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A THEORY OF THE OPTIMAL DESIGN OF STRAIGHT-AXIS
MINIMUM-B MIRROR CONFINEMENT CONFIGURATIONS

Laurence S. Hall

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A THEORY OF THE OPTIMAL DESIGN OF STRAIGHT-AXIS
MINIMUM-B MIRROR CONFINEMENT CONFIGURATIONS

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ABSTRACT

The design of modern straight-axis linked-mirror plasma-confinement configurations involves a balance between many competing requirements. The dipole and quadrupole components of magnetic induction required in one confinement region often do not match onto the fields of an adjacent region without complications that seriously affect particle drifts or confinement stability. Here, the relevant factors are set down together with the techniques for analytical optimization of the design of a general configuration. A general sufficient condition for the stability of an arbitrary guiding-center MHD equilibrium is derived. This condition makes explicit the stabilizing qualities of "good" normal curvature and "diamagnetic" axial current. The instability drive depends on two terms: one carries the sign of normal curvature and the other relates to the relative signs of geodesic curvature and geodesic torsion. The theory is applied to low-beta, large-aspect-ratio equilibria for which analytic expressions for the confining magnetic fields are known. Two optimizations are required to specify the arbitrary features of the quadrupole and dipole fields. One optimization is nonlinear and can be performed by the ordinary calculus of variations; the second optimization is linear and subject to the rules of game theory. Appropriate quality factors are obtained, thus giving the designer quantitative measures with which to balance design trade-offs.

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I. INTRODUCTION

With the usual caveats that go with rules-of-thumb, the design of a stable, efficient, straight-axis, minimum-B, quadrupole magnetic mirror (or linked-mirror) confinement configuration can be characterized by how well one has satisfied two critical geometric conditions in the important containment region(s). The first condition relates to the ordinary (normal) curvature of a magnetic line of force; its sign should be mostly "good",¹ and it should be appreciable.² The second condition relates to the geodesic curvature (the component of the curvature lying in the principal flux surface) and the geodesic torsion, another property both of the line of force and of the principal flux surface. The geodesic curvature, in turn, relates to the quantity and direction of flow of plasma currents along a line of force. These axial currents should be of "good" sign, as determined by the geodesic torsion, but not too appreciable.³

Subject to some restrictions, these rules-of-thumb can be given quantitative measure, principally as a consequence of the requirements of stability. In the following, this quantification will be written in terms of appropriate averages of the geometric factors ω , Eq. (9), a mixed but predominantly normal curvature term, and λ , Eq. (7), which measures the geodesic curvature. In the final analysis, the restrictions mean that our detailed theory applies only to small gyroradius, large-aspect-ratio, low-beta equilibria--although some of its features can be carried well beyond these limitations. [In particular, we find a general sufficient condition for stability that applies to arbitrary guiding-center magnetohydrodynamic (MHD) equilibria.]

In spite of the formal restrictions of our detailed theory, we argue that our results also can be of value to finite-beta design. In the final design stages, the prudent designer naturally will want to utilize the elaborate numerical codes now available that were constructed to evaluate fully the effects of finite beta and, to some extent, finite gyroradius. But such codes are complicated and expensive to run; they require special expertise, and they are hardly intuitive. During much of any design phase, it is useful to have a more flexible and interactive computational tool that can help to estimate the quality of equilibrium. In addition, the earliest finite-beta computations already showed that most configurations tend not to

distort themselves too greatly until an unstable boundary is approached.⁴ The most recent calculations of tandem-mirror systems seem to support this observation.⁵ Moreover, we can couch our analysis in terms of just those geometric factors whose vacuum values are the most well preserved at finite beta; so doing, we try to carry our results as far as possible beyond their strict low-beta limitations. Of course, the proof of success at finite beta will lie in the correspondence of our results with those of the bigger codes.

We will review only our design theory in this present paper. Numerical work will be reported later.

We will first derive in Section II the general finite beta, arbitrary aspect ratio, guiding-center, MHD foundations of our theory. This section points out how the fundamental geometrical elements enter the general theory and gives one confidence that, if strictly necessary, our results could be reformulated for finite-beta, arbitrary-aspect-ratio application.

In Section III, we step to low-beta, large-aspect-ratio equilibria in order to introduce analytic expressions for the magnetic field and its characteristic geometric parameters. These parameters are then used to evaluate our own sufficient condition for stability (derived in Section II).

With the critical stability parameters in hand, we pause in Section IV to discuss our general design philosophy, which is then implemented in Sections V-VII. We perform first (Section VI) and second (Section VII) optimizations to determine both the quadrupole and dipole components of magnetic field. It is of interest that one of these optimizations is necessarily linear and, therefore, a problem of game theory⁶ (see Sections V and VII). After both optimizations we compute in Section VIII an upper limit to the confined plasma pressure for stability. This upper limit is defined in terms of our sufficient condition for stability (Sections II and III). Finally, our summary and conclusions are given in Section IX.

II. GUIDING-CENTER MHD STABILITY OF THE MIRROR WELL

We begin with the full three-dimensional theory of an anisotropic, finite-beta, guiding-center plasma equilibrium.^{7,8} Assuming omnigenous symmetry as the basis of long-term equilibrium,⁸ the pressures can be assumed to be isorhopic, $p_{\perp,\parallel} = p_{\perp,\parallel}(\psi, B)$, and the Kruskal-Oberman energy principle for the determination of stability can be written:⁷

$$\Delta W = \Delta W_0 + \Delta W_M + \Delta U, \quad \Delta W_M \geq 0, \quad \Delta U \geq 0.$$

