BNL 51302

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LONGITUDINAL SYMMETRIC COUPLED BUNCH MODES

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December 1980

ACCELERATOR DEPARTMENT

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UPTON, NEW YORK 11973

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Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161
Price: Printed Copy \$5.00; Microfiche \$3.50

ABSTRACT

The theory of longitudinal coupled bunch instabilities is discussed. Although some of the results included are not new¹, it is useful to summarize them here in a unified fashion.

I. VLASOV EQUATION

Let $\psi^\mu(\phi, \dot{\phi}, t)$ be the distribution function of the μ -th bunch, $\mu = 1, 2, 3 \dots h$, in phase space, where ϕ = azimuthal angle $-\omega_0 t$, and $\dot{\phi} = d\phi/dt$. The total distribution function $\psi(\phi, \dot{\phi}, t)$ is related to $\psi^\mu(\phi, \dot{\phi}, t)$ by

$$\psi(\phi, \dot{\phi}, t) = \sum_{\mu=1}^h \psi^\mu(\phi, \dot{\phi}, t). \quad (1)$$

The Vlasov equation for the μ -th bunch can be written as

$$\frac{\partial \psi^\mu}{\partial t} + \frac{\partial \psi^\mu}{\partial \phi} - \omega_s^2 (\phi - \phi_0^\mu) \frac{\partial \psi^\mu}{\partial \dot{\phi}} - \frac{\eta \omega_0}{p} F(\phi, t) \frac{\partial \psi^\mu}{\partial \dot{\phi}} = 0, \quad (2)$$

where $\phi_0^\mu \equiv (2\pi/h)\mu$ describes the position of the center of the bunch, and η is the frequency slip factor. The last term in Eq. (2) describes the interaction among the bunches. The function $F(\phi, t)$ will be given below.

Define the polar coordinates (r_μ, θ_μ) by the following:

$$\begin{aligned} \phi - \phi_0^\mu &= r_\mu \cos \theta_\mu, \\ \dot{\phi} &= -\omega_s r_\mu \sin \theta_\mu. \end{aligned} \quad (3)$$

For symmetric coupled bunch modes, we can express the distribution function ψ^μ in terms of the polar variables as

$$\psi^\mu(\phi, \dot{\phi}, t) = \psi_0(r_\mu) + \psi_1(r_\mu, \theta_\mu) e^{-i\Omega t}, \quad \mu = 1, 2, \dots, h, \quad (4)$$

where Ω is the coherent frequency. The unperturbed distribution function ψ_0 is the same in form for all μ , since we are dealing with identical bunches.

If we ignore the term which describes the change in the equilibrium (time independent) distribution of the bunches due to their interaction with the environment, then, to the first order in F , Eq. (2) becomes

$$-i\Omega \psi_1^\mu + \omega_s \frac{\partial \psi_1^\mu}{\partial \theta_\mu} + \frac{\eta \omega_0}{p} F(\phi_0^\mu + r_\mu \cos \theta_\mu) \frac{\sin \theta_\mu}{\omega_s} \psi_0(r_\mu) = 0. \quad (5)$$

The function $F(\phi)$ is defined to be the force which the beam exerts on itself through the environment. It is proportional to the sum of the products of the Fourier components of the line density and the

impedance at the corresponding frequencies. Thus, in our notation,

$$F(\phi) = - \frac{\omega_o^2}{4\pi^2 \beta c} \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} d\phi'' \sum_{\nu=1}^h \psi_1^\mu(\phi', \phi'') \sum_{n=-\infty}^{\infty} Z_n e^{in(\phi-\phi')}, \quad (6)$$

$$= - \frac{\omega_o^2 s}{4\pi^2 \beta c} \sum_{\nu} \sum_n Z_n \int_0^{2\pi} r' dr' \int_0^{2\pi} d\theta' \psi_1^\nu(r', \theta') e^{in(\phi-r' \cos \theta')} \quad (7)$$

where we have used the abbreviation

$$Z_n \equiv Z(n\omega_o + \Omega),$$

$Z(\omega)$ being the impedance at frequency ω .

