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TITLE: DETERMINATION OF ϵ, β, α FROM AN ARBITRARY NUMBER OF BEAM SIZE MEASUREMENTS*

This note describes how the emittance ϵ and the betatron functions β and α of a beam in an uncoupled high energy linac or transport line are extracted from an arbitrary number (≥ 3) of beam size measurements separated by various betatron phase advances. It is assumed that the beam sizes have no contributions from dispersion, intentional or otherwise.

This analysis derives the beam properties at the location of the most upstream profile monitor. These properties can then be propagated anywhere else in the accelerator. Let $\epsilon_1, \beta_1, \alpha_1$ be the beam properties at the first monitor. The measured size there would be $\sigma_1 = \sqrt{\epsilon_1 \beta_1}$. The measured beam size at any downstream monitors labelled "j" would be $\sigma_j = \sqrt{\epsilon_1 \beta_j}$. β_j can be obtained from β_1 and α_1 using the transport matrix elements from position 1 to j, $R_{11}(j), R_{12}(j)$

$$\beta_j = R_{11}^2(j)\beta_1 - 2R_{11}(j)R_{12}(j)\alpha_1 + R_{12}^2(j)\gamma_1. \quad (1)$$

where γ_1 satisfies the equation $\beta_1 \gamma_1 = 1 + \alpha_1^2$. $R_{11}(j)$ and $R_{12}(j)$ can be rewritten as functions of the lattice parameters $\tilde{\beta}_j, \tilde{\alpha}_j, \tilde{\gamma}_j$ and $\Delta\tilde{\psi}_j$ where $\tilde{\beta}, \tilde{\alpha}, \tilde{\gamma}$ are the TWISS parameters of the lattice at position j given fixed but arbitrary initial values, and $\Delta\tilde{\psi}_j$ is the integrated phase advance starting at position 1.

$$R_{11}(j) = \sqrt{\tilde{\beta}_j / \tilde{\beta}_1} \left(\cos \Delta\tilde{\psi}_j + \tilde{\alpha}_1 \sin \Delta\tilde{\psi}_j \right) \quad (2)$$

$$R_{12}(j) = \sqrt{\tilde{\beta}_j \tilde{\beta}_1} \sin \Delta\tilde{\psi}_j \quad (3)$$

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All quantities here are known, given drifts, accelerating structures, and quadrupoles. The beam size measurements are scaled to the betatron functions at the first monitor.

$$\sigma_j^2 \text{ scaled} = \frac{\tilde{\beta}_1}{\tilde{\beta}_j} \cdot \sigma_j^2 = \frac{\tilde{\beta}_1}{\tilde{\beta}_j} \epsilon_1 \beta_j \quad (4)$$

Inserting Eqs. (2) and (3) into Eq. (1), and then inserting Eq. (1) into Eq. (4).

$$\begin{aligned} \sigma_j^2 \text{ scaled} = \epsilon_1 \left[\right. & \beta_1 \cos^2 (\Delta \tilde{\psi}_j) \\ & + 2 (\beta_1 \tilde{\alpha}_1 - \alpha_1 \tilde{\beta}_1) \cos (\Delta \tilde{\psi}_j) \sin (\Delta \tilde{\psi}_j) \\ & \left. + (\beta_1 \tilde{\alpha}_1^2 - 2\alpha_1 \tilde{\alpha}_1 \tilde{\beta}_1 + \gamma_1 \tilde{\beta}_1^2) \sin^2 \Delta \tilde{\psi}_j \right] \quad (5) \end{aligned}$$

The only unknown quantities on the right hand side are the beam parameters $\epsilon_1, \beta_1, \alpha_1$ and γ_1 . Three independent measurements and $\beta\gamma = 1 + \alpha^2$ will determine all the unknowns. The question here is how can additional measurements be used to improve the errors.

The form of Eq. (5) suggests a method to include more than three measurements. Note that all the coefficients of the cosine and sine terms on the right hand side depend only on TWISS parameters at the first profile monitor. Therefore, $\sigma_j^2 \text{ scaled}$ can be fit to the form.

$$\begin{aligned} \sigma_j^2 \text{ scaled} (\Delta \tilde{\psi}_j) = & A \cos^2 \Delta \tilde{\psi}_j + B \cos \Delta \tilde{\psi}_j \sin \Delta \tilde{\psi}_j \\ & + C \sin^2 \Delta \tilde{\psi}_j \quad (6) \end{aligned}$$

The values of A, B, and C can be determined using a least square fit and as many measurements as desired. Once A, B, and C are determined with their errors, Eq. (5) can be used to determine ϵ_1, β_1 , and α_1 .

$$\epsilon_1 \beta_1 = A \quad (7)$$

$$\epsilon_1 \alpha_1 = \frac{A \tilde{\alpha}_1 - \frac{B}{2}}{\tilde{\beta}_1}, \quad (8)$$

$$\epsilon_1 \gamma_1 = \frac{C - A \tilde{\alpha}_1^2 + 2\epsilon_1 \alpha_1 \tilde{\alpha}_1 \tilde{\beta}_1}{\tilde{\beta}_1^2}, \quad (9)$$

and

$$\epsilon_1^2 = (\epsilon_1 \beta_1)(\epsilon_1 \gamma_1) - (\epsilon_1 \alpha_1)^2, \quad (10)$$

The errors can be determined by propagation.

The form of Eq. (5) also suggests an interesting format to display the data. The format is obtained by transforming Cartesian coordinates into polar coordinates using the betatron phase advance as the rotation angle. In general, an ellipse has the form $a = bx^2 + cxy + dy^2$ with constants a, b, c, and d. If $x = r \cos \theta$ and $y = r \sin \theta$, then the ellipse equation becomes

$$a = r^2(b \cos^2 \theta + c \cos \theta \sin \theta + d \sin^2 \theta). \quad (11)$$

Eq. (5) can be transformed into this form if σ_j^2 scaled is taken to the other side of the equation. A appropriate choice for a polar display of measured beam sizes would be

$$\begin{aligned} \theta_j &= \Delta \tilde{\psi}_j \\ \text{and} \quad r_j &= \left(\frac{\sigma_1}{\sigma_j \text{ scaled}} \right) \end{aligned} \quad (12)$$

The resulting fit to Eq. (5) would be an ellipse in this plot.

In the limit of letting $\beta_1 = \tilde{\beta}_1$ and $\alpha_1 = \tilde{\alpha}_1$, Eq. (5) reduces to

$$\sigma_j^2 \text{ scaled} = \epsilon_1 \tilde{\beta}_1 \quad (13)$$

independent of $\Delta \tilde{\psi}_1$ and implies a circle in the display for a correctly matched beam.

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