The non-negative integrals ΔW_M and ΔU are quadratic functionals of $\nabla \cdot \underline{\xi}$ and $\underline{\kappa} \cdot \underline{\xi}$, and

$$\begin{aligned} \Delta W_0 = (8\pi)^{-1} \int d\psi d\alpha d\ell \{ (2\pi B)^{-1} (h/B) (x' \nabla \alpha - y' \nabla \psi)^2 \\ - (T/B) [(\underline{\kappa} \times \underline{b} \cdot \nabla \alpha) x - (\underline{\kappa} \times \underline{b} \cdot \nabla \psi) y] x - (h/B) (x'y - y'x) \}, \quad (1) \end{aligned}$$

where (ψ, α) are the Clebsch coordinates of the induction $\underline{B}$ along which distance ℓ is measured, $\underline{B} = (2\pi)^{-1} \nabla \psi \times \nabla \alpha$, $B = |\underline{B}|$, $\underline{b} = B^{-1} \underline{B}$; $\underline{\kappa} = \underline{b} \cdot \nabla \underline{b}$ is the curvature vector whose components along $\nabla \psi$ and in the principal flux surface, $\psi = \text{const}$, are the "normal" and "geodesic" curvatures, respectively;⁹
 $h = B + 4\pi B^{-1} (p_\perp - p_\parallel)$, $T = -4\pi B^{-1} [(\partial p_\parallel / \partial \psi) + (h/B)(\partial h / \partial B)^{-1} (\partial p_\perp / \partial \psi)]$;
 $\iota = \underline{b} \cdot \nabla x \underline{b}$, where ι is the rotational shear, which also measures the axial flow of plasma current since in equilibrium $\underline{b} \cdot \nabla x \underline{b} = 4\pi c^{-1} (j_\parallel / B)$; $x = (\underline{\xi} \cdot \nabla \psi / 2\pi)$,
 $y = (\underline{\xi} \cdot \nabla \alpha / 2\pi)$, where $\underline{\xi}$ is the usual displacement vector; and the prime denotes the directional derivative along a line of force, $x' = \underline{b} \cdot \nabla x$. We assume that both $h > 0$ to avoid firehose instability, and $\partial h / \partial B > 0$ to avoid the mirror instability.⁷

Clearly, it is sufficient for stability if $\Delta W_0 > 0$. Indeed, for large-aspect-ratio equilibria, consideration of ΔW_M and ΔU requires that $\nabla \cdot \underline{\xi} = 0$ for admissible perturbations, and then the requirement that ΔW_0 be nonnegative is both necessary and sufficient for stability.¹⁰ The same holds for all perturbations (ballooning) localized to a line of force.¹¹

A looser but considerably simpler sufficient stability-condition can be derived from Eq. (1); indeed, the associated modified form of ΔW_0 is instructive for its general application. We define $\underline{n} = |\nabla \psi|^{-1} \nabla \psi$ as the outward unit vector normal to the principal flux surface, and introduce $\underline{\tau} = \underline{b} \cdot \nabla \underline{n}$ as a vector tangent to the ψ surface. That is, $\underline{\tau} \cdot \underline{n} = 0$ trivially, and equally obviously $\underline{\kappa} \cdot \underline{n} + \underline{b} \cdot \underline{\tau} = 0$ (Rodrigues' formula). The component of $\underline{\tau}$ perpendicular to $\underline{b}$ has been called the "geodesic torsion".⁹ That is,

$$\text{geodesic torsion} = \tau_g = \underline{\tau} \cdot \underline{n} \times \underline{b} = -(|\nabla \psi| / 2\pi B) \underline{\tau} \cdot \nabla \alpha.$$

A less obvious but very important identity involving both the rotational shear and the geodesic torsion also follows solely from geometrical considerations:³

$$1 = -2\tau_g - (|\nabla\psi|^2/2\pi B)\psi', \quad \psi = |\nabla\psi|^{-2}(\nabla\psi \cdot \nabla\alpha). \quad (2)$$

Proof: $\underline{b} \cdot \nabla \times \underline{b} = (2\pi B)^{-1} (\nabla\psi \times \nabla\alpha) \cdot \nabla \times \underline{b} = (2\pi B)^{-1} [\nabla\psi \cdot (\nabla \underline{b}) \cdot \nabla\alpha - \nabla\alpha \cdot (\nabla \underline{b}) \cdot \nabla\psi]$,

$$= (2\pi B)^{-1} [-\nabla\psi \cdot (\nabla \nabla\alpha) \cdot \underline{b} + \nabla\alpha \cdot (\nabla \nabla\psi) \cdot \underline{b}] ,$$

$$= (2\pi B)^{-1} [\underline{b} \cdot \nabla (|\nabla\psi|^{-1} \nabla\psi)] \cdot (|\nabla\psi| \nabla\alpha) - (2\pi B)^{-1} [\underline{b} \cdot \nabla (|\nabla\psi|^{-1} \nabla\alpha)] \cdot (|\nabla\psi| \nabla\psi) ,$$

$$= (|\nabla\psi|^2/2\pi B) [2|\nabla\psi|^{-1} \nabla\alpha \cdot (\underline{b} \cdot \nabla \underline{n}) - \underline{b} \cdot \nabla (|\nabla\psi|^{-2} \nabla\psi \cdot \nabla\alpha)] , \quad \text{Q.E.D.}$$

With a bit of algebra and some integrations by parts, we can now write

$$\Delta W_0 = \Delta W_s + \Delta W_t , \quad \Delta W_t \geq 0 ,$$

where

$$\Delta W_t = (1/4) \int d\psi d\alpha d\ell \, h |\nabla\psi|^{-2} [(|\nabla\psi|^2/2\pi B) (y - vx)' - 2\tau_g x]^2 \quad (3a)$$

and

$$\Delta W_s = (1/4) \int d\psi d\alpha d\ell \, h |\nabla\psi|^{-2} \{x'^2 + [(T/h) \kappa \cdot \nabla\psi + 2\tau_g] x^2\} . \quad (3b)$$

Observe that $y - vx = (\underline{\xi} \cdot \nabla_\psi \alpha / 2\pi)$ where $\nabla_\psi = \nabla - \underline{n} \underline{n} \cdot \nabla$. Also, notice that since $\kappa \cdot \nabla\psi = -|\nabla\psi| \underline{b} \cdot \underline{\tau}$, the curvature can be completely eliminated from ΔW_0 in terms of τ and 1 . But there is no need to do that.

The sufficient requirement $\Delta W_s > 0$ is related to the generalization of Newcomb's¹⁰ sufficiency condition for large-aspect-ratio equilibria. There is an essential difference, however, because of our recognition and use of Eq. (2).

