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EMBEDDINGS OF SU_3^c IN UNIFYING GAUGE GROUPS*

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ABSTRACT

Hypothetical models that attempt to unify electromagnetic, weak, and strong interactions into a simple, compact gauge group G are discussed. The problem of embedding the strong group SU_3^c in any larger simple group is solved, and a complete classification of theories where the color in some representation is restricted to $\underset{\sim}{1}$, $\underset{\sim}{3}$, and $\underset{\sim}{\bar{3}}$ is given.

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* [illegible]

I. INTRODUCTION

It is an attractive speculation to suppose that each of Nature's fundamental interactions may be described mathematically by a quantum field theory based on a local symmetry. The generalization of quantum electrodynamics to include the weak interactions (through a local $SU_2 \times U_1$ or perhaps a larger group like $SU_3 \times U_1$) has provided a framework for organizing huge quantities of experimental data.[1] Although the precise form of the theory is not settled completely, the strategy of assuming a Yang-Mills Lagrangian for understanding weak and electromagnetic phenomena appears to be correct. Not so well explored is quantum chromodynamics, which is the candidate theory of the strong interactions based on a local, unbroken SU_3^c.[2] (The label c , for color, denotes the strong gauge group, and is used in order to avoid confusion with other gauged and ungauged SU_3 's that appear in particle physics.) The 8 bosons (gluons) that mediate the strong interactions are assumed not to carry electric or weak charges. Although critical tests of QCD are still lacking, its successes inspire enough confidence that we shall assume here that SU_3^c is the correct theory of the strong interactions. Thus the smallest gauge theory that can describe electromagnetic, weak, and strong interactions is based on a group G_0 where

$$G_0 = SU_2 \times U_1 \times SU_3^c \quad . \tag{1}$$

The $SU_2 \times U_1 \times SU_3^c$ theory contains three gauge couplings and, in order to accomodate present phenomenology, there must be many multiplets of quarks and leptons that may also couple to scalar and pseudoscalar fields. In the context of contemporary model building it is necessary to appeal to huge quantities of experimental data to determine the arbitrary parameters and to assign fields to representations of $SU_2 \times U_1 \times SU_3^c$. At best such a theory is a bit awkward. However, at present all efforts to overcome those problems must be viewed as highly speculative, including the general category of models to be discussed here.

The simplest (and probably most naive) scheme for unifying the electromagnetic, weak, and strong interactions is the proposal that $SU_2 \times U_1 \times SU_3^c$ is a proper subgroup of some grand local symmetry group G,[3] where

$$G \supset G^{flavor} \times SU_3^c \supset SU_2 \times U_1 \times SU_3^c \quad . \qquad (2)$$

G^{flavor} is <u>defined</u> by the requirement that $G^{flavor} \times SU_3^c$ is a <u>maximal</u> subgroup decomposition of G. We shall assume that G is a simple, compact Lie group; G must be one of the classical groups, SU_n, SO_n, Sp_{2n}, or one of the 5 exceptional groups G_2, F_4, E_6, E_7, or E_8. Some of the advantages of such a gauge theory are: (1) Many $SU_2 \times U_1 \times SU_3^c$ representations are grouped into comparatively few representations of G; (2) There is only one gauge coupling among the gauge bosons; (3) Some of the parameters that are arbitrary in the $SU_2 \times U_1 \times SU_3^c$ theory become computable in a theory based on G. (All of these advantages are true of $G \times G$ theories, G simple, where the two factors are related by a discrete symmetry. The classification of $G \times G$ theories is similar to the one for G simple; they are considered elsewhere.[4])

The first step in constructing all possible unified models based on a simple G is to solve the embedding problem posed by Eq.(2). The main purpose of this talk is to classify all possible embeddings of SU_3^c in any simple G; that is, to find all possible forms of G^{flavor}. (We shall make one simplification in doing this.) A more detailed exposition of the physics derived from this classification is included in Ref. 4. Before proceeding to the technical discussion, I think it is important to review why the unification provided by Eq. (2) should be considered quite speculative and probably premature. (Of course the investigation of those models may provide some insight into a better form of unification and so the present exercise may be more than academic.) Here are some reasons for the pessimism that some simple G provides the correct unification of $SU_2 \times U_1 \times SU_3^c$:

(1) A theory based on a simple G is likely to contain huge mass scales.[5] The gauge couplings to the unbroken sectors of the theory are measured at mass scales of several GeV to be very different: $\alpha_e \approx \frac{1}{137}$ and $\alpha_s \approx 1/3$. For unification there must exist a reference mass, which will be of order of the largest mass in the broken theory, above which $\alpha_s(M) \approx \alpha_e(M)$. Renormalization group arguments show us that $\alpha(M)$ varies logarithmically, and so, roughly, unification should occur at masses of order $M \sim$ (few GeV) exp (α_s/α_e). (See Ref. 5 for details.) Although the arguments that grand unification *must* involve masses above $10^{10} - 10^{20}$ GeV are not completely rigerous, it is hard to avoid them in grand theories based on a simple G. The dangers of extrapolating present knowledge to such huge energies is obvious. Even if the correct theory is in this class of models, there is need for experimental guidance in choice of gauge group and of particle representations; even the selection between economical or more Rococo models is unclear.

(2) Those huge mass scales may overlap with those of gravity, which is also described by a kind of gauge theory. (The Planck mass m_h is 2×10^{19} GeV.) Proper unification may require including gravity; extended supergravity is one such proposal and is not considered here.[6] This suggests that the limitation of the local algebraic structure to a simple Lie algebra may be overly restrictive; this possibility opens up limitless visitas for speculation.

(3) It is somewhat difficult to understand the origin of conservation

laws not associated with long range forces. For example, in the simplest unified models the proton is unstable, although the decay rate may be exceedingly small due to the huge mass scales. This raises the question, are baryon number, muon number, etc., exact conservation laws, or are they merely approximate? Again adequate experimental guide is not available, and it is not clear whether exact conservation laws should be required. [It is possible to do so, but there is a price to pay.[4]]

(4) The symmetry breakdown from G to $U_1 \times SU_3^c$ must be quite complicated. If this is done by explicit Higgsism, then there will be scalar fields that carry color and scalar-gluon couplings that are not asymptotically free. Moreover one Higgs representation breaks G only a little way, so more representations are necessary. If the symmetry breakdown is dynamical, it is even harder to assess the situation with the present state of understanding.

It is not yet possible to make a firm contact with the hypothetical reality conjectured by these theories. We now turn to the well-defined mathematical problem posed by Eq. (2): What are all possible embeddings of SU_3^c in a simple Lie group G?

