformations compensate exactly for the gauge transformations of the left- and right-handed fermi fields respectively. In our analysis, we may, without loss of generality, drop the couplings of the gauge fields A_μ to the right-handed fermions and set the scalar fields $g^R(n)$ to unity. Then, the fermionic part

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of the gauged AFK action is

$$\begin{aligned}
S_{AFK} &= S_{\text{Dirac}}(\bar{\psi}, \psi, U) + S_{\text{Wilson}}(\bar{\psi}, \psi, g), \\
&= -a^d \sum_{n,\mu} \bar{\psi}(n) \gamma_\mu \frac{1}{2a} \left[(U_\mu(n) P_L + P_R) \psi(n + a\hat{\mu}) \right. \\
&\quad \left. - (U_\mu^\dagger(n - a\hat{\mu}) P_L + P_R) \psi(n - a\hat{\mu}) \right] \\
&\quad + a^d \sum_{n,\mu} \bar{\psi}(n) \frac{1}{2a} \left[P_L (g(n + a\hat{\mu}) \psi(n + a\hat{\mu}) \right. \\
&\quad \left. + g(n - a\hat{\mu}) \psi(n - a\hat{\mu}) - 2g(n) \psi(n)) \right. \\
&\quad \left. + g^\dagger(n) P_R (\psi(n + a\hat{\mu}) + \psi(n - a\hat{\mu}) - 2\psi(n)) \right], \tag{2.1}
\end{aligned}$$

where the $U_\mu(n) = \exp(ieaA_\mu(n))$ denote the usual link variables, e denotes the gauge field coupling constant, $P_{L,R} = (1 \pm \gamma_5)/2$, and we have dropped the superscript on $g^L(n)$. Since, as AFK show explicitly, (2.1) is gauge invariant, we expect the lattice theory defined by this action to be anomaly free. Aoki points out that the action (2.1) can be constructed from the gauged version of (1.1) by a change of variables:

$$\begin{aligned}
\psi(n) &= (g(n) P_L + P_R) \psi'(n), \\
\bar{\psi}(n) &= \bar{\psi}'(n) (g^\dagger(n) P_R + P_L), \\
U_\mu(n) &= g(n) U'_\mu(n) g^\dagger(n + a\hat{\mu}). \tag{2.2}
\end{aligned}$$

Owing to the fact that the fermion measure, $\int [d\bar{\psi}][d\psi]$, is invariant under (2.2), the partition function for the AFK action is identical to that of the original gauged Wilson theory. We note, however, that the Green's functions evaluated in the two cases are, in general, quite different. That is,

$$\begin{aligned}
&\int [d\bar{\psi}][d\psi][dU] \mathcal{O} \exp(-S_{\text{Wilson}}(\bar{\psi}, \psi, U)) \\
&\neq \int [d\bar{\psi}][d\psi][dU][dg] \mathcal{O} \exp(-S_{\text{AFK}}(\bar{\psi}, \psi, U, g)) \tag{2.3}
\end{aligned}$$

for $\mathcal{O}$ an arbitrary operator. For example, matrix elements of the axial-vector current in the Wilson theory are anomalous, that is, they have a nonvanishing divergence. On the other hand, since the AFK action is chirally invariant, the axial-vector current is conserved in that theory. Evidently, the additional contributions to the axial-vector current arising from the scalar fields cancel those arising from the fermion fields. Hence, the AFK scheme with a single fermion flavor is nonanomalous. This is to be contrasted with the continuum theory, in which one can cancel the anomaly only through

the introduction of additional flavors. Another way to see that the Wilson and AFK theories may lead to different physics is to integrate over the scalar fields. In the AFK theory, the resulting effective action contains four-fermi interactions that are not present in the Wilson theory.² Although these four-fermi interactions vanish in the naive continuum limit (tree amplitudes), they could potentially contribute to amplitudes involving loops.

In light of these examples, one might ask whether the AFK theory actually corresponds to chiral QED in the continuum limit. Further, one might question whether the amplitudes of the AFK theory have the appropriate (renormalizable) behavior in the continuum limit. In their work, AFK attempt to address these questions by applying their scheme to the case of chiral QED in two dimensions, that is, the chiral Schwinger model. They calculate $\mathcal{W}(A)$, the effective gauge field action that one obtains by integrating over the fermion and scalar fields. The AFK scheme successfully reproduces the well-known mass gap of the continuum theory.

In order to assess the significance of this result, let us review briefly the properties of the continuum chiral Schwinger model. Two basic variants of the model have been studied. Jackiw and Rajaraman⁶ have considered the anomalous version in which the apparent chiral gauge invariance—present in the classical Lagrangian—is broken by quantum corrections. The spectrum of the theory contains a massive vector particle, whose mass is governed by the renormalization ambiguity of the theory. The second variant of the chiral Schwinger model^{7,8} contains an additional Wess-Zumino term which restores the chiral gauge invariance and renders the quantum field theory nonanomalous. However, its spectrum is identical to that of the Jackiw-Rajaraman version. In fact, Falck and Kramer⁸ have concluded that the Jackiw-Rajaraman variant is a gauge-fixed version of the Wess-Zumino variant.