Equations (3) point up the important factors that govern stability. Since $T > 0$ for the normal situation in which the pressures fall off radially because of their explicit dependence upon ψ , $\kappa \cdot \nabla\psi > 0$ contributes to stability. "Good" normal curvature, directed away from the pressure gradient, is stabilizing.¹

The last term of Eq. (3b) involves the geometrical coefficient τ_g ; this parameter, the geodesic torsion, tells us how the ψ surface twists locally as

one travels along a line of force. In the usual quadrupole configurations, the sign of τ_g is preserved for macroscopic distances; $\tau \cdot \nabla \alpha$ gives us the relative strength and orientation of the quadrupole assembly. (Think of a line of force in the "twisted bow tie" ψ -surface of a single minimum-B cell.) In particular, for vacuum fields at least, τ_g has the sign opposite to that of the current in the nearest equivalent axial (Ioffe) element of the external coils. At finite beta τ_g will also have a similar interpretation if the configuration has not undergone a geometric transition--such as to reversed-field equilibrium.¹² (However, the sign of τ_g would certainly be destroyed by crinkling of the ψ surface at the onset of ballooning instability.)

It is clear from Eq. (3b) that since i is proportional to the component of the plasma current along $\underline{B}$, it is best that such currents flow in the opposite sense to those of the external coils. Diamagnetic axial currents are stabilizing.³

How relevant to stability is the form ΔW_S ; is it so pessimistic that its trends may not reflect the trends of ΔW_0 ?

It is certainly true that if $T \kappa \cdot \underline{n} > 0$ and $i \tau_g > 0$ inside the plasma, stability is guaranteed! This is already important news. However, it is not too difficult to find some additional information that tells us about trends as well. Thus, it is useful to consider the special case of a flat-topped radial-pressure profile, especially for what it can say about ΔW_t and ΔW_s .

We recall⁸ it is a condition of equilibrium that $(h_1/B)' = - (T/B) \kappa \times \underline{b} \cdot \nabla \psi$. Hence if $(T/h) \kappa \cdot \nabla \psi + 2 i \tau_g$ is not everywhere positive, the instability drive is proportional to T . In turn, T is proportional to $B\beta/\epsilon\psi$ where β is the usual ratio of plasma pressure to magnetic pressure. The factor ϵ^{-1} measures the strength of the radial-pressure gradient, i.e., the steepness of the derivative of the pressure with respect to its explicit dependence upon ψ . If $\Delta\psi$ is the increment in ψ for which there is an appreciable falloff of pressure, then $\epsilon = \Delta\psi/\psi$. When gradients are confined to a boundary layer, ϵ is small inside the layer and T is large. Since β at the limit of stability is proportional to ϵ , such systems are always low-beta configurations. Finite-beta stability requires that radial (flux-) gradient lengths must be at least of the order of the plasma radius.

Suppose for the moment, however, that radial pressure gradients are confined to a boundary layer; β and ϵ are small but the ratio β/ϵ can be anything. Since T vanishes outside the boundary layer, only contributions from

within the layer contribute to $\Delta W_0 = \Delta W_t + \Delta W_s$. Because β is small, the confining field is the vacuum field, and because of the geometry of the configuration, $\kappa \cdot \underline{n}$ and τ_g are bounded. However ϵ and [from Eq. (2)] v' are $O(\beta/\epsilon)$, which is unbounded. Indeed, the term in square brackets in Eq. (3a) can be rewritten

$$u = (i\nabla\psi)^2/2\pi B (y - vx)' - 2\tau_g x = (i\nabla\psi)^2/2\pi B (y' - vx') + ix.$$

Hence u is also $O(\beta/\epsilon)$ within the relative order of x , which we can choose to be unity. Therefore, $\Delta W_t = O(\beta^2/\epsilon^2)$, while the instability drive in ΔW_s is $O(\beta/\epsilon)$. But then ΔW_t dominates the energy integral unless $u = 0$, in which case $\Delta W_t = 0$. Hence, for large β/ϵ , stability is determined by ΔW_s (subject to the constraint $u = 0$).

III. ANALYTIC EQUILIBRIUM FIELDS

At low beta and in the large-aspect-ratio limit, the magnetic field can be written down analytically.^{13,14} The required result expresses $\underline{B}$ in terms of the induction on-axis, $B(z) = B(-z)$, and the (flaring) factor, $c(z) = -c(-z)$. In turn, $c(z)$ measures the eccentricity of the $\psi = \text{const}$ surfaces at positions z along the axis, provided the flux surfaces in question intersect the midplane, where $c = 0$, in circles. (Then, the ratio of major to minor elliptical axes is $e^{|c|}$ at any plane $z = \text{const.}$)

Through first order in the inverse aspect ratio,

$$\underline{B} = \underline{B}_D + \underline{B}_Q, \quad \underline{B}_D = B e_z - (r/2) (dB/dz) e_r,$$

$$\underline{B}_Q = (rB/2) \{dc/dz\} (e_r \cos 2\theta - e_\theta \sin 2\theta), \quad (4)$$

where (r, θ, z) are the usual cylindrical coordinates. For definiteness, we will adhere to the convention that $c = 0$ and $dc/dz = \mathcal{K} > 0$ at the origin.

Notice that for the vacuum system³

$$-\tau_g = (1/2) (\cosh c - \sinh c \cos 2\alpha)^{-1} (dc/dz) \sin 2\alpha.$$

Hence, from the right-hand rule and the expression for $\underline{B}_Q$, it is easy to verify that $-\tau_g$ has the sign of the current in the nearest (equivalent) axial

quadrupole conductor of the external coils (just as we stated in Section II). In other words, dc/dz measures the relative strength and orientation of the quadrupole field, and determines the sign of τ_g in a given quadrant. The dipole field B_D , on the other hand, is completely determined by $B(z)$.