II. EMBEDDINGS OF SU_3^c IN A SIMPLE, COMPACT LIE GROUP

We assume that the unifying gauge group G has a maximal subgroup decomposition of the form

$$G \supset G^{flavor} \times SU_3^c \quad . \tag{3}$$

We shall list the embeddings of SU_3^c in G and classify the structure of G^{flavor}. We note here that if G^{flavor} is larger than $SU_2 \times U_1$, then the new generators are coupled to bosons that mediate interactions not yet observed. The supplementary flavor bosons are either of high mass or else not coupled significantly to transitions between light, familiar particles. Other new bosons implied by this kind of unification can include diotons, which are color octets that also carry flavor; leptoquarks, which change quarks into leptons; and diquarks, which change quarks into antiquarks. It should be obvious that an unstable proton is often predicted in this kind of unified theory.

The fields appearing in the Lagrangian are assigned to representations of G. The spin ½ fermion representation $\underline{f}$ must include leptons $(\underline{1}^c)$, quarks $(\underline{3}^c)$ and antiquarks $(\bar{\underline{3}}^c)$; at present experiment does not suggest the existence of new fermions transforming as higher SU_3^c representations. Theoretically there is no objection to having fermions in more complicated color representations; indeed such fermions are commonplace in supersymmetric theories. Nevertheless, it is usually the case that some set of fermions (which might have spin 3/2, for example) belong to a representation containing only $\underline{1}^c$, $\underline{3}^c$ and $\bar{\underline{3}}^c$. The embedding procedure can be simplified if we make the proviso that there must exist at least one non-trivial representation of G containing at most $\underset{\sim}{1}^c$, 3^c and $\underset{\sim}{\bar{3}}^c$. This means that *some* representation $\underset{\sim}{f}$ of G, under the $G^{flavor} \times SU_3^c$ decomposition, has the form

$$\underset{\sim}{f} = (\underset{\sim}{n}_1', \underset{\sim}{1}^c) + (\underset{\sim}{n}_3', \underset{\sim}{3}^c) + (\underset{\sim}{n}_{\bar{3}}', \underset{\sim}{3}^c) \quad , \qquad (4)$$

where $\underset{\sim}{n}_1'$, $\underset{\sim}{n}_3'$, and $\underset{\sim}{n}_{\bar{3}}'$ are representaions of G^{flavor}. The following theorem provides the rationale for our embedding procedure: (The theorem will be proved in the next section.)

If any representation $\underset{\sim}{f}$ of $\underset{\sim}{G}$ decomposed according to Eq. (3) is of the form Eq. (4) then the fundamental representation also contains at most $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$ and $\underset{\sim}{\bar{3}}^c$. (The fundamental representaions of the simple Lie groups are: $\underset{\sim}{n}$ of SU_n ; $\underset{\sim}{n}$ of SO_n ; $\underset{\sim}{2n}$ of Sp_{2n} ; $\underset{\sim}{7}$ of G_2 ; $\underset{\sim}{26}$ of F_4; $\underset{\sim}{27}$ of E_6 ; $\underset{\sim}{56}$ of E_7 ; and $\underset{\sim}{248}$ of E_8 .)

Thus, we may study the embedding in terms of the fundamental representation $\underset{\sim}{n}$, which is not necessarily the fermion representation, with a $G^{flavor} \times SU_3^c$ decomposition of the form,

$$\underset{\sim}{n} = (\underset{\sim}{n}_1, \underset{\sim}{1}^c) + (\underset{\sim}{n}_3, \underset{\sim}{3}^c) + (\underset{\sim}{n}_{\bar{3}}, \underset{\sim}{\bar{3}}^c) \ , \qquad (5)$$

where $\underset{\sim}{n}_1, \underset{\sim}{n}_3$ and $\underset{\sim}{n}_{\bar{3}}$ are representations of G^{flavor}.

The generators of G belong to the adjoint representation Adj(G), and its color content is easily obtained from that of the fundamental representation (see Table 1). With the $G^{flavor} \times SU_3^c$ embedding, Adj(G) has the form,

$$\mathrm{Adj}(G) = (\mathrm{Adj}(G^{flavor}), \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{8}^c) + \text{cross terms} \ , \qquad (6)$$

where the cross-terms correspond to the generators of G that mix flavor and color, except in one case for SU_n. With n_3 and $n_{\bar{3}}$ both different from zero, Adj(G) includes two $(\underset{\sim}{1}, \underset{\sim}{8}^c)$'s, which generate an $SU_3 \times SU_3$ subgroup of SU_n. The sum of the corresponding generators generates SU_3^c. There is then a temptation to enlarge the color group to $SU_3 \times SU_3$, although doing so is optional. We identify G^{flavor} by explicit examination. It is then straightforward to determine the other representations of G that satisfy our color restrictions.

The whole procedure can be generalized by considering embeddings $G \supset G^{\text{flavor}} \times SU_3^c$ with progressively more and more relaxed color restrictions on the fundamental representations and on the others to be used for the fermions, but we do not do this here.

The structure of G^{flavor} falls into one of four classes. A list of the simple groups, their fundamental representations, and the construction of the adjoint representation are shown in Table 1.

Class I: $G^{\text{flavor}}=G_\ell \times G_q \times U_1$, where G_ℓ is a nontrivial simple factor that transforms only the color singlets and G_q is another nontrivial simple factor that transforms only the color triplets (and antitriplets) of the fundamental representation.[7] The U_1 distinguishes $\underset{\sim}{1}^c$ from $\underset{\sim}{3}^c$ and (or) $\underset{\sim}{\bar{3}}^c$. This embedding occurs only if G is a classical group, i.e., if G is SU_n (unitary), SO_n (orthogonal), or Sp_{2n} (symplectic). Only the fundamental representations of G, which are shown in Table 2, satisfy the color restriction: $\underset{\sim}{n}$ of SU_n ; $\underset{\sim}{n}$ of SO_n; or the $\underset{\approx}{2n}$ of Sp_{2n}. Thus, the color singlets of the fundamental representation may be identified with the leptons, and the color triplets with quarks. Since the quarks and leptons have commuting flavor groups, the observed universality of leptonic and quark electromagnetic and weak charges must come from the symmetry breaking mechanism. The $\underset{\sim}{n}$ of SO_n and the $\underset{\approx}{2n}$ of Sp_{2n} are self-conjugate representations, and therefore contain equal numbers of $\underset{\sim}{3}^c$ and $\bar{\underset{\sim}{3}}^c$. In this embedding the $\underset{\sim}{n}$ of SU_n, which is complex, contains $\underset{\sim}{1}^c$ and $\underset{\sim}{3}^c$ only. We now carry out the explicit construction of the results summarized in Table 2.