Let us consider the lattice chiral Schwinger model. Since the addition of a Wess-Zumino term does not alter the spectrum of the continuum model, we would expect this to be the case for the lattice version as well. That is, in two dimensions we would not expect the gauge invariance of the lattice theory to play a crucial role in determining the spectrum. Hence, a

successful application of the AFK scheme to the chiral Schwinger model does not constitute a stringent test of the scheme. We also note that, since two-dimensional gauge theories are super-renormalizable, the lattice chiral Schwinger model does not provide a test whether the Wess-Zumino construction could preserve the renormalizability property of chiral gauge theories. In order to provide a further test of the AFK scheme, we apply it to four-dimensional chiral QED.

3. CHIRAL QED

3.1. Diagrammatic Method in Two Dimensions

In order to illustrate a diagrammatic technique which generalizes readily to the four-dimensional case, we repeat the AFK calculation of the effective gauge field action $W(A)$ in the chiral Schwinger model. The effective action is defined by

$$\exp W(A) = \int [d\theta] \exp(S_{gauge} + \Gamma[A, \theta]), \quad (3.1a)$$

where

$$\Gamma[A, \theta] = \ln \int [d\bar{\psi}][d\psi][d\theta] \exp(S_{AFK}(\bar{\psi}, \psi, A, \theta)), \quad (3.1b)$$

$$g(n) = e^{i\theta(n)}, \quad (3.1c)$$

and S_{gauge} is the usual gauge-field action. In evaluating $W(A)$ we first carry out the integration over the fermion fields. AFK have used the properties of $\Gamma[A, \theta]$ under gauge transformations to show that $\Gamma[A, \theta]$ can be rewritten as follows:

$$\begin{aligned} \Gamma[A, \theta] &= \ln \int [d\bar{\psi}][d\psi][d\theta] \exp\left(S_{AFK}\left(\bar{\psi}, \psi, A - \frac{\nabla\theta}{e}, 0\right)\right) \\ &= \Gamma\left[A - \frac{\nabla\theta}{e}, 0\right]. \end{aligned} \quad (3.2)$$

As illustrated in Figure 1, $\Gamma[A, 0]$ has a diagrammatic expansion in which each term consists of an n -point Green's function involving a single fermion loop and n gauge-field factors. Solid lines indicate fermion propagators, dots indicate fermion-photon vertices, and we use the convention that the (suppressed) Lorentz index of an expression adjacent to a vertex is contracted with the Lorentz index of that vertex. (In the case of the lattice theory, we use nonseagull diagrams of the type shown in Figure 1 to symbolize the sum of the seagull

and nonseagull contributions.)

$$\frac{1}{2} A \text{---} \bigcirc \text{---} A + \frac{1}{4} \begin{array}{c} A \text{---} \bigcirc \text{---} A \\ A \text{---} \bigcirc \text{---} A \end{array} + \dots$$

FIGURE 1

Diagrammatic expansion of $\Gamma[A, 0]$. Loops containing an odd number of vertices vanish by Furry's theorem.

The first term in the expansion of $\Gamma[A, 0]$ is $\frac{1}{2} A_\mu \Pi_{\mu\nu} A_\nu$, where $\Pi_{\mu\nu}$ denotes the vacuum polarization amplitude in the theory with the scalar fields θ set to zero. In the continuum chiral Schwinger model, the vacuum polarization gives the only nonvanishing contribution to $\Gamma[A, 0]$. This is also the case in the continuum limit of the lattice model. Hence, using (3.2), we see that the result of performing the integrations over the fermion fields is

$$\Gamma[A, \theta] = \int d^2x (A_\mu - \frac{\partial_\mu \theta}{e}) (\frac{1}{2} \Pi_{\mu\nu}) (A_\nu - \frac{\partial_\nu \theta}{e}). \quad (3.3)$$

Next we perform the θ integration. Following AFK, we make the usual weak-coupling approximation of extending the range of integration of θ to $\pm\infty$.⁹ Rather than using the direct integration method of AFK, we make use of diagrammatic techniques in which we expand the integrand $\exp(\Gamma[A, \theta])$ formally in a perturbation series. We treat the terms with no θ fields and the term that is bilinear in the θ fields exactly and expand the remaining terms in a power series. Hence, the term in (3.3) that is bilinear in the θ fields yields the θ propagator, and the terms linear in the fields A and θ , yield the A - θ vertices. In Figure 2 we show the fermion loop diagrams that contribute to the θ propagator and vertices. The momentum-space expressions for the propagator and the vertices are

$$S_\theta = \frac{-e^2}{k_\mu \Pi_{\mu\nu} k_\nu}, \quad (3.4a)$$

and

$$V_\theta = \frac{-i \Pi_{\mu\nu} k_\nu}{e}, \quad (3.4b)$$

respectively.

