EFFECT OF THE EQUILIBRIUM MAGNETIC FIELD ON PLASMA EDGE FLUCTUATIONS*

Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831-8071

ABSTRACT

The magnetic field structure reflected in the safety factor \( q \) profile can play an important role in defining some of the basic structures of the magnetohydrodynamic (MHD) turbulence: it plays a role in the generation of spatial structures in the initial nonlinear phase, and it gives a radial dependence to the spectral average of the square of the poloidal mode number, \( \langle m^2 \rangle^{1/2} \), that can modify the explicit dependences of an analytically derived expression for turbulence-induced anomalous losses.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

† Computing and Telecommunications Division, Martin Marietta Energy Systems, Inc.
I. INTRODUCTION

A great deal of attention has been focused on the study of fluctuations and anomalous transport at the edge of toroidally confined plasmas. Understanding the transport properties of edge plasmas is essential for any magnetic confinement device. In recent years, abundant data on edge fluctuations have been gathered, mostly from the Texas Experimental Tokamak (TEXT)\textsuperscript{1} but also from other devices such as the University of Wisconsin Tokapole\textsuperscript{2} and the Advanced Toroidal Facility (ATF).\textsuperscript{3} Theoretical models have also been developed for edge plasmas. It is now possible to undertake a detailed modeling of the plasma edge turbulence and compare these models with the experimental results. The use of fluid-type equations to study these plasmas facilitates the computational tasks.\textsuperscript{4} Many of the available theoretical models are based on magnetohydrodynamic (MHD) turbulence. An example considered in this paper is the model proposed by Thayer and Diamond,\textsuperscript{5} which is based on the coupling of convective cells driven by line radiation and the resistivity-gradient-driven turbulence. Many features of the resistivity-gradient-driven turbulence model with thermal instability drive agree with experimental tokamak edge fluctuation measurements. In particular, comparisons with TEXT experimental results show that the model can explain features of the data such as \(\langle e\Phi/T_e\rangle > 50\%\), \(\langle e\Phi/T_e\rangle > \langle \bar{n}/n_0 \rangle\), and \(\langle m \rangle_{\text{rms}} \approx 15\) to 40.

A number of properties of the radial dependence of the average poloidal mode number and the real-space turbulence structures observed in the numerical calculations are common to most of the MHD turbulence modes. They depend only on the underlying magnetic field structure, and they have a general nature. These features are the ones we discuss in this paper.

In Sec. II, after a short discussion of the model, the radially integrated (over the global spectral structures) spectrum of the turbulence is described, showing the localization of the energy spectrum in \((m,n)\) space. In real space, mode coupling structures are observed in the nonlinear calculations. They are related to the underlying magnetic field structure and described in Sec. III. The local spectrum depends on the local value of the safety factor \(q\), which induces a radial dependence of the local spectrum that is discussed in Sec. IV. Finally, in Sec. V, the conclusions are stated.

II. EQUATIONS AND GLOBAL SPECTRAL STRUCTURE

Although the results discussed in this paper are general to many MHD turbulence models, to focus the discussion we use the model of Thayer and Diamond.\textsuperscript{5} This model couples the resistivity-gradient-driven turbulence, for which the underlying instability is the rippling mode,\textsuperscript{6} with the thermal instability drive. The basic equations are Ohm's law,

\[
\frac{\partial \Psi}{\partial t} = -R_0 \nabla_{\parallel} \Phi + R_0 \frac{d\eta}{dT} \frac{E_{|| \eta_0}}{\eta_0} + R_0 \eta_0 \Phi_{||} + R_0 \frac{d\eta}{dT} \Phi_{||},
\]

the momentum balance equation,

\[
\frac{\partial \left( \nabla^2_{||} \Phi \right)}{\partial t} + \vec{v} \cdot \nabla \left( \nabla^2_{||} \Phi \right) = \frac{B^2_{\parallel}}{\rho_m} \Phi_{||} + \mu \nabla^4 \Phi,\]