II. MULTIPOLE EXPANSION

We now examine the a -th multipole mode of coupled bunch oscillations:

$$\psi_1^\mu(r_\mu, \theta_\mu) = \xi^\mu R(r_\mu) e^{ia\theta_\mu}, \quad (8)$$

where a is any integer, and $a = \pm 1$ corresponds to the dipole modes, $a = \pm 2$ to the quadrupole modes, etc. It can be shown that, if the frequency shift $\Delta\Omega = \Omega - a\omega_s$ is small compared to the synchrotron frequency ω_s , the different multipole modes approximately decouple, and therefore it is justifiable to consider different multipole modes separately.

The form of the function R in Eq. (8) is independent of μ , since we are considering symmetric coupled bunch modes. The relative phases of the bunches are described by the constants ξ^μ , $|\xi^\mu|$ being independent of μ .

Substituting (8) into (7) and (5), we obtain

$$\begin{aligned} (\Omega - a\omega_s) R(r) \xi^\mu &= -ia \frac{\eta \omega_o^3}{2\pi p \beta c} \frac{\psi_o'(r)}{r} \sum_{\nu, n} \xi^\nu \frac{Z_n}{n} e^{in(\phi_o^\mu - \phi_o^\nu)} \\ &\quad \times J_a(nr) \int_0^\infty r' dr' R(r') J_a(nr'). \end{aligned} \quad (9)$$

In obtaining Eq. (9), we used the following identities:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta' e^{-inr' \cos \theta'} e^{ia\theta'} = (-i)^a J_a(nr')$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \sin \theta e^{inr \cos \theta} e^{-ia\theta} = -a(-i)^a \frac{J_a(nr)}{nr}.$$

Equation (9) defines the vector $\vec{\xi} \equiv (\xi^1, \xi^2, \dots, \xi^h)$ to be an eigenvector of an $h \times h$ dimensional matrix. The matrix is a cyclic matrix since $e^{in(\phi_0^\mu - \phi_0^\nu)} = \exp[i2\pi n(\mu - \nu)/h]$. The solutions to such an equation are²

$$\xi^\mu = e^{i \frac{2\pi}{h} s\mu}, \quad (10)$$

where the parameter s , $s = 1, 2, \dots, h$, characterizes different solutions. Substituting (10) into (9), and summing over ν , one obtains:

$$(\Omega - a\omega_s) R(r) = -ia \frac{\eta\omega_o^3 h}{2\pi p\beta c} \frac{\psi_o'(r)}{r} \sum_{n=-\infty}^{\infty} \frac{Z_{nh+s}}{nh+s} J_a[(nh+s)r] \\ \times \int_0^\infty r' dr' R(r') J_a[(nh+s)r']. \quad (11)$$

Remember that the impedance Z_n satisfies the following identity:

$$Z_n^* = Z^* (n\omega_o + \Omega) = Z(-n\omega_o - \Omega^*). \quad (12)$$

Using Eq. (12) we observe that the solutions of Eq. (11) corresponding to the modes $(-a, -s)$ and (a, s) are the complex conjugates of each other. We shall therefore consider only the modes with positive a .

There is little hope of solving (11) for Ω and $R(r)$ without some approximations. We adopt the small bunch approximation:

$$J_a(nr) \approx \frac{1}{2^a a!} (nr)^a. \quad (13)$$

Let us first use (13) for the Bessel function in (11) outside of the integral. The result is

$$(\Omega - a\omega_s) R(r) = -i \frac{\eta\omega_o^3 h}{2\pi p\beta c} \frac{1}{2^a (a-1)!} r^{a-1} \psi_o'(r) \sum_h (nh+s)^{a-1} Z_{nh+s} \\ \times \int_0^\infty r' dr' R(r') J_a[(nh+s)r']. \quad (14)$$

Equation (14) is the basis of the discussions in the following sections.