$$\begin{aligned}
& \left(-\frac{\partial\theta}{e} \right) \cdot \text{[circle with dots]} \cdot \left(-\frac{\partial\theta}{e} \right) = \text{---} \text{---} \text{---} \text{---} \quad (a) \\
& \left\{ \begin{array}{c} A \\ -\frac{\partial\theta}{e} \end{array} \right\} \cdot \text{[circle with dots]} \cdot \left\{ \begin{array}{c} -\frac{\partial\theta}{e} \\ A \end{array} \right\} = \left\{ \begin{array}{c} A \text{---} \theta \\ \theta \text{---} A \end{array} \right\} \quad (b)
\end{aligned}$$

FIGURE 2

Contributions to (a) the θ -field propagator and (b) the A - θ vertices. The left-hand sides of the equalities indicate the origins of these quantities in terms of the fermion-loop amplitudes. The right-hand sides show the diagrammatic notations that we use.

In order to obtain $W(A)$, we evaluate all of the diagrams containing only external gauge fields and no external θ fields. The only contribution is the exponential of the diagram in Figure 3b.

FIGURE 3

The leading contributions to the effective gauge-field action.

Hence, the effective action is given by Figure 3a and Figure 3b, where the diagram in Figure 3a corresponds to the term in $\Gamma[A, \theta]$ that is quadratic in the gauge fields. Evaluating these diagrams using (3.4a) and (3.4b), we find that the effective action is

$$W(A) = \frac{1}{2} \int d^2x A_\mu \left(\Pi_{\mu\nu} - \frac{k_\alpha \Pi_{\mu\alpha} \Pi_{\beta\nu} k_\beta}{k_\rho \Pi_{\rho\sigma} k_\sigma} \right) A_\nu. \quad (3.5)$$

The term in parentheses in (3.5) is the "effective" vacuum polarization for the AFK effective action in the chiral Schwinger model. We note that this quantity is transverse:

$$k_\mu \left(\Pi_{\mu\nu} - \frac{k_\alpha \Pi_{\mu\alpha} \Pi_{\beta\nu} k_\beta}{k_\rho \Pi_{\rho\sigma} k_\sigma} \right) = 0. \quad (3.6)$$

Hence, the effective vacuum polarization satisfies the

usual Ward identity; that is, the gauge field couples to a conserved chiral current.

3.2. Generalization to Four Dimensions

In the case of four-dimensional chiral QED, the analysis is complicated by the fact that the higher order diagrams no longer vanish. From Figure 1 we see that, in four dimensions, $\Gamma[A, \theta]$ contains contributions from any n -point amplitude with n even. Although we cannot evaluate $\Gamma[A, \theta]$ exactly, we can determine the leading divergent behavior in the limit $a \rightarrow 0$. Consider the first term in the series for $\Gamma[A, \theta]$, namely, the vacuum polarization $\Pi_{\mu\nu}$. Simple dimensional power counting indicates that $\Pi_{\mu\nu}$ can, at worst, be quadratically divergent in the limit $a \rightarrow 0$. In QED, the presence of the constraints imposed by the Ward identities leads to a reduction of this quadratic divergence to a logarithmic one. However, in AFK version of *chiral* QED, the Ward identities are not satisfied by the fermionic sector of the theory alone. (Recall that AFK introduced the scalar fields in order to restore the Ward identities for the effective theory.) Hence, we expect $\Pi_{\mu\nu}$ to display a quadratic divergence in the limit $a \rightarrow 0$. We have verified that this expectation is correct by an explicit calculation of $\Pi_{\mu\nu}$ for the case of zero external momentum. We find that

$$\Pi_{\mu\nu}(0) \approx -0.054 \frac{e^2}{a^2} \delta_{\mu\nu}. \quad (3.7)$$

A power counting analysis of the remaining amplitudes that can contribute to $\Gamma[A, \theta]$ reveals that they are all sub-leading—logarithmically divergent at worst.