where \(\Psi\) is the magnetic potential, \(\Phi\) is the electric potential, \(R_0\) is the safety factor, \(\eta_0\) is the resistivity, \(E_{\parallel}\) is the electric field along the magnetic field, \(B_{\parallel}\) is the magnetic field along the plasma current, \(\rho_m\) is the magnetic field line density, and \(\mu\) is the viscosity. The momentum equation describes the transport of momentum in the plasma, taking into account the effects of viscosity and magnetic Reynolds number. The energy equation describes the transport of energy in the plasma, taking into account the effects of resistivity and magnetic Reynolds number. These equations are used to study the turbulence in the edge plasma and to compare with experimental data.
and the electron temperature equation,
\[
\frac{\partial \tilde{T}}{\partial t} + \vec{V} \cdot \nabla \tilde{T} = \chi_\parallel \nabla^2_\parallel \tilde{T} - \vec{V}_r \frac{dT_0}{dr} - I_Z(T) + \chi_\perp \nabla^2_\perp \tilde{T} .
\] (3)

Here $\Psi$ is the poloidal magnetic flux, which allows one to write the magnetic field as $\vec{B} = \frac{1}{R} \nabla \Psi \times \hat{\zeta} + \vec{B}_0 \hat{\zeta}$; $\Phi$ is the electrostatic potential; $T$ is the electron temperature; $\vec{V} \equiv \nabla \frac{\Phi}{\vec{B}_0} \times \hat{\zeta}$ is the flow velocity; the resistivity is $\eta$; the cross-field transport coefficient is $\chi_\perp$; the parallel conductivity is $\chi_\parallel$; the viscosity is $\mu$; and the mass density is $\rho_m$. The intensity of the cooling caused by line radiation is given by $I_Z$, which is a function of the temperature. In most of our calculations, we have used the functional dependence on $T$ given by the coronal radiation model, but the level has been normalized to experimental measurements. The parallel current $J_\parallel$ is related to the poloidal magnetic flux by
\[
J_\parallel = \frac{1}{\mu_0 R_0} \nabla^2_\perp \tilde{\Psi} .
\] (4)

The main nonlinearities are $E \times B$ nonlinearities in the convective terms and the magnetic nonlinearity in the parallel derivative operator
\[
\nabla_\parallel \tilde{\Phi} = \frac{1}{|\vec{B}|} \vec{B} \cdot \nabla \tilde{\Phi} .
\] (5)

In these equations, the perturbed quantities are indicated by a tilde and the equilibrium quantities by the subindex 0.

These equations lead to a total energy conservation condition
\[
\frac{d}{dt} (E_M + E_K + E_T) = - \int dV \left( \eta J^2_\parallel + \chi_\perp \left| \nabla_\perp \tilde{T} \right|^2 + \mu \left| \nabla^2_\perp \tilde{T} \right|^2 \right)
- \int dV \left[ I_Z(T) + \vec{V}_r \frac{dT_0}{dr} \right] \tilde{T} - \int dV \chi_\parallel \left| \nabla_\parallel \tilde{T} \right|^2 .
\] (6)

The first three terms on the right-hand side of the equation are the collisional dissipation terms, which in general are small in comparison with the other terms. The dominant terms are the next two terms, which can provide the drive through the temperature gradient and the line radiation, and the last term, which provides the main stabilization mechanism, through parallel conduction. By expanding the function $I_Z$ around the equilibrium temperature value, we have $I_Z(T) \approx (dI_Z/dT) \tilde{T}$. Therefore, in the region where $dI_Z/dT < 0$, line radiation drives the instability. The corresponding energies are the magnetic energy
\[
E_M = \int dV \frac{\vec{B}^2}{2\mu_0} = \int dV \frac{\left| \nabla_\perp \tilde{\Psi} \right|^2}{2R_0^2\mu_0} ,
\] (7)

the kinetic energy
\[
E_K = \int dV \frac{\vec{V}_r^2}{2} = \int dV \frac{\left| \nabla_\parallel \tilde{\Phi} \right|^2}{2B_0^2} .
\] (8)
and a measure of the thermal energy

$$E_T = \int dV |\tilde{T}|^2$$

These nonlinear equations are solved numerically as an initial value problem. All fields are Fourier expanded in poloidal, $\theta$, and toroidal, $\zeta$, angles. A finite difference representation is used for the radial coordinate $r$. The numerical scheme used in these calculations treats all linear terms as fully implicit. The nonlinear terms are always treated explicitly. Details on the numerical scheme and its implementation are given in Ref. 7.