With these analytic simplifications, then

$$\Delta W_s = (16\pi)^{-1} \int \psi^{-1} d\psi d\alpha dz \{ \eta x'^2 + A[\eta\gamma (X - 2Y \cos 2\alpha) + \eta^2 \Lambda (dc/dz) \sin^2 2\alpha] x^2 \} , \quad (5)$$

where $h = B$ and $z = z$ in this approximation. We have also written $A\gamma(z) = \psi T/2B \sim O(\beta/\epsilon)$, and $\gamma(z=0) = 1$, i.e., A is the appropriate normalization constant. In addition,

$$X = (dc/dz)^2 + (B^{-1} dB/dz)^2 - 2(d/dz)(B^{-1} dB/dz) ,$$

$$Y = (B^{-1} dB/dz)(dc/dz) - (d^2 c/dz^2) , \quad (6)$$

so that $X(-z) = X(z)$, $Y(-z) = -Y(z)$,

$$\eta = (\cosh c - \sinh c \cos 2\alpha)^{-1} > 0 ,$$

and

$$\Lambda(z) = \int_z^L \gamma \lambda dz = \Lambda(-z) , \quad \lambda(z) = X \sinh c - 2Y \cosh c = -\lambda(-z) . \quad (7)$$

Thus $[\Lambda \text{ sign } (d\alpha/dz)]$ measures the diamagnetic axial current, since

$$i = - (AB\psi/2h) \Lambda \sin 2\alpha .$$

Notice that $\eta^2 dc/dz = (d/dz)(\eta \sinh c)$; this is just the result of Eq. (2), since $i = 0$ for the vacuum fields.

We also have

$$\kappa \cdot \nabla \psi = (\psi/2) (X - 2Y \cos 2\alpha) , \quad \kappa \times \underline{b} \cdot \nabla \psi = -(\psi/2) \lambda \sin 2\alpha .$$

This identifies $X - 2Y \cos 2\alpha$ with the normal curvature and λ with the geodesic curvature.

If the axial currents are predominantly diamagnetic, the regions of greatest instability tend to be pushed toward the symmetry planes ($\alpha = 0$ or $\alpha = \pi/2$). Thus we are led to examine

$$\delta W_S^0 = \int_{-L}^L dz \{ e^{\pm c} x'^2 + A\gamma (\omega \mp \lambda) x^2 \} , \quad (8)$$

where

$$\omega = X \cosh c - 2Y \sinh c , \quad \omega(-z) = \omega(z) . \quad (9)$$

We see from Eq. (8) that $\delta W_S^0 > 0$ if $\omega > |\lambda|$. In turn, this implies $X > 2|Y|$ so that certainly $X > 2|Y| \cos 2\alpha$ for any α . Clearly, therefore, if $\omega > |\lambda|$ and $\Delta c/dz > 0$ everywhere, then $\Delta W_S > 0$. Stability is guaranteed.

Unfortunately, one cannot require $\omega > |\lambda|$ and $\Delta c/dz > 0$ everywhere in the kinds of traps one wants to build today. Either $B(z)$ or dc/dz , or both, tend to increase without limit away from the midplanes. The same behavior characterizes the individual condition $\omega > |\lambda|$, good normal curvature everywhere. For linked-cell systems (tandem mirrors), we must relax our criteria. Before we go on, however, we should make a further point with regard to the geodesic curvature (and axial currents).

Stupakov¹⁵ showed that, whatever the sign, the distortion of the magnetic field by axial currents creates considerable problems for tandem mirror systems. Various workers^{3,16,17} independently suggested the desirability of keeping λ , and thus Λ , small. However, if we try to set $\lambda = 0$ everywhere, we encounter most of the same kinds of design problems that enter in trying to keep $\omega > |\lambda|$ everywhere. The best we can really do is to keep $\Delta c/dz$ small and positive in some average sense.

Finally, it is already clear from Eq. (5) that at least one average over the geometric parameters must be preserved. If we consider flute modes ($x' = 0, y' = 0$), then an integration by parts changes the term in square brackets into $\gamma (\omega - 2\lambda \cos 2\alpha)$, and since λ is antisymmetric

$$\Delta W_S + (16\pi)^{-1} \int \psi^{-1} d\psi d\alpha dz x_0^2 A\gamma\omega .$$

Hence, on each line of force (we are assuming $A > 0$),

$$\Omega_0 = \frac{1}{2} \int_{-L}^L dz \gamma\omega = \int_0^L dz \gamma\omega > 0 , \quad (10)$$

if $\Delta W_s > 0$. [As is well known, Eq. (10), in general, is also a necessary condition for stability.]

It is obviously in our interest to maximize Ω_0 , consistent with our engineering constraints. We also expect that

$$\Lambda_0 = \Lambda(0) = \int_0^L dz \gamma \lambda \quad (11)$$

must be constrained in some fashion, consistent with the general desirability of small λ and positive $\Delta dc/dz$. We will return to these points after the next section.

IV. DESIGN PHILOSOPHY

Now that we have become acquainted with the important stability factors and before we write down the actual equations to be solved, we need to pause long enough to set up a rudimentary design philosophy. We want to end up analytically somewhere between our present limitations for which, on the one hand, we know some easily written criteria sufficient for stability ($\omega > |\lambda|$, $\Delta dc/dz > 0$) that nevertheless cannot be satisfied in an actual system and, on the other hand, an essentially exact stability condition ($\Delta W_0 > 0$) that must be satisfied but is too complicated as it stands for intuitive interaction.

Whatever one's design philosophy, that philosophy will necessarily motivate some of the choices to be made in the remainder of this report. We hope that the reader who may wish to make some alterations of our philosophy for specific applications, or even personal predilection, will find it easier to do so, knowing our motivations. We also hope that the reader who wants to proceed with our results as a design tool will find our considerations flexible enough to meet that basic need without redoing our analysis.

One of the more difficult design problems today is determining how to build a "transition". How does one go from a quadrupole well section to an axisymmetric well section, say, without introducing bad curvature in a way that significantly affects the properties of one or both of the sections being connected? In a day in which one expects to hook up many special-purpose regions in tandem, we especially ought to know how, for example, to terminate

a given quadrupole section effectively. We ought to understand how to make $dc/dz \neq 0$ [Eq. (4)] without spoiling stability wholesale.

Our own predilection is toward the design of the modular cell. Each region between mirror peaks should be designed to be as stable as possible, of course, depending upon its particular function. But it should keep its bad curvatures to itself--unfavorable properties should not be passed on from well section to well section.

Based on the conditions for omnigenity for the most well-trapped particles,⁸ the fundamental symmetries of the usual tandem-mirror systems are such that the sign of the geodesic torsion near the minimum of a well alternates from cell to cell; necessarily, the sign of "good" current will then do the same. We believe these sign changes should occur at the mirror peaks. Otherwise "bad" normal curvature will spring up in an interactive fashion wherever τ_g is corrected. Likewise, net axial currents of transiting particles that might be of good sign in one section are bad in the next. Both of these behaviors lead to problems that can be avoided.