Consider SU_n, for which the fundamental representations is complex. The simplest form of $\underset{\sim}{n}$ is

$$\underset{\sim}{n} = (\underset{\sim}{n_1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{n_3}, \underset{\sim}{3}^c) \quad , \tag{7}$$

where $n = n_1 + 3n_3$ and n_1 and n_3 are integers greater than 1. The notation in Eq. (7) reflects the fact that this will be a Class I embedding. Since this is a special example of Eq. (5) with $n_{\bar{3}} = 0$, it will not yield the only embedding. However, as we shall see, the general case has new features that should be discussed separately. The adjoint representation of SU_n, which is constructed from $\underset{\sim}{n} \times \bar{\underset{\sim}{n}} - \underset{\sim}{1}$, provides the list of the generators needed to identify the embedding:

$$\begin{aligned} \underset{\sim}{n^2 - 1} = {} & (\underset{\sim}{n_1^2-1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{n_3^2-1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{8}^c) \\ & + (\underset{\sim}{n_1}, \overline{\underset{\sim}{n_3}}, \overline{\underset{\sim}{3}^c}) + (\overline{\underset{\sim}{n_1}}, \underset{\sim}{n_3}, \underset{\sim}{3}^c) + (\underset{\sim}{1}, \underset{\sim}{n_3^2-1}, \underset{\sim}{8}^c) \quad . \end{aligned} \tag{8}$$

Note that Eq. (8) includes no flavor cross terms that are color singlets, so that G^{flavor} is indeed given by

$$G^{\text{flavor}} = SU_{n_1} \times SU_{n_3} \times U_1 \quad . \tag{9}$$

The U_1 distinguishes $\underset{\sim}{1}^c$ and $\underset{\sim}{3}^c$ in the fundamental representation.

For $n_3 > 1$, only the $\underset{\sim}{n}$ of SU_n satisfies the restriction that no more than $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$ and $\bar{\underset{\sim}{3}}^c$ occur in the fermion representation; the leptons are assigned to $(\underset{\sim}{n}_1, \underset{\sim}{1}, \underset{\sim}{1}^c)$ and the quarks to $(\underset{\sim}{1}, \underset{\sim}{n}_3, \underset{\sim}{3}^c)$ of Eq. (7). The assignment must be vectorlike in order to avoid divergences from triangle anomalies.[10]

The fundamental representation of SO_n is the vector representation $\underset{\sim}{n}$, where n may be even or odd. Since $\underset{\sim}{n}$ is self-conjugate, $\underset{\sim}{3}^c$ and $\bar{\underset{\sim}{3}}^c$ must appear symmetrically, and Eq. (5) must be of the form,

$$\underset{\sim}{n} = (\underset{\sim}{n}_1, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{n}_3, \underset{\sim}{3}^c) + (\underset{\sim}{1}, \bar{\underset{\sim}{n}}_3, \bar{\underset{\sim}{3}}^c) \ , \qquad (10)$$

where $n = n_1 + 6n_3$ and $(\underset{\sim}{n}_1, \underset{\sim}{1})$ is a self-conjugate representation of G^{flavor}. Here we consider n_1 and n_3 greater than 1; as our notation indicates, this is a Class I embedding. G^{flavor} is identified from the adjoint representation,

$$\begin{aligned}(\underset{\sim}{n} \times \underset{\sim}{n})_A = {} & \left(\underset{\sim}{\tfrac{1}{2}n_1(n_1-1)}, \underset{\sim}{1}, \underset{\sim}{1}^c\right) + (\underset{\sim}{1}, \underset{\sim}{n_3^2-1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{8}^c) \\ & + (\underset{\sim}{n}_1, \underset{\sim}{n}_3, \underset{\sim}{3}^c) + (\underset{\sim}{n}_1, \bar{\underset{\sim}{n}}_3, \bar{\underset{\sim}{3}}^c) + \\ & + \left(\underset{\sim}{1}, \underset{\sim}{\tfrac{1}{2}n_3(n_3+1)}, \bar{\underset{\sim}{3}}^c\right) + \left(\underset{\sim}{1}, \overline{\underset{\sim}{\tfrac{1}{2}n_3(n_3+1)}}, \underset{\sim}{3}^c\right) \qquad (11) \\ & + \left(\underset{\sim}{1}, \underset{\sim}{\tfrac{1}{2}n_3(n_3-1)}, \underset{\sim}{6}^c\right) + \left(\underset{\sim}{1}, \overline{\underset{\sim}{\tfrac{1}{2}n_3(n_3-1)}}, \bar{\underset{\sim}{6}}^c\right) \ ,\end{aligned}$$

and the flavor group is

$$G^{\text{flavor}} = SO_{n_1} \times SU_{n_3} \times U_1 \ . \qquad (12)$$

The explicit U_1 in Eq. (12) counts $\underset{\sim}{3}^c$'s minus $\bar{\underset{\sim}{3}}^c$'s in Eq. (10), and has zero eigenvalue for the color singlet part of $\underset{\sim}{n}$. There are no other representations of SO_n $(n_3 > 1)$ that satisfy our color restriction.

The most general form of $\underset{\sim}{2n}$ for Sp_{2n} consistent with Eq. (5) is

$$\underset{\sim}{2n} = (\underset{\sim}{2n_1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{n_3}, \underset{\sim}{3}^c) + (\underset{\sim}{1}, \underset{\sim}{\bar{n}_3}, \underset{\sim}{\bar{3}}^c) \;, \qquad (13)$$

where $n = n_1+3n_3$. Because both $\underset{\sim}{3}$ and $\underset{\sim}{\bar{3}}^c$ appear in $\underset{\sim}{2n}$, all higher representations have at least $\underset{\sim}{8}^c$ and are excluded for fermions by our color restriction. The adjoint representation is obtained from $(\underset{\sim}{2n} \times \underset{\sim}{2n})_s$:

$$\begin{aligned} \underset{\sim}{n(2n+1)} = {} & (\underset{\sim}{n_1(2n_1+1)}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{n_3-1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{8}^c) \\ & + (\underset{\sim}{2n_1}, \underset{\sim}{n_3}, \underset{\sim}{3}^c) + (\underset{\sim}{2n_1}, \underset{\sim}{\bar{n}_3}, \underset{\sim}{\bar{3}}^c) \\ & + \left(\underset{\sim}{1}, \underset{\sim}{\tfrac{1}{2}n_3(n_3+1)}, \underset{\sim}{6}^c\right) + \left(\underset{\sim}{1}, \overline{\underset{\sim}{\tfrac{1}{2}n_3(n_3+1)}}, \underset{\sim}{\bar{6}}^c\right) \qquad (14) \\ & + \left(\underset{\sim}{1}, \underset{\sim}{\tfrac{1}{2}n_3(n_3-1)}, \underset{\sim}{\bar{3}}^c\right) + \left(\underset{\sim}{1}, \overline{\underset{\sim}{\tfrac{1}{2}n_3(n_3-1)}}, \underset{\sim}{3}^c\right) \\ & + (\underset{\sim}{1}, \underset{\sim}{n_3^2-1}, \underset{\sim}{8}^c) \;, \end{aligned}$$

which implies the Class I embedding,

$$G^{\text{flavor}} = Sp_{2n_1} \times SU_{n_3} \times U_1 \;. \qquad (15)$$

Class II: $G^{\text{flavor}} = G_\ell \times G_q \times G_{\bar{r}} \times U_1 \times U_1$. This is possible only for $G = SU_n$, with fermions assigned to the $\underset{\sim}{n}$, where $\underset{\sim}{n}$ contains $\underset{\sim}{3}^c$ q-type quarks and $\underset{\sim}{\bar{3}}^c$ $\bar{r}$-type antiquarks. The two U_1's distinguish among $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$, and $\underset{\sim}{\bar{3}}^c$. This embedding is quite similar to Class I, but it contains some additional interest because there is a temptation to enlarge the color group to $SU_3 \times SU_3$.

The coil will be placed below the surface of the earth where the compressive stresses in the rock are larger than the tensile stresses produced by the magnet. The magnetic forces can be contained by rock to keep the cost of the system low. The cost of steel bands to take the force would far exceed that of other types of storage systems. A set of struts and rods, as shown in Fig. 7, is required to transmit the forces from the coil at 1.8 K to the rock at about 300 K.

The stresses and deflections associated with the thermally induced contraction of the coil during cooldown and the magnetic Lorentz forces on the conductor are taken up by rippling as shown at the top of the figure. Axial loads are allowed to accumulate until they reach the allowable stress in the conductor, about 138 MPa (20 000 psi), then they are transmitted through struts to the rock. This is accomplished if the coil is segmented and the central segments are stepped inward to a slightly smaller radius.

Conductor

Superconductor for a SMES coil must be reliable (this includes but is not limited to stability considerations), must cost as little as possible, must be capable of being fabricated with existing techniques or extensions of those techniques, and must be flexible enough to be wound into a magnet in a 3-m-wide tunnel.

Operation at 1.8 K rather than at 4 to 6 K and the use of NbTi rather than Nb_3Sn keep the total system cost low. The use of high purity aluminum instead of copper as the current stabilizer is more economical and reduces the size of the conductor.

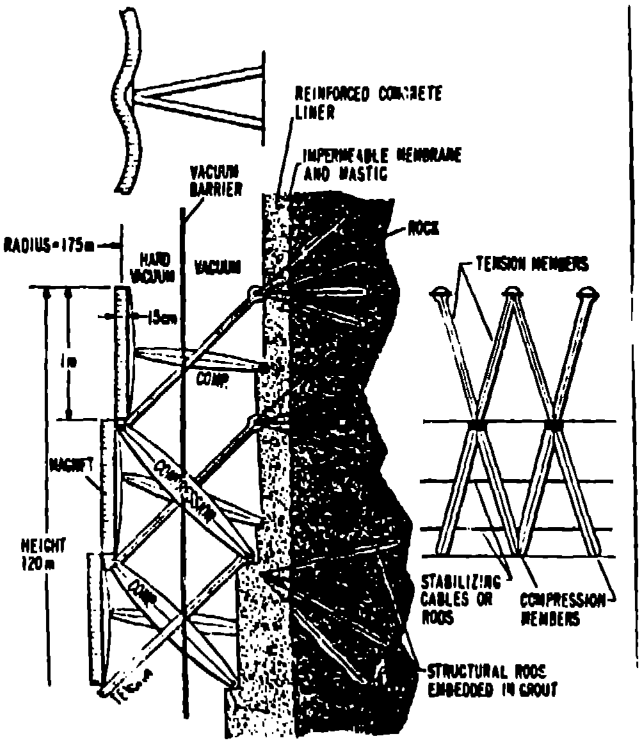

Fig. 7.
Coil and support structure cross section of a 10-GWh SMES unit.

If both n_3 and $n_{\bar{3}}$ of Eq. (5) are non-zero, the adjoint contains two color octets and the most natural embedding is not really of the form Eq. (3) since the two λ's are generators of separate SU_3's. The embedding should be written initially as

$$SU_n \supset (SU_{n_1} \times SU_{n_3} \times SU_{n_{\bar{3}}} \times U_1 \times U_1) \times SU_3^{c'} \times SU_3^{c''} \tag{16}$$

where the flavor group has the structure of a Class II embedding

$$G^{flavor} = G_\ell \times G_q \times G_{\bar{r}} \times U_1 \times U_1 \ . \tag{17}$$

Only the fundamental representation satisfies the color restrictions. The n contains n_1 leptons, n_3 q quarks and $n_{\bar{3}}$ $\bar{r}$ antiquarks, with $n = n_1 + 3n_3 + 3n_{\bar{3}}$. Since n of SU_n is unsafe from triangle anomalies, the fermion assignment must be vectorlike; both $SU_3^{c'}$ and $SU_3^{c''}$ are generated by vector currents. Consequently there is a temptation to enlarge the color group.

So that both q and r quarks be confined, the conventional color generators must be sums of the corresponding $SU_3^{c'}$ and $SU_3^{c''}$ generators. The eight SU_3^c generators are conserved, but there are two distinct possibilities for the remaining eight $SU_3^{c'} \times SU_3^{c''}$ generators: either they are all broken, or they are all conserved. If only SU_3^c is conserved we obtain the usual strong interaction gauge group: the q and r quarks would be confined by the same set of gluons, and the hadron spectrum would include qqr and $q\bar{r}$ states. If the unbroken strong [illegible] group were the full $SU_3 \times SU_3$, then the q and r quarks would be bound together by different sets of gluons. Consequently $q\bar{r}$ states, which transform as $(3, \bar{3})$ of color, would be confined; similar considerations apply to [illegible] qqr states. The qqq and $q\bar{q}$ hadrons would be quite distinct from the rrr and $r\bar{r}$ ones. Bosons of the unified theory transforming as $(3,$ [illegible] $(\bar{3}, \bar{3})$ of color would couple q to r. [illegible]