The analytical theory\(^8,^9\) shows that the turbulence saturation condition is the result of the enhanced parallel diffusion by the radial turbulent transport balancing the gradient and thermal instability drives. These analytical results have been confirmed by the numerical calculations.

The calculations discussed here are for a driven turbulence situation. Since the driving term is the radiation cooling, they were done with the radiation profile frozen in time. Therefore, in these calculations, the quasilinear modification of the line radiation profile is cancelled. This hypothesis is equivalent to the assumption that the heating is instantaneous and the impurity density of the profile is maintained. The reason for such an assumption is to have a steady-state turbulence with a prescribed radiation profile. This implies that the source of free energy for the instabilities is constant during the calculation and feeds the long-scale instabilities. The sink of energy is essentially the parallel conduction term. To reach a steady state, the turbulence level must adjust itself in such a way that the nonlinear broadening of the modes is balanced by the parallel conduction that restricts its radial extent. In this way, the dissipation balances the input energy that feeds the instability. Many convergence tests of the saturation level of turbulence as a function of the number of modes and radial grid size have been done.

$\Psi$ and $\tilde{\Phi}$ can be written as a Fourier expansion on the poloidal and toroidal angles

$$\Psi = \sum_{m,n} \Psi_{mn}^e \cos(m\theta + n\zeta) \quad (10)$$

$$\tilde{\Phi} = \sum_{m,n} \tilde{\Phi}_{mn}^e \sin(m\theta + n\zeta) \quad (11)$$

Here, $m$ and $n$ are the poloidal and toroidal mode numbers, respectively. All quantities can be written in terms of the Fourier expansion. It is interesting to consider first the expansion of the energies and their spectral distribution in the $(m, n)$ plane.

Because the driving term of the instability, the radiation cooling, is radially localized in an interval $(r_1, r_2)$ where $dI_\zeta/dT < 0$, the relevant resonance surfaces are bounded by the values $q_1 = q(r_1)$ and $q_2 = q(r_2)$. The resonant components dominate the spectrum; hence the $(m, n)$ components most relevant for these calculations fall into a wedge in the $(m, n)$ plane delimited by the values of $q_1$ and $q_2$. Therefore, the corresponding energy spectrum is localized in the $(m, n)$ plane. This localization of the global energy spectrum has been tested numerically. The kinetic energy spectrum at constant $n$ shows a sharp falloff, indicating the localization of the spectrum in a wedge in the $(m, n)$ space. The
falloff of the energy spectrum with $n$ along a fixed helicity is less sharp than at constant $m$.

This localization of the energy spectrum in the $(m,n)$ plane is one of the consequences of the underlying magnetic structure effects on the characterization of the MHD turbulence. It has important consequences in facilitating the plasma turbulence calculations, as discussed in Ref. 4.

### III. MODE COUPLING IN REAL SPACE

In these turbulence calculations, apart from the statistical analysis of the numerical results to determine average mode number and spectral decay index, the analysis is essentially visual. Sometimes this is the only way to understand some of the basic physics involved in the nonlinear processes. High-resolution contour plots of the fields (Fig. 1) are useful for this analysis. In this figure, the contours of constant electrostatic potential fluctuations are plotted in a rectangular box, the horizontal axis being the $r$ range ($0.80 < r/a < 0.92$) in which $dI/dT < 0$ and the vertical axis being the poloidal angle between $0^\circ$ and $180^\circ$. The structures observed in the electrostatic potential fluctuations are very similar to the structures in the temperature fluctuations.