The simplest tandem systems have $dc/dz = 0$ at the mirror peaks and the fanning amplitude, the value of $|c|$ at the mirror peaks, is standardized. (This standardization of $|c|$ is independent of the individual mirror ratios, for example). With such modularity the "transition" is built into each cell; it can be designed optimally with respect to individual cell characteristics. Cells can be buckled together as pieces of pipe, with minimal adverse interaction.

While the modular cell is our own preference, we will not make it a requirement of our analysis; our optimizer can be used more generally. Nevertheless, our procedural choices will make it easy to analyze the modular cell, and our discussions will center on this model.

Generally speaking, we shall choose the overall parameters of a system as follows: The dipole field is characterized by its central induction B_0 , its half-length L , and its mirror ratio R , $B(\pm L) = RB_0$. The induction is a minimum at $z = 0$, a maximum at $z = \pm L$, and symmetric about $z = 0$. For the modular cell, $B(z)$ is monotonic between $z = 0$ and $z = L$; for more general systems there may be various maxima and minima--with B specified at these discreet points. Otherwise, $B(z)$ is subject to optimization.

For B_0 , we take $c = 0$ at $z = 0$, where $dc/dz = \mathcal{K} > 0$. Thus $\mathcal{K}$ determines the radial well depth. At $z = \pm L$, we set $dc/dz = 0$ and $c(\pm L) = \pm a$. (If the

actual system to be examined has nonzero dc/dz at the ends, we just consider that an extra length is added on. The plasma pressure vanishes in this extra length for which dc/dz is reduced from the given value to zero).

The plasma properties enter through $A\gamma(\psi, B)$; γ includes the effects of all particles in the region--those in transit as well as those trapped within the cell. For design purposes it sometimes can be convenient to guarantee stability in terms of a particular plasma model $\gamma = (B_0/B)^{2+p}$. When $p = 0$ this model represents an isotropic pressure distribution. For larger positive values of p , this pressure model corresponds to particle distributions that are confined more and more closely to the center of the well. In a quadrupole-stabilized well, increasing p tends to increase the stability of the increasingly well-trapped anisotropic ion pressure distribution. Conversely, the complementary stability of the field configuration will decrease with increasing p as measured by A at the stability limit--other things all being equal. We use A as the measure of stability; the larger A can be, the more stable the system.

Finally, we recognize there is some tendency in optimization to try to localize "bad" properties to domains of small spatial extent. This can lead to requests for very steep axial field gradients, for example, since (roughly) the overall length $2L$ is to be given. Thus, we introduce $L\delta$ as a kind of working-minimum scale length. Indeed, δ will be used here as the general dimensionless measure of how small we will let a "practicality-bounded" quantity become. Likewise, δ^{-1} will measure practical upper limits. Of course, we could use different limits for each different bound that might be encountered, but the reader can easily supply that generalization (in the simplex tableau of Section VII).

V. GENERAL OPTIMIZATION

The discussion of general optimization can be carried out in terms of the modular cell; the introduction of the appropriate intermediate constraints of a multiple-region cell can be left to the end. Thus, we have

$$B(0) = B_0, \quad B(\pm L) = RB_0, \quad dB(0)/dz = dB(\pm L)/dz = 0, \quad (12)$$

$$c(0) = 0, \quad c(\pm L) = \pm a, \quad dc(0)/dz = \mathcal{X} > 0, \quad dc(\pm L)/dz = 0. \quad (13)$$

Also, $B(-z) = B(z)$, $c(-z) = -c(z)$, and $B(z)$ is monotonic in the range $0 \leq z \leq L$. Otherwise, within the bounds of engineering constraints characterized by the parameter δ , both $B(z)$ and $c(z)$ are arbitrary. Hence, it will be necessary to optimize twice in order to specify both $B(z)$ and $c(z)$.

Suppose first, for simplicity, that we just want to maximize Ω_0 , Eq. (10). To perform the first maximization, we select some combination of $B(z)$ and $c(z)$ that we consider initially fixed. Then, we take the variation of Ω_0 with respect to the remaining free function and solve the associated Euler-Lagrange equation. Of course, our choice of the combination that is initially held constant makes no difference. Transformations among different combinations may be made either before or after maximization, with identical results. At this point, therefore, we can choose the first variation to suit our convenience.

Once we have the answer from the first step, we might expect to substitute the result back into Eq. (10), and once again maximize Ω_0 with respect to the remaining function to obtain the final answer. But there is a surprise. The second maximization cannot be performed by the variational calculus at all! It turns out that this process of double optimization is linear in one step. Thus, one optimization must always be done in accordance with the techniques of linear (game) theory.⁶

We continue the example of Ω_0 to show how the linearity enters. Suppose $B(z)$ is considered fixed and that we have optimized $c(z)$ first. If we write $B = B_0 \exp[f(z)]$, then after some integrations by parts, using Eqs. (12) and (13), we have

$$\begin{aligned} \Omega_0 = \int_0^L dz \{ [2(d\gamma/dz)(df/dz) + \gamma(df/dz)^2 - \gamma(dc/dz)^2] \cosh c \\ - 2(d\gamma/dz)(dc/dz) \sinh c \} \end{aligned} \quad (14)$$

Now, let us change our variable of integration from z to f . Writing $df/dz = g(f)$, Eq. (14) becomes

$$\begin{aligned} \Omega = \int_0^{LNR} df g(f) \{ [2(d\gamma/df) + \gamma - \gamma(dc/df)^2] \cosh c \\ - 2(d\gamma/df)(dc/df) \sinh c \} \end{aligned} \quad (15)$$

It is now supposed that $c(f)$ is known, then Ω_0 is a linear form in g . Maximization of Ω_0 against g is a problem of linear programming.

More fully, of course, if $c(f)$ is first obtained by solving the Euler-Lagrange equation from Eq. (14) [with $f(z)$ fixed], the result depends upon $g(f)$. The natural process then is to proceed iteratively, alternately solving Eq. (14) for its nonlinear variation and Eq. (15) linearly.