[illegible]

<u>Class III</u>: $G^{\text{flavor}} = G_{q+\ell} \times U_1$, where $G_{q+\ell}$ transforms the color singlet piece of the fundamental representation, but the fermions are in a different representation such that the same simple[7] factor $G_{q+\ell}$ transforms both quarks and leptons. There are two cases:

If $G = SU_n$, we may ignore the trivial case $n_1 = 1$ in Eq. (7) for which all leptons have the same electric charge, but an interesting special case occurs for $n_3 = 1$ and $n_1 > 1$. Then $\underline{n}$ becomes

$$\underline{n} = (\underline{n}_1 , \underline{1}^c) + (1 , \underline{3}^c) \ , \tag{18}$$

with $n_1 = n-3$. The adjoint representation is

$$\underline{n^2-1} = (\underline{n_1^2-1}, \underline{1}^c) + (\underline{1}, \underline{1}^c) + (\underline{1}, \underline{8}^c) + (\bar{\underline{n}}_1, \underline{3}^c) + (\underline{n}_1, \bar{\underline{3}}^c) \ , \tag{19}$$

which implies the embeding,

$$G^{\text{flavor}} = SU_{n_1} \times U_1 \ . \tag{20}$$

Equation (18) by itself is not an interesting candidate for the fermions. However, our color restriction is satisfied for the representations of dimension $\binom{n}{k}$, obtained by antisymmetrizing $\underline{n}$ k times,

$$(\underline{n}^k)_A = \left\{(\underline{n}_1^k)_A , \underline{1}^c\right\} + \left\{(\underline{n}_1^{k-1})_A , \underline{3}^c\right\} + \left\{(\underline{n}_1^{k-2})_A , \bar{\underline{3}}^c\right\} \tag{21}$$
$$+ \left\{(\underline{n}_1^{k-3})_A , \underline{1}^c\right\} .$$

The last term is omitted for $k < 2$. If n is even and $k = n/2$, $(\underline{n}^k)_A$ is self-conjugate. [For example, the $\underline{20}$ of SU_6 is $(\underline{6}^3)_A$, which is equivalent to $(\bar{\underline{6}}^3)_A$.] Otherwise, each of these representations is complex and can be made triangle anomaly divergenceless. There are no other representations of SU_n that satisfy our color restriction.

The second case has $G = SO_n$, with $n_3 = 1$ in Eq. (10). We briefly review the structure of the spinor representation. There is one self-conjugate spinor for SO_{2n+1} of dimension 2^n. SO_{4n} has two inequivalent self-conjugate spinors, each of dimension 2^{2n-1}, and the two spinors of SO_{4n+2} are complex and conjugate to one another. Each has dimension 2^{2n}.

Although the structure of the spinor representations of SO_n differs for n even and odd, the characterization of their color content is similar enough to treat them together. Recall that we embed SU_3^c through the fundamental representation, Eq. (16). The requirement that the spinor representation contain $\underline{1}^c$, $\underline{3}^c$ and $\bar{\underline{3}}^c$ only implies the same for the fundamental representation. We first show that $n_3 = 1$ in Eq. (10), so $G^{flavor} = SO_{n-6} \times 1_3$; we then give the $SO_{n-6} \times SU_3^c$ decomposition of the spinors.

First consider SO_{2n+1}. Since the spinor of SO_{2n+1} has only $\underline{1}^c$, $\underline{3}^c$, $\bar{\underline{3}}^c$, the color content of $\underline{s} \times \underline{s}$ cannot go beyond $\underline{1}^c$, $\underline{3}^c$, $\bar{\underline{3}}^c$, $\underline{6}^c$, $\bar{\underline{6}}^c$, $\underline{8}^c$. However, the color content of $\underline{s} \times \underline{s}$ for the [illegible] can be computed directly from $\underline{s} \times \underline{s} = \sum_{k=0}^{n} (\underline{n}^k)_A$. Specifically, if $n_3 > 1$, $(\underline{n}^3)_A$ contains a $\underline{15}^c$, so $\underline{s}$ can satisfy the color restriction only if $n_3 = 1$ in Eq. (10). Since the factor SO_{n_3} disappears for $n_3 = 1$, the flavor group is

$$G^{flavor} = SO_{\text{[illegible]}} \times 1_3 . \qquad (2\text{[illegible]})$$

The decomposition of the SO_{2n+1} spinor into [illegible] $\times\ SO_{2j}$ [illegible] tions is particularly [illegible]:

$$\text{[illegible]} (\ \ , \ \ + (\ \ , \ \ )) .$$

where [illegible] the [illegible] are the two SO_{2j} [illegible] [illegible] 1, [illegible] 4 [illegible] 4 of [illegible] SO_4 [illegible] SO_4 [illegible] $1_3 \times SU_3^c$ [illegible] $\underline{1}^c + \underline{3}^c$ [illegible] decomposition [illegible]

$$\sigma = (\xi,1^c) + (\xi,3^c) + (\xi,1^c) + (\xi,\bar{3}^c) \quad . \qquad (23)$$

The proof that the color restriction on σ requires $n_3 = 1$ works also for the spinors of SO_{2n}. For n odd, σ is complex and $\sigma' = \bar{\sigma}$, so that neither $\sigma \times \bar{\sigma}$ nor $\sigma \times \sigma$ can have color representations of dimension greater than 8, which is possible only if $n_3 = 1$ in Eq.(10). The same argument applies if n is even, although σ and σ' are then self-conjugate spinors and only $\sigma \times \sigma$ needs to satisfy the color restriction. The flavor group is then

$$G^{\text{flavor}} = SO_{2n-6} \times U_1 \ , \qquad (24)$$

and we have a Class III embedding.

The $SO_{2n-6} \times SO_6$ decompositions of the SO_{2n} spinors are

$$\sigma = (\xi,4) + (\xi',\bar{4}) \ ,$$

$$\sigma' = (\xi',4) + (\xi,\bar{4}) \ ,$$

and the $G^{\text{flavor}} \times SU_3^c$ decompositions are

$$\sigma = (\xi,\ 1^c) + (\xi,\ 3^c) + (\xi',\ 1^c) + (\xi',\ \bar{3}^c) \qquad (25)$$

$$\sigma' = (\xi',\ 1^c) + (\xi',\ 3^c) + (\xi,\ 1^c) + (\xi,\ \bar{3}^c) \ , \qquad (26)$$

where ξ and ξ' are SO_{2n-6} spinors. As before, the U_1 of Eq. (24) appears in the decomposition $SO_6 \supset SU_3^c \times U_1$.