At the beginning of the nonlinear phase in some of the turbulence calculations, well-defined radial patterns are apparent for the electrostatic potential fluctuations (Fig. 1a). Deep into the nonlinear regime, a memory of the radial patterns seems to remain. It is, therefore, important to understand the cause of such structures. As we will discuss, these patterns reflect the structure of the lowest $m$ numbers associated with the rational surfaces.

To calculate the lowest $m$ number for each rational surface, we can generate the rationals between two given surfaces in a way similar to a Fibonacci sequence. Let us consider the case for which the unstable region is between the $q_1 = 2$ and $q_2 = 3$ surfaces. The lowest $m$ modes at these two surfaces are $(m_1 = 2, n_1 = 1)$ and $(m_2 = 3, n_2 = 1)$, respectively. The next lowest $m$ mode in this $q$ interval is obtained by the beating of these two modes, that is, $(m_1 + m_2, n_1 + n_2) = (5, 2)$. This mode is associated with $q = (m_1 + m_2)/(n_1 + n_2) = 5/2$. We can proceed in this way and generate sequences of rational numbers such that the numerators and denominators are obtained by adding the numerators and denominators, respectively, of the two previous terms in the sequence. For the case considered here, an example of a three-step sequence is

$$
\begin{align*}
3/1 & \quad 5/2 & \quad 2/1 \\
3/1 & \quad 8/3 & \quad 7/3 & \quad 2/1 \\
3/1 & \quad 11/4 & \quad 13/5 & \quad 12/5 & \quad 9/4 & \quad 2/1
\end{align*}
$$

Here, the rationals are ordered in a descending order which would correspond to a radial distribution of the rational flux surfaces. These sequences give the lowest relevant $m$ at each radial position. The lowest $m$ as a function of $r$, $m(r)$, is a Cantor set, and the correlation between $r$ and $m$ is rather complicated.

The set of $m$ numbers for the fifth step in this sequence has been represented graphically in Fig. 1b. This has been done by assigning at each radial position a number of cells corresponding to the $m$ value. These cells schematically represent the structure of a given $m$ mode localized at the resonant surface. For neighboring modes, the nonlinear interaction is maximized at a few $\theta$ values. The phase-reinforcing lines form a well-defined pattern in the radial direction. These patterns are very similar to those obtained in the numerical calculation for the electrostatic potential contours (Fig. 1a). This indicates that the observed
Fig. 1. Comparison of (a) lowest $m$ values at the resonance surfaces between $q = 2$ and $q = 3$ and (b) observed structures in the numerical calculations using Eqs. (1)–(3).
patterns in real space correspond to directions of maximum overlap between neighboring modes.

These structures are less likely to occur in more realistic models. The existence of a differential rotation between neighboring surfaces, because of diamagnetic effects or sheared electric fields, will change these patterns and can substantially change the nonlinear interactions. Their relevance for MHD turbulence remains when the growth of the instability is larger than the differential rotation.

IV. RADIAL DEPENDENCE OF THE SPECTRUM

The important role that the lowest $m$ mode in each magnetic surface plays in the nonlinear interaction can be expected to affect the local mode spectrum. Numerical results show that the $m$ spectrum at a fixed radius depends on the lowest $m$ value associated with the nearest resonant surface. In Fig. 2, the $m$-spectra at two different radial positions have been plotted. These two positions correspond to $q$ values of $8/3$ and $20/7$, respectively. The peak of the spectrum shifts as the lowest dominant $m$ value changes, being lower at the lowest $m$ value. The value of the spectral average of $m^2$ also changes with the radial position. In general, the spectrum-averaged value of $m$ is proportional to the lowest local $m$ value. The proportionality factor is between 1.25 and 1.5, depending on the particular problem. Therefore, determination of the lowest relevant $m$ at a given radial position is important in determining the expected properties of the spectrum. To estimate the radial dependence of the spectral average of $m^2$, $(m^2)^{1/2}$,
we calculate the function \( m = M(r) \) as follows: (1) first, we calculate the lowest \( m \) at each radial position as described in Sec. III; (2) each rational surface is associated with radial width, \( W(m) \), given by the nonlinear theory (in the case of the resistivity-gradient-driven turbulence this nonlinear width scales as \( m^{-1/3} \)); and (3) in radial regions where two widths overlap, the lowest \( m \) value is taken to be the dominant local \( m \).