VI. THE NONLINEAR FIRST OPTIMIZATION

Having made an appreciable investment now in formulating our analytical machinery, the reader is probably anxious for us to get on with its use. Shall we combine the likes of Eqs. (10) and (11) with Lagrange multipliers and solve the resultant Euler-Lagrange equations? Unfortunately, that just does not work out. Once again the solutions we find are incompatible with the kinds of linked-mirror systems we would like to build. Once again we must step back from the simplest results, sharpening our constraints.

The difficulty is this. Suppose, for example, that we solve the variational equation for given $B(z)$ in order to find the best $c(z)$. The second-order nonlinear ordinary differential equation that results permits us to choose $c = 0$ at $z = 0$ and $dc/dz = 0$ at $z = L$, but that exhausts the boundary conditions. Both $\mathcal{K} = dc(0)/dz$ and $a = \cdot(L)$ end up as eigenvalues, most frequently not at all what we would like them to be. It is conceivable, of course, that the addition of other constraints of similar form (linear in ω and λ) will permit more attractive values of $\mathcal{K}$ and a to be found from the elimination of the additional Lagrange multipliers. We have not seen how to do this satisfactorily, however, and believe that the difficulties are more basically associated with the linear averages of ω and λ . Our next logical step, therefore, is simply to abandon our derived expressions and go on to quadratic forms. Indeed, faced with the same problem, Myra, D'Ippolito, and Catto¹⁸ also made use of a variational approach based on a quadratic form--essentially the average of λ^2 .

But there are at least two disadvantages to the quadratic forms. First, quadratic forms alone say nothing about the signs of ω and λ ; our controls relate primarily to their relative magnitudes. Second, we lose the exact constraint $\Omega_0 > 0$, which in fact must be satisfied physically for interchange stability. Of course, sign-dependent holonomic constraints (e.g., $\Lambda_0 = 0$) can

be tacked on with Lagrange multipliers, but a nonholonomic constraint such as $\Omega_0 > 0$ is more elusive. On the other hand, we have two optimizations to perform. It makes sense to go ahead with use of quadratic forms for the first optimization (augmented, if desired, by holonomic sign-dependent constraints) and return to our earlier sign-sensitive forms for the second. Agreeing on this, we have just two more instructions before getting to the first variational equation.

First, instead of $f(z)$ as variable of integration, let us choose some other function in Eq. (15), $s(z)$, say, $0 \leq s \leq 1$. Then

$$\Omega_0 = \int_0^1 ds \gamma(s) \tilde{\omega}(s) \quad (16a)$$

where

$$\begin{aligned} \tilde{\omega}(s) = & [\sigma(dc/ds)^2 + \sigma(df/ds)^2 - 2(d/ds)(\sigma df/ds)] \cosh c \\ & - 2[\sigma(dc/ds)(df/ds) - (d/ds)(\sigma dc/ds)] \sinh c, \end{aligned} \quad (16b)$$

and $\sigma(s) = ds/dz$. It is convenient to introduce the auxiliary function $s(z)$, even if $f(z)$ is basically the quantity to be fixed in the first optimization. Use of f itself as a coordinate is disadvantageous. Even if f varies monotonically in the region $0 < z < L$, infinities crop up unnecessarily since df/dz vanishes at the end points. Instead, for the modular cell, we put

$$f = 2s^2(1 - s^2/2) \ln R, \quad (17)$$

which gives us the appropriate behavior of f without the end-point difficulty. For the multiple-region cell, the equivalent of Eq. (17) can be used piecewise.

We now go on to quadratic forms and certainly it is natural to take $\int_0^1 \gamma \tilde{\omega}^2 ds$ as a first measure of quality. In addition, we have already remarked that it can be important to keep $\Lambda(z)$, and thus $\lambda(z)$, small; the reasons go beyond those of stability. Using s as coordinate, the relevant measure of λ , analogous to Eq. (16b), simply interchanges $\cosh c$ and $\sinh c$. We have

$$\begin{aligned} \tilde{\lambda}(s) = & [\sigma(dc/ds)^2 + \sigma(df/ds)^2 - 2(d/ds)(\sigma df/ds)] \sinh c \\ & - 2[\sigma(dc/ds)(df/ds) - (d/ds)(\sigma dc/ds)] \cosh c. \end{aligned} \quad (18)$$

We are finally ready to set up the variation. The quality factor

$$q = (1 - \theta) \tilde{\omega}^2 - \theta \tilde{\lambda}^2, \quad 0 \leq \theta \leq 1, \quad (19)$$

measures the strength of ω and the weakness of λ . The constant relative weight θ can be chosen according to the importance the designer attaches to the smallness of λ . The first optimization is then accomplished by variation of

$$Q = \int_0^1 ds \gamma^2 q. \quad (20)$$

At last we have the flexibility we need. The Euler-Lagrange equation is now a fourth-order nonlinear differential equation. Considering $\sigma(s)$ to be given [$f(s)$ is given by Eq. (17)] then $c(s)$ is to be found; the boundary conditions are

$$c(0) = 0, \quad dc(0)/ds = \tilde{\mathcal{K}}, \quad d^2c(0)/ds^2 = 0, \quad dc(L)/ds = 0, \quad (21)$$

where $\tilde{\mathcal{K}} = \mathcal{K}/\sigma(0)$. If $\omega(0) > 0$, so that $X(0) > 0$, and so that there is good normal curvature at $z = 0$, then $\tilde{\mathcal{K}} > 2\sqrt{2} \ln R$. Solution of the Euler-Lagrange equation, with or without constraints in addition to $\delta Q = 0$, iterates with the determination of $\sigma(s)$ from the second optimization. Thus $c(L) = a$ is determined parametrically by $\tilde{\mathcal{K}}$.

In passing, it is worth mentioning that the calculations of Myra, D'Ippolito, and Catto show that their results, which minimized essentially $\int_0^L \gamma \lambda^2 dz$, closely match those corresponding coil designs for MFTF-B at Lawrence Livermore National Laboratory for which small λ was the major design criterion.¹⁸

Finally, to treat a multiregion cell instead of the modular cell, we set up the first optimization in exactly the same way as just demonstrated, individually considering the regions separated by mirror peaks. The Euler-Lagrange equation is solved across the whole length of the multiregion cell by matching boundary conditions at the interfaces. In this way, the most general configurations can be accommodated.