III. FREQUENCY SHIFT

In this section, we assume that the synchrotron angular frequency ω_s is independent of the synchrotron amplitude r :

$$\omega_s(r) = \omega_{so},$$

where ω_{so} is the synchrotron frequency at vanishing amplitude. Under this assumption, the frequency shift $\Delta\Omega \equiv \Omega - a\omega_{so}$ can easily be calculated.

By observation, the solution to Eq. (14) is

$$R(r) = \text{constant} \times r^{a-1} \psi'_0(r), \quad (15)$$

or equivalently,

$$\psi^\mu(r_\mu, \theta_\mu) = \text{constant} \cdot \xi^\mu r_\mu^{a-1} \psi'_0(r_\mu) e^{ia\theta_\mu}. \quad (15')$$

Substituting (15) into (14), we obtain

$$\Delta\Omega = -i \frac{\eta \omega_o^3}{2\pi E} \frac{1}{2^a (a-1)!} \sum_n (nh+s)^{a-1} Z_{nh+s} \int_0^\infty dr r^a \psi'_0(r) J_a[(nh+s)r]. \quad (16)$$

The above results can easily be understood. Equation (15') describes the situation where the a -th multiple pole component of the distorted bunches is rotating rigidly in the phase space. The rotating distorted bunches excite electromagnetic field in the ring through the impedance Z , which in turn causes the frequency shift as given by Eq. (16). Note that the integral in (16) is nothing but the $(nh+s)$ -th Fourier component of the line charge density corresponding to the phase space distribution (15').

Let us now specialize to the case of a Gaussian bunch:

$$\psi_0(r) = \frac{N_B}{2\pi\omega_{so} L^2} e^{-r^2/2L^2}, \quad (17)$$

where L is the rms half bunch length in units of radians, and N_B is the number of particles per bunch, $N_B \equiv N/h$.

The integration in (16) can be done, giving

$$\Delta\Omega = i \frac{\eta I_o \omega_o^2}{2\pi E \omega_{so}} \frac{1}{2^a (a-1)!} L^{2a-2} Z_{\text{eff}}, \quad (18)$$

where

$$Z_{\text{eff}} \equiv \sum_{n=-\infty}^{\infty} (nh + s)^{2a-1} Z[(nh + s)\omega_o + \Omega] e^{-\frac{1}{2}(nh+s)^2 L^2}, \quad (19)$$

and $I_o \equiv (1/2\pi) N\omega_o$ is the total average current.
In the limit $L \rightarrow 0$, Eq. (19) becomes

$$Z_{\text{eff}} = \sum_{n=-\infty}^{\infty} (nh + s)^{2a-1} Z[(nh + s)\omega_o + \Omega]. \quad (20)$$

The approximation of (19) by (20) is equivalent to approximating the Bessel function in (16) by (13).

IV. LANDAU DAMPING

When the synchrotron frequency ω_s is a function of the synchrotron amplitude r , the resulting frequency spread provides Landau damping.

To take into account the Landau damping, all we have to do is to regard ω_s in Eq. (14) as a function of r ; thus we have

$$R(r) = -i \frac{\eta\omega_o^3 h}{2\pi E} \frac{1}{2^a (a-1)!} \frac{r^a \psi_o'(r)}{\Omega - a\omega_s(r)} \sum_n (nh + s)^{a-1} Z_{nh+s} \\ \times \int_0^\infty r' dr' R(r') J_a[(nh + s)r']. \quad (21)$$

From observation of (21), one obtains immediately

$$R(r) = \text{constant} \cdot r^{a-1} \psi_o'(r) / [\Omega - a\omega_s(r)]. \quad (22)$$

With use of Eq. (22), Eq. (21) becomes

$$1 = -i \frac{\eta\omega_o^3 h}{2\pi p\beta c} \frac{1}{2^a (a-1)!} \sum_n (nh + s)^{a-1} Z_{nh+s} \\ \times \int_0^\infty dr \frac{r^a \psi_o'(r)}{\Omega - a\omega_s(r)} J_a[(nh + s)r]. \quad (23)$$