The resulting function \( M(r) \) has a complicated structure (Fig. 3). If the nonlinear width \( W(m) \) is increased for each mode, many of the details of the radial structure of \( M(r) \) are washed out. For a given \( W(m) \), \( (m^2)^{1/2} \) scales as \( q \) in a radially averaged sense. Since \( q(a) \approx 3 \) in TEXT and \( q(a) \approx 1 \) in ATF, we expect \( (m^2)^{1/2} \) for ATF to be about a factor of three lower than for TEXT. Because of the complicated dependence on \( r \), the factor of three is only relevant in a radial averaged sense. Recent results from ATF\(^3\) seem to verify this scaling. However, at present, low-temperature results from TEXT are difficult to compare with those from ATF because the TEXT results correspond to plasmas in the shadow of the limiter. But it will be interesting to continue the experimental evaluations of the radial dependence of \( (m^2)^{1/2} \).

The numerical results for \( (m^2)^{1/2} \) seem to follow the structure of the calculated \( M(r) \) function (Fig. 4). The results plotted in Fig. 4 correspond to a

![Graph](ORNL-DWG 90M-2594 FED)

Fig. 3. Lowest \( m \) value resonant at a given \( q \) surface as a function of radius for \( q \) profiles relevant to TEXT and ATF.
nonlinear calculation using the model described in Sec. II with parameters corresponding to TEXT edge plasmas. The numerical results have been averaged over the steady-state turbulence regime and over a small radial interval $\Delta r = 0.05a$.

For MHD turbulence, the induced thermal and particle losses are, in general, a function of $\langle m^2 \rangle^{1/2}$. In comparing the analytical results with experimental data in transport simulations, the term $\langle m^2 \rangle^{1/2}$ is normally taken to be a given constant. Therefore, the radial dependence discussed here is never included. Its inclusion will have obvious implications, such that low values of $\langle m^2 \rangle^{1/2}$ imply high levels of losses, and high values of $\langle m^2 \rangle^{1/2}$ imply low levels of losses. This hidden $q$ dependence in $\langle m^2 \rangle^{1/2}$ could explain the step-like structures in the measured electron temperature profiles in the Tokamak Fusion Test Reactor (TFTR).\textsuperscript{10} It also could explain the increased losses near the low rational values of $q$ at the plasma edge. It will have an effect similar to the plasma edge mechanism proposed by R. Waltz et al.\textsuperscript{11} to degrade confinement at high $q$.

In stellarators, a dependence of confinement on $q(a)$ has been experimentally established by the Wendelstein VIIA group.\textsuperscript{12} This dependence is attributed to the existence of resonant magnetic islands, but could also come through electrostatic fluctuation-induced transport in which the basic scale length verifies the pattern shown in Fig. 3.
V. CONCLUSIONS

These results indicate that even for cases in which the anomalous particle and thermal losses are dominated by electrostatic turbulence, the confinement can depend on the structure of the $q$ profile. The magnetic field structure reflected in the $q$ profile can play an important role in defining some of the basic structures of the MHD turbulence. It plays a role in the generation of spatial structures in the initial nonlinear phase that can modify the character of the turbulence in the steady state. It also plays a role in giving a complicated radial dependence to $(m^2)^{1/2}$. That is, the basic scales of the turbulence can change significantly near low-$m$ resonant surfaces, enhancing the loses at these $q$ values. The dependence of $(m^2)^{1/2}$ on the $q$ value, and therefore on the radial position, can modify the explicit dependences of analytically derived anomalous transport coefficients.

REFERENCES