There are no other representations of SO_r that satisfy the color restriction, since σ, σ' and r each contain both 3^c and $\bar{3}^c$.

[illegible] charges is natural since [illegible]. However, the relation between [illegible] the symmetry breaking, __Class IV__: $G^{\text{flavor}} = G_{q+\ell}$. This embedding, which contains no U_1 factor that distinguishes $\underset{\sim}{1}^c$ from $\underset{\sim}{3}^c$, is possible only for the exceptional groups. Three of the five exceptional groups satisfy our assumptions, and in each case, only the fundamental representation satisfies the color restriction. For F_4, where $G^{\text{flavor}} = SU_3$, the fundamental representation is the $\underset{\sim}{26}$, which is self conjugate. The flavor group for E_6 is $SU_3 \times SU_3$ and the fermions are assigned to the $\underset{\sim}{27}$, which is complex. For E_7, $G^{\text{flavor}} = SU_6$ and the fermions are assigned to the $\underset{\sim}{56}$, which is self conjugate.

The restrictiveness of the exceptional groups, both in number and in internal structure, makes them quite attractive for model building. The universality of the quark and leptonic weak and electromagnetic charges is a consequence of the group structure, as is the 1/3 integral charge structure of the quarks if the leptons have charges ±1 and 0 only.

Two of the exceptional groups fall outside our assumptions. G_2 has rank 2, and SU_3 alone is a maximal subgroup; thus G^{flavor} is trivial, lacking even a U_1 for electromagnetism. E_8 has rank 8 and 248 generators. It is the only Lie group for which the smallest representation is the adjoint; there are no representations satisfying our color restriction.

F_4 has rank 4 and 52 generators. SU_3^c is embedded by

$$F_4 \supset SU_3 \times SU_3^{\,c} \,, \qquad (27)$$

so that $G^{\text{flavor}} = SU_3$. Only the smallest nontrivial representation satisfies the color restrictions. Like all other representations of F_4, it is self conjugate:

$$\underset{\sim}{26} = (\underset{\sim}{8}, \underset{\sim}{1}^c) + (\underset{\sim}{3}, \underset{\sim}{3}^c) + (\bar{\underset{\sim}{3}}, \bar{\underset{\sim}{3}}^c) \,. \qquad (28)$$

If the other SU_3 were the color group, there would be no F_4 representations satisfying our color constraint, which is proved in next section.
$SU_3 \times SU_3^c$ decomposition of the adjoint representation is

$$\underset{\sim}{52} = (\underset{\sim}{8}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{8}^c) + (\bar{\underset{\sim}{6}}, \underset{\sim}{3}^c) + (\underset{\sim}{6}, \bar{\underset{\sim}{3}}^c) . \qquad (29)$$

E_6 has rank 6 and 78 generators. Its fundamental representation is complex, and decomposes as

$$\underset{\sim}{27} = (\underset{\sim}{3}, \bar{\underset{\sim}{3}}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{3}, \underset{\sim}{3}^c) + (\bar{\underset{\sim}{3}}, \underset{\sim}{1}, \bar{\underset{\sim}{3}}^c) \qquad (30)$$

under the maximal subgroup decomposition,

$$E_6 \supset (SU_3 \times SU_3) \times SU_3^c . \qquad (31)$$

Either of the other SU_3's could be identified as color.
The adjoint representation is

$$\underset{\sim}{78} = (\underset{\sim}{8}, \underset{\sim}{1}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{8}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{1}, \underset{\sim}{8}^c) \qquad (32)$$

$$+ (\underset{\sim}{3}, \underset{\sim}{3}, \bar{\underset{\sim}{3}}^c) + (\bar{\underset{\sim}{3}}, \bar{\underset{\sim}{3}}, \underset{\sim}{3}^c) .$$

The $\underset{\sim}{27}$ and $\overline{\underset{\sim}{27}}$ are the only representations with $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$ and $\bar{\underset{\sim}{3}}^c$ only.

E_7 has rank 7 and 133 generators. The color can be embedded by

$$E_7 \supset SU_6 \times SU_3^c . \qquad (33)$$

Only the $\underset{\sim}{56}$ satisfies the color restrictions: its $SU_6 \times SU_3^c$ decomposition is

$$\underset{\sim}{56} = (\underset{\sim}{20}, \underset{\sim}{1}^c) + (\underset{\sim}{6}, \underset{\sim}{3}^c) + (\underset{\sim}{6}, \underset{\sim}{3}^c) . \qquad (34)$$

The decomposition of the adjoint representation is

$$\underset{\sim}{133} = (\underset{\sim}{35}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{8}^c) + (\underset{\sim}{\overline{15}}, \underset{\sim}{3}^c) + (\underset{\sim}{15}, \underset{\sim}{\bar{3}}^c) \quad . \tag{35}$$

It is also possible that the SU_3^c is embedded in the SU_6 subgroup of E_7. This could happen in two ways: (1) If $SU_6 \supset SU_3 \times SU_3 \times U_1$, then the $\underset{\sim}{56}$ decomposes to $\underset{\sim}{27} + \underset{\sim}{\overline{27}} + \underset{\sim}{1} + \underset{\sim}{1}$ of E_6. Only the 56 satisfies our color restrictions; (2) If $SU_6 \supset SU_3 \times SU_2$, then no E_7 representation satisfies our color restrictions, which is proven in the next section

There are no other embeddings of SU_3^c in any simple G for which there is at least one representation satisfying the $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$, $\underset{\sim}{\bar{3}}^c$ color restriction.

The classification of the fermion representations is not complete until we analyze their helicity structure. Fermions of a given chirality are transformed among themselves under G, which we continue to assume to be simple.