VII. THE LINEAR SECOND OPTIMIZATION

The second optimization is drawn from Eqs. (10) and (11). Thus, we want to maximize Ω_0 consistent with our engineering constraints and the requirement

$$(1 - \theta_0) \Omega_0 \geq \theta_0 \Lambda_0 \geq 0, \quad 0 \leq \theta_0 \leq 1. \quad (22)$$

Here, Θ_0 is a given weight constant chosen together with Θ , the corresponding weight constant of Eq. (19). Putting $\Theta_0 = 1$ requires $\Lambda_0 = 0$. Normally, we would expect just to put $\Theta_0 = \Theta$. In terms of the variable s , and after some integrations by parts,

$$\Omega_0 = \int_0^1 \sigma \hat{\omega} ds, \quad \Lambda_0 = \int_0^1 \sigma \hat{\gamma} ds - 2\mathcal{H}, \quad (23)$$

where

$$\hat{\omega} = \left[2 \left(\frac{d\gamma}{ds} \right) \left(\frac{df}{ds} \right) + \gamma \left(\frac{df}{ds} \right)^2 - \gamma \left(\frac{dc}{ds} \right)^2 \right] \cosh c - 2 \left(\frac{d\gamma}{ds} \right) \left(\frac{dc}{ds} \right) \sinh c, \quad (24a)$$

$$\hat{\lambda} = \left[2 \left(\frac{d\gamma}{ds} \right) \left(\frac{df}{ds} \right) + \gamma \left(\frac{df}{ds} \right)^2 - \gamma \left(\frac{dc}{ds} \right)^2 \right] \sinh c - 2 \left(\frac{d\gamma}{ds} \right) \left(\frac{dc}{ds} \right) \cosh c, \quad (24b)$$

and $\hat{\omega}$ and $\hat{\lambda}$ are known from the first optimization. Only $\sigma(s)$ remains to be determined, and our discussion from this point now encompasses both the modular cell and the multiregion cell.

Now, Ω_0 is the value of a game; it is to be maximized against σ subject to constraints. Such problems of linear optimization may be somewhat unfamiliar to the plasma physicist, but this is not the first game to be encountered in the theory of stable mirror confinement. Hall, Hooper, and Newcomb¹⁹ used game theory to solve the problem of optimal placement of injectors for a beam-generated mirror-trapped plasma against mirror-mode instability limitations. The computer code of Walker and Hall,²⁰ developed in that context for implementation of the simplex algorithm, is suitable as well for our present work.

The procedure is as follows: For Ω_0 and each constraint involving an integral over s , the integral is to be rewritten as a finite sum. Using Simpson's rule,

$$\int_0^1 \phi(s) \sigma(s) ds = \sum_{j=0}^N \epsilon_j \phi_j \sigma_j, \quad \phi_j = \phi(j/N), \quad (25)$$

where N is even and $\epsilon_j = (2/3N)$ if j is even, except that $\epsilon_0 = \epsilon_N = (1/3N)$. If j is odd, $\epsilon_j = (4/3N)$.

Our unknowns now are the $(N+1)$ quantities $\sigma_j = \sigma(j/N)$, $\sigma_j \geq 0$. The constraints are a set of linear inequalities or linear equalities in the σ_j .

and Ω_0 is a linear sum over the nonnegative σ_j . This set of constraints plus the expression for Ω_0 make up the starting "simplex tableau." The starting tableau itself is entered according to specific rules into a standard computer program for optimization.^{5,19,20} The accuracy of the result and its difficulty of accomplishment increase with N .

Let us now derive the starting simplex tableau. Our first engineering constraint requires that

$$\sigma \leq (L_0 \delta)^{-1} , \quad (26)$$

where L_0 is a given measure of the overall length. We use

$$L_0^{-1} = \int_0^L dz (ds/dz)^2 = \int_0^1 \sigma(s) ds , \quad (27)$$

insisting that this normalization be preserved during optimization. When σ is found, the conversion back to laboratory coordinates is

$$z(s) = \int_0^s ds/\sigma(s) . \quad (28)$$

Hence, for example, our actual overall length $2L$ is

$$2L = 2L_0 \left[\left(\int_0^1 ds/\sigma \right) \cdot \left(\int_0^1 \sigma ds \right) \right] . \quad (29)$$

If $\sigma = \text{const}$, then $2L = 2L_0$.

The quadrupole field, see Eq. (4), is proportional to dc/dz and thus to σ . Therefore, to maintain continuity of the quadrupole field, σ must vary smoothly; we shall impose limits on the derivatives of σ as well as upon σ itself. Thus,

$$-(L_0 \delta)^{-1} \leq d\sigma/ds \leq (L_0 \delta)^{-1} , \quad (30)$$

according to our agreed engineering criteria.

Our full set of constraints include Eqs. (22), (26), and (30), which are all inequalities, and Eq. (27), the normalization, which is an equality constraint. Our game is to maximize Eq. (23a).

Rewriting our constraints, using Eq. (25), we have

$$(1 - \theta_0) \Omega_0 - \theta_0 \Lambda_0 = [2\theta_0 \tilde{\chi} - \epsilon_0 (1 - \theta_0) \tilde{\chi}^2] \sigma_0 + \sum_{j=1}^N \epsilon_j [(1 - \theta_0) \hat{\omega}_j - \theta_0 \hat{\lambda}_j] \sigma_j \geq 0, \quad (31a)$$

$$\Lambda_0 = (-2\tilde{\chi}) \sigma_0 + \sum_{j=1}^N \epsilon_j \hat{\lambda}_j \sigma_j \geq 0, \quad (31b)$$

from Eqs. (22) and (23); and

$$-\sigma_j + (L_0 \delta)^{-1} \geq 0, \quad 0 \leq j \leq N, \quad (31c)$$

from Eq. (26); and

$$-\sigma_{j-1} + \sigma_j + (NL_0 \delta)^{-1} \geq 0, \quad 1 \leq j \leq N, \quad (31d)$$

$$\sigma_{j-1} - \sigma_j + (NL_0 \delta)^{-1} \geq 0, \quad 1 \leq j \leq N, \quad (31e)$$

from Eq. (30). In addition, from Eq. (27),

$$\sum_{j=0}^N \epsilon_j \sigma_j - L_0^{-1} = 0, \quad (32)$$

and from Eq. (23)

$$\Omega_0 = \sum_{j=0}^N \epsilon_j \hat{\omega}_j \sigma_j. \quad (33)$$

In the notation of Refs. 19 and 20, there are $m = 3N + 3$ inequality constraints, Eqs. (31), and $\ell = 1$ equality constraints, Eq. (32). There are $n = N + 1$ basic coordinates σ_j . The corresponding starting simplex tableau is given in Fig. 1. (At the start, $\xi_j = \sigma_{j-1}$.)