Now approximate the Bessel function in (23) by using (13). The result is

$$1 = -i \frac{\eta\omega_o^3 h}{2\pi p\beta c} a \left[\frac{1}{2^a a!} \right]^2 Z_{\text{eff}} \int_0^\infty dr \frac{r^{2a} \psi_o'(r)}{\Omega - a\omega_s(r)}. \quad (24)$$

Assume that all the non-linearity of the synchrotron frequency is provided by the "natural" rf voltage,

$$V(t) = \hat{V} \sin(h\omega_0 t), \quad (25)$$

and that the bunch length is so small that it is sufficient to approximate Eq. (25) by

$$V(t) = \hat{V} \left[h\omega_0 t - \frac{1}{6} (h\omega_0 t)^3 \right],$$

then the amplitude dependence of the synchrotron frequency is given by³

$$\omega_s(r) = \omega_{so} \left(1 - \frac{1}{16} r^2 \right). \quad (26)$$

For Gaussian bunches, we substitute Eqs. (17) and (26) into Eq. (24); we obtain as a result

$$1 = i \frac{\eta I_0}{\pi p \beta c} \left(\frac{\omega_0}{\omega_{so}} \right)^2 2^{2-a} (a!)^{-2} L^{2a-4} Z_{\text{eff}} [G_a(\Omega)]^{-1} \quad (27)$$

where

$$[G_a(\Omega)]^{-1} \equiv \int_0^\infty dx \frac{x^a e^{-x}}{x - y}, \quad (28)$$

and

$$y \equiv \frac{8}{a\omega_{so} L^2} (a\omega_{so} - \Omega). \quad (29)$$

The threshold curve of the instability corresponds to the solution of Eq. (27) in the limit of $\text{Im}(\Omega) \rightarrow 0^+$. In Figs. 1 and 2, $\text{Re}(G_a)$ is plotted against $\text{Im}(G_a)$ for the cases $a = 1$ and $a = 2$ respectively.

The threshold impedance, $Z_{\text{eff}}^{\text{th}}$, in the sense of Keil and Schnell corresponds to the minimum of $|G_a|$ along this curve. Thus

$$F_a = \frac{\eta I_0}{\pi p \beta c} \left(\frac{\omega_0}{\omega_{so}} \right)^2 2^{2-a} (a!)^{-2} L^{2a-4} |Z_{\text{eff}}^{\text{th}}|, \quad (30)$$

where

$$F_1 = 0.78,$$

$$F_2 = 0.56.$$

Fig. 1. Dipole mode. Stability boundary.

Fig. 2. Quadrupole mode. Stability boundary.

It often happens that the effective impedance Z_{eff} is predominantly inductive. Namely,

$$\text{Re } Z_{\text{eff}} > 0, \text{ Im } Z_{\text{eff}} < 0, \text{ and}$$

$$\left| \frac{\text{Re } Z_{\text{eff}}}{\text{Im } Z_{\text{eff}}} \right| \ll 1.$$

Such a phenomenon happens for the resistive wall contributions at ISABELLE. In a circumstance like this, we should take $F_1 = F_2 = 1$.

Let us now rewrite the stability condition

$$|Z_{\text{eff}}| < |Z_{\text{eff}}^{\text{th}}|, \quad (31)$$

in terms of the frequency shift $\Delta\Omega$ as given by Eq. (18) and the synchrotron frequency spread $\delta\omega_s$ defined as

$$\delta\omega_s = \frac{1}{16} \omega_{\text{so}} \cdot (2.5L)^2. \quad (32)$$

Compare (32) and (26); we are defining the "total" frequency spread $\delta\omega_s$ as the 2.5σ value.

The stability criterion (31) becomes

$$\frac{|\Delta\Omega|}{\delta\omega_1} < 0.32a(a!) F_a. \quad (33)$$

ACKNOWLEDGEMENT

The author thanks C. Pellegrini for discussions.

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