We first study the case where a scalar fermion number is defined, so that fermions and antifermions are initially distinguished. Suppose the left-handed fermions are assigned to $\underset{\sim}{f}_L$ of G and the right-handed fermions to $\underset{\sim}{f}_R$ of G. Then all left-handed states are in $\underset{\sim}{f}_L + \overline{\underset{\sim}{f}}_R$, and all right-handed states are in $\underset{\sim}{f}_R + \overline{\underset{\sim}{f}}_L$. If the quark-gluon couplings are to conserve parity, there must be a discrete symmetry that relates the quarks in $\underset{\sim}{f}_L$ to those in $\underset{\sim}{f}_R$, and also relates any antiquarks that may be in $\underset{\sim}{f}_L$ to those in $\underset{\sim}{f}_R$. This same discrete symmetry will relate the leptons in $\underset{\sim}{f}_L$ and $\underset{\sim}{f}_R$, if we ignore the possibility of adding G singlets to either $\underset{\sim}{f}_L$ or $\underset{\sim}{f}_R$. Consequently, $\underset{\sim}{f}_L$ and $\underset{\sim}{f}_R$ are either equivalent or else related by group conjugation. Theories in which $\underset{\sim}{f}_R$ is equivalent to $\underset{\sim}{f}_L$ are called vectorlike.[8] If $\underset{\sim}{f}_R$ is equivalent to $\overline{\underset{\sim}{f}}_L$, we call the theory flavor chiral.[9]

We now examine the case where a scalar fermion number cannot be defined; $\underset{\sim}{f}_L$ contains all the left-handed fermions and antifermions of the theory, and $\underset{\sim}{f}_R$ contains all the right-handed ones. Parity conservation in the quark-gluon couplings then implies that the quarks in $\underset{\sim}{f}_L$ and $\underset{\sim}{f}_R$ are related by a discrete symmetry. As before the theory is either vectorlike or flavor chiral. In the latter case ($\underset{\sim}{f}_R$ equivalent to $\overline{\underset{\sim}{f}}_L$), there exists a pseudoscalar quantum number that initially distinguishes $\underset{\sim}{f}_L$ and $\underset{\sim}{f}_R$.

Further limitations on $\underset{\sim}{f}_R$ given $\underset{\sim}{f}_L$ follow from the renormalizability of the theory. The theory must not have divergences due to Adler, Bell and Jackiw triangle anomalies. The fermion representation falls into one of three categories[8]

(1) If $\underset{\sim}{f}_L$ is a self-conjugate representation, there will never be any problem with triangle anomalies. Such theories are always vectorlike.

(2) If $\underset{\sim}{f}_L$ is a complex representation but G is not a unitary group, there is again no problem with triangle anomalies. These theories are based on G = E_6 ($\underset{\sim}{f}$ = $\underset{\sim}{27}$) or SO_{4n+2} ($\underset{\sim}{f}$ = spinor), and may be vectorlike or flavor chiral.

(3) The complex representations of SU_n ($n \geq 3$) are unsafe, but may be used in a vectorlike theory, or in a nonvectorlike theory if there is an accidental cancellation of right- and left-handed anomalies separately. (In the latter case $\underset{\sim}{f}_R$ is equivalent to $\overline{\underset{\sim}{f}}_L$, where $\underset{\sim}{f}_R$ is reducible.) When the cancellation does take place, $\underset{\sim}{f}_L$ often appears as the branching of a safe representation of a larger group. For example, the anomalies from the $\underset{\sim}{\bar{5}}$ and $\underset{\sim}{10}$ of SU_5 cancel, where the decomposition of the spinor of SO_{10} into SU_5 representations is given as $\underset{\sim}{1} + \underset{\sim}{\bar{5}} + \underset{\sim}{10}$. (The singlet does not contribute to the anomaly.)

The results of this classification of the chiral structure of the fermion representations are summarized in Table 6. Except for gauge groups permitting a flavor chiral theory, it is most natural to assume a vectorlike theory.

III. THE THEOREM[4]

We state and prove here the theorem that justifies the embedding procedure followed in Sec. II. Consider any embedding of SU_3^c in a simple Lie group G for which there is at least one representation $\mathbf{f}$ with color content restricted to $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. Then the fundamental representation of G must also be limited to $\mathbf{1}^c$, $\mathbf{3}^c$, and $\bar{\mathbf{3}}^c$. In other words, the condition tha~ the fermion representation contains color singlets, triplets, and possibly antitriplets <u>implies</u> Eq. (5) for the fundamental representation. The fundamental representations of the simple Lie groups are:
$\mathbf{n}$ of SU_n; $\mathbf{n}$ of SO_n; $\mathbf{2n}$ of Sp_{2n}; $\mathbf{7}$ of G_2; $\mathbf{26}$ of F_4; $\mathbf{27}$ of E_6; $\mathbf{56}$ of E_7; and $\mathbf{248}$ of E_8.

The proof for the classical groups merely requires finding the color content of the group generators, which is explicitly displayed by the adjoint representation. Let $\mathbf{c}$ be a set of generators forming an irreducible representation of SU_3^c. Since each group generator must transform $\mathbf{f}$ within the representation it is necessary that $\mathbf{c}$ acting on any SU_3^c representation in $\mathbf{f}$ contain at least one of the color representations in $\mathbf{f}$. If $\mathbf{f}$ has only color singlets, triplets and antitriplets, then $\mathbf{c} \times \mathbf{1}^c$, $\mathbf{c} \times \mathbf{3}^c$, <u>or</u> $\mathbf{c} \times \bar{\mathbf{3}}^c$ must contain a $\mathbf{1}^c$, $\mathbf{3}^c$ <u>or</u> $\bar{\mathbf{3}}^c$. This is true <u>only if</u> $\mathbf{c}$ is $\mathbf{1}^c$, $\mathbf{3}^c$, $\bar{\mathbf{3}}^c$, $\mathbf{6}^c$, $\bar{\mathbf{6}}^c$ or $\mathbf{8}^c$. Thus $\mathbf{f}$ can satisfy the color restriction only if each of the color representations in the adjoint G has dimension less than or equal to 8. The proof is completed by constructing the adjoint representation, which must satisfy this condition, from the fundamental representation.

Suppose the $\mathbf{n}$ of SU_n violates our color restriction, so that it contains a set of operators $\mathbf{d}$ transforming as some higher representation (dimension greater than 3) of SU_3^c. The adjoint representation of SU_n is $\mathbf{n} \times \bar{\mathbf{n}} - \mathbf{1}$, so it includes generators transforming under SU_3^c as the representations in $d \times \bar{d}$, which always contains a $\underset{\sim}{27}^c$. The theorem then follows for SU_n. The proof is similar for the orthogonal and symplectic groups. If the $\underset{\sim}{n}$ (vector representation) of SO_n contains a set $\underset{\sim}{d}$ as defined above, then the adjoint representation, constructed from $(\underset{\sim}{n} \times \underset{\sim}{n})_A$, must include sets of generators that transform as the representations in $(\underset{\sim}{d} \times \underset{\sim}{d})_A$ under SU_3. The theorem follows since $(\underset{\sim}{d} \times \underset{\sim}{d})_A$ always has at least one representation of dimension greater than 8. The adjoint representation of Sp_{2n} is constructed from $(\underset{\sim}{2n} \times \underset{\sim}{2n})_s$, and includes color operators in $(\underset{\sim}{d} \times \underset{\sim}{d})_s$. As before, $\underset{\sim}{d}$ must be empty if $(\underset{\sim}{2n} \times \underset{\sim}{2n})_s$ is to have no sets of generators that transform under SU_3^c as a representation of dimension greater than 8.