	ξ_1	ξ_2	...	ξ_{k+1}	ξ_{k+2}	...	ξ_{n-1}	ξ_n	1
$-n_1$	$\{2\theta_0\tilde{\omega} - \theta_0\theta_c\tilde{\omega}^2\}$	$\xi_1\{\theta_c\tilde{\omega}_1 - \theta_0\hat{\lambda}_1\}$	...	$\xi_k\{\theta_c\tilde{\omega}_k - \theta_0\hat{\lambda}_k\}$	$\xi_{k+1}\{\theta_c\tilde{\omega}_{k+1} - \theta_0\hat{\lambda}_{k+1}\}$	...	$\xi_{n-2}\{\theta_c\tilde{\omega}_{n-2} - \theta_0\hat{\lambda}_{n-2}\}$	$\xi_{n-1}\{\theta_c\tilde{\omega}_{n-1} - \theta_0\hat{\lambda}_{n-1}\}$	0
$-n_2$	$(-2\tilde{\omega})$	$(\xi_1\hat{\lambda}_1)$	...	$(\xi_k\hat{\lambda}_k)$	$(\xi_{k+1}\hat{\lambda}_{k+1})$	...	$(\xi_{n-2}\hat{\lambda}_{n-2})$	$(\xi_{n-1}\hat{\lambda}_{n-1})$	0
$-n_3$	-1	0	...	0	0	...	0	0	$(L_0\delta)^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-n_{k+2}$	0	0	...	-1	0	...	0	0	$(L_0\delta)^{-1}$
$-n_{k+3}$	0	0	...	0	-1	...	0	0	$(L_0\delta)^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-n_{2+n}$	0	0	...	0	0	...	0	1	$(L_0\delta)^{-1}$
$-n_{3+n}$	-1	1	...	0	0	...	0	0	$(NL_0\delta)^{-1}$
$-n_{4+n}$	1	-1	...	0	0	...	0	0	$(NL_0\delta)^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-n_{1+n+2k}$	0	0	...	-1	1	...	0	0	$(NL_0\delta)^{-1}$
$-n_{2+n+2k}$	0	0	...	1	-1	...	0	0	$(NL_0\delta)^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-n_{3n-1}$	0	0	...	0	0	...	-1	1	$(NL_0\delta)^{-1}$
$-n_{3n}$	0	0	...	0	0	...	1	-1	$(NL_0\delta)^{-1}$
$-n_{3n+1}$	ξ_0	ξ_1	...	ξ_k	ξ_{k+1}	...	ξ_{n-2}	ξ_{n-1}	$(-L_0)^{-1}$
w	$(\xi_0\tilde{\omega}_0)$	$(\xi_1\tilde{\omega}_1)$	...	$(\xi_k\tilde{\omega}_k)$	$(\xi_{k+1}\tilde{\omega}_{k+1})$	...	$(\xi_{n-2}\tilde{\omega}_{n-2})$	$(\xi_{n-1}\tilde{\omega}_{n-1})$	0

Figure 1. The starting simplex tableau. [Note: $\theta_c = (1 - \theta_0)$].

VIII. RELATIVE STABILITY

Once the optimizations have been performed, how stable is the resulting configuration? The answer will certainly change with the external parameters. Here, we have chosen to use ΔW_S , Eq. (5), as the relevant bounding functional; ΔW_S , for us, measures stability. But of course the sign of the worst ΔW_S will depend on how much plasma is present, provided that either Eq. (10) is satisfied or that flute modes are proscribed by (conducting) boundary conditions at one or both ends.

We have already agreed to use $A = (\sqrt{T}/2B)|_{z=0} \sim B/\epsilon$ as the characteristic parameter measuring stability. We seek the maximum value of A for which the minimum value of ΔW_S is nonnegative; A is the smallest positive eigenvalue of

$$(\eta x')' - AGx = 0, \quad G = \gamma[\eta(X - 2Y \cos 2\alpha) + \eta^2 \Lambda (dc/dz) \sin^2 2\alpha] \quad (34)$$

Obviously, the value of A depends on the boundary conditions. In general, $\Delta W_S = 0$ if $xx' = 0$ at the ends.

For the isolated system, the (insulating) external flute boundary conditions, $x'(\pm L) = 0$ are least stable. Let A_I be the smallest eigenvalue of Eq. (34) for these conditions. We are interested in values of A_I for individual mirror plugs, for example, which are MHD-dominant entities, and for multiregion cells. For other cases, such as the central cell of a simple tandem system, the relevant physical model has "tied" ends; the boundary conditions are $x(\pm L) = 0$. Let A_D be the smallest eigenvalue of Eq. (34) for such dependent, nonisolated systems. In general, for a system composed of several modular wells, the designer will probably want to know both A_I and A_D for each member of the set, depending on the functions of the individual cells. There is, of course, little difficulty in providing both numbers.

IX. SUMMARY

We derive a sufficient condition for stability of an anisotropic, finite-beta, arbitrary-aspect-ratio guiding-center MHD equilibrium: $\Delta W_S > 0$, Eq. (3b). The condition shows the importance of "good" normal curvature on the one hand and "good" axial currents (which involves the interplay between geodesic curvature and geodesic torsion) on the other hand.

We apply our ideas to the development of design criteria for low-beta, large-aspect-ratio equilibria, for which analytic magnetic fields can be written down. We show that simultaneous optimization of both quadrupole and dipole magnetic-field profiles is a problem that involves both nonlinear variational techniques and linear variational techniques. We have written down the appropriate procedures. Overall design quality is measured by A_I and A_D , the appropriate eigenvalues of Eq. (34).

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