Now, it remains to integrate over the θ fields. As in the two-dimensional case, we compute the gauge-field effective action by making a formal diagrammatic expansion of the partition function. The Feynman rules for the θ propagator and the lowest-order A - θ vertex are as given in (3.4a) and (3.4b) respectively. Of course, in the four-dimensional case, the Feynman rules include higher-order A - θ vertices, formed from the n -point amplitudes that contribute to $\Gamma[A, \theta]$. However, in working out the leading behavior in the limit $a \rightarrow 0$, we need only retain a finite number of terms in the perturbation series. It is easy to see from power-counting arguments that the diagrams which yield the leading divergence in a are the ones depicted in Figure 3. All diagrams that involve higher-order A - θ vertices are sub-leading. The reason is that every higher-order vertex is smaller by

at least a power of a (neglecting logarithms) than the vertex that arises from $\Pi_{\mu\nu}$ and every θ propagator brings in a power of a^2 as well. In fact, only a finite number of diagrams with internal θ lines contribute to $W(A)$ in the continuum limit. This is because n -point amplitudes with $n \geq 6$ are non-anomalous, that is, transverse in the limit $a \rightarrow 0$. Hence, if one attempts to build an A - θ vertex from such an amplitude, that vertex will vanish. Another way to look at this is that the Wess-Zumino construction introduces "correction terms" to remove the fermion anomalies. Since only a finite number of diagrams are anomalous, only a finite number of corrections arise.

Having worked out the leading behavior of the effective action in the limit $a \rightarrow 0$, we can now compute the leading behavior of various amplitudes in the theory. In particular, let us consider $\Pi_{\mu\nu}^{eff}$, the effective vacuum polarization for the AFK version of chiral QED. The diagrams which contribute to the leading behavior of this amplitude are those depicted in Figure 3. Hence, using (3.4a) and (3.4b) and substituting the leading behavior for $\Pi_{\mu\nu}$ from (3.7), we find that the leading behavior of $\Pi_{\mu\nu}^{eff}$ is given by

$$\begin{aligned} \Pi_{\mu\nu}^{eff} &\sim \left(\Pi_{\mu\nu} - \frac{k_\alpha \Pi_{\mu\alpha} \Pi_{\beta\nu} k_\beta}{k_\rho \Pi_{\rho\sigma} k_\sigma} \right) \\ &\sim -0.054 \frac{e^2}{a^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \end{aligned} \quad (3.8)$$

We see from (3.8) that the leading part of $\Pi_{\mu\nu}^{eff}$ is transverse. The Wess-Zumino construction has yielded, for the effective vacuum polarization, a particular projection of the ordinary non-transverse vacuum polarization. However, the leading behavior of $\Pi_{\mu\nu}^{eff}$, in the limit $a \rightarrow 0$, is quadratically divergent. Thus, the Wess-Zumino construction has restored the gauge-invariance of the theory, but has failed to control the ultraviolet (UV) behavior.

4. CONCLUSIONS

As we have seen, the lattice Wess-Zumino construction proposed by AFK restores the chiral gauge-invariance of the Wilson lattice-fermion theory. Consequently, the chiral Ward identities are satisfied and, in particular, all of the contributions to the effective gauge-field action are transverse. In two dimensions, the transversality of these contributions is a convenience, since it precludes the generation of longitudinal

modes for the gauge field. However, it is not essential to formulation of a satisfactory chiral gauge theory in two dimensions.¹⁰

In four dimensions, the AFK model is less satisfactory because the transversality of the contributions to the effective action is insufficient to ensure that the loops contained in these contributions are well behaved in the UV. Usually the transversality of an amplitude implies that its degree of divergence is reduced from the value that one obtains through simple power counting. However, the Wess-Zumino mechanism renders the contributions to the effective action transverse by forming certain projections of non-transverse amplitudes. Consequently, it does nothing to control the UV behavior of the loop integrals themselves. We have shown that, in the continuum limit, the leading divergence in the effective action originates in the vacuum polarization diagram and is proportional to $1/a^2$. That is, the gauge field develops a quadratically divergent "transverse mass." In order to obtain a sensible theory, one would need to add a counterterm to cancel at least this quadratic divergence. (The light-by-light-scattering diagram would probably develop a logarithmic divergence that would also require a counterterm.) Since the leading divergence of $\Pi_{\mu\nu}^{eff}$ contains a transversality projector, the corresponding counterterm would be non-local (at least at the level of the gauge-field effective action). In contrast, four-dimensional continuum chiral QED and lattice chiral QED with Wilson fermions require only local counterterms.

We conclude that, in four-dimensions, the AFK model differs from the corresponding continuum theory. In order to obtain a sensible theory in the continuum limit, one would need, at least, to add counterterms beyond those that appear in the continuum renormalization program.

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10. Since the longitudinal modes of the gauge-field propagator would go like $k_\mu k_\nu / e^2$, they could potentially spoil the power-counting arguments that one usually uses to eliminate higher-order fermion loops in an analysis of the Schwinger model. (We wish to thank Jan Smit for drawing our attention to this potential difficulty.) However, it can be shown, by a making use of a variant of the non-renormalization theorem for the axial anomaly (S. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517), that such higher-order contributions actually vanish in the continuum limit.