There exist embeddings of SU_3^c in the exceptional groups where the fundamental representation is not restricted to $\underset{\sim}{1}^c$, $\underset{\sim}{3}^c$ and $\bar{\underset{\sim}{3}}^c$, but where the adjoint contains no color representations of dimension greater than 8. The previous proof must then be supplemented with some information about the commutation relations of the group generators. We consider each exceptional group individually.

Three cases are trivial. Since the $\underset{\sim}{7}$ of G_2 is self-conjugate, it must decompose to $\underset{\sim}{1}^c + \underset{\sim}{3}^c + \bar{\underset{\sim}{3}}^c$. The decomposition of E_6 into $SU_3 \times SU_3 \times SU_3$ is essentially symmetrical, and any SU_3 may be color, as is clear from Eqs. (30) and (32). E_8 is hopeless since the fundamental representation is the adjoint, which must have an $\underset{\sim}{8}^c$.

We might use the other SU_3 of Eq. (27) as the color subgroup of F_4. The adjoint representation then has the decomposition,

$$\underset{\sim}{52} = (\underset{\sim}{1}, \underset{\sim}{8}^c) + (\underset{\sim}{8}, \underset{\sim}{1}^c) + (\underset{\sim}{3}, \bar{\underset{\sim}{6}}^c) + (\bar{\underset{\sim}{3}}, \underset{\sim}{6}^c) ,$$

as can be seen from Eq. (29). A color octet does appear in the fundamental representation, Eq. (28), for this embedding, so we must prove that no higher representations satisfy our color restrictions.

Consider the action of the generators transforming as $(3, \overline{6}^c)$ on the supposed higher representation of the form, $(y, 1^c) + (x, 3^c) + (\bar{x}, \bar{3}^c)$, x and y any representations of the flavor SU_3. Since these generators annihilate the pieces transforming as $(y, 1^c)$ and $(\bar{x}, \bar{3}^c)$, they must not annihilate $(x, 3^c)$. However, we prove that they do. The generators in $(\bar{3}, 6^c)$ must annihilate $(x, 3^c)$. All the commutation relations among these generators yield generators in $(3, \overline{6}^c)$. Therefore the $(3,\overline{6}^c)$ generators must also annihilate $(x,3^c)$ and the representation cannot be of the supposed form. This completes the theorem for F_4.

Suppose SU_3^c is embedded in the SU_6 subgroup of E_7, Eq. (33). If the color is identified with one of the SU_3's in $SU_6 \supset SU_3 \times SU_3 \times U_1$, then the fundamental 56 of E_7, under the decomposition $E_7 \supset E_6 \times U_1$, is $56 = 27 + \overline{27} + 1 + 1$. This has no higher color representations. The other possibility is to obtain the SU_3^c from $SU_6 \supset SU_2 \times SU_3^c$. Here, higher color representations do occur in the 56, since the 20 of SU_6 decomposes as $(2,8^c) + (4,1^c)$. Again, we must prove that no other E_7 representation satisfies the color restrictions. Under $SU_6 \supset SU_2 \times SU_3^c$, the only SU_6 representations restricted to 1^c, 3^c, $\bar{3}^c$ are 1, 6 and $\bar{6}$; the $SU_3 \times SU_6$ content of the supposed higher representation must be $(y, 1) + (x, 6) + (\bar{x}, \bar{6})$. The generators of E_7 include a set $(\bar{3}, 15)$, which annihilates $(y, 1)$ and $(x, 6)$, since 15×6 does not have 1, 6 or $\bar{6}$. Moreover the set of generators formed by the commutators in $(3, \overline{15})$ with itself also fall into the $(\bar{3}, 15)$ class, so that $(\bar{3}, 15)$ also annihilates $(\bar{x}, \bar{6})$. Thus there is no faithful representation of the supposed type.

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7. We often discuss $SO_4 \approx SU_2 \times SU_2$ as if it were a simple (sub) group.
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9. The name "flavor chiral" is appropriate for E_6 and SO_{10}, where the flavor groups are $SU_3 \times SU_3$ and $SU_2 \times SU_2$, respectively, and for the simplest flavor chiral assignment the factors act chirally on the quarks. For SO_{14}, SO_{18}, ..., this name is less appropriate.
10. "Vectorlike" refers to theories where the representations of the left- and right-handed fermions are equivalent; in flavor chiral theories, where the representation is complex, they are conjugate to one another. This classification is discussed near the end of Sec. II.

TABLE 1. Group Taxonomy

Group	Rank	Order	Fundamental representation	Construction of adjoint
SU_n	$n - 1$	$n^2 - 1$	$\underline{n}$	$\underline{n} \times \underline{\bar{n}} - \underline{1}$
$SO_{2n + 1}$	n	$n(2n + 1)$	$\underline{2n + 1}$	$(\underline{2n + 1})^2_A$
SO_{2n}	n	$n(2n - 1)$	$\underline{2n}$	$(\underline{2n} \times \underline{2n})_A$
SP_{2n}	n	$n(2n + 1)$	$\underline{2n}$	$(\underline{2n} \times \underline{2n})_S$
G_2	2	14	$\underline{7}$	$(\underline{7} \times \underline{7})_A = \underline{7} + \underline{14}$
F_4	4	52	$\underline{26}$	$(\underline{26} \times \underline{26})_A = \underline{1} + \underline{52} + \underline{273}$
E_6	6	78	$\underline{27}, \underline{\overline{27}}$	$\underline{27} \times \underline{\overline{27}} = \underline{1} + \underline{78} + \underline{650}$
E_7	7	133	$\underline{56}$	$(\underline{56} \times \underline{56})_S = \underline{133} + \underline{1463}$
E_8	8	248	$\underline{248}$	$(\underline{248} \times \underline{248})_A = \underline{248} + \underline{30380}$

TABLE 2

Class I $G^{flavor} = G_\ell \times G_q \times U_1$

$(G \supset G^{flavor} \times SU_3^c)$

SU_n:

$\underset{\sim}{n} = (\underset{\sim}{n_1}, \underset{\sim}{1}, \underset{\sim}{1^c}) + (\underset{\sim}{1}, \underset{\sim}{n_3}, \underset{\sim}{3^c})$ $\qquad n_3 > 1$

$SU_n \supset (SU_{n_1} \times SU_{n_3} \times U_1) \times SU_3^c$ $\qquad (n = n_1 + 3n_3)$

SO_n:

$\underset{\sim}{n} = (\underset{\sim}{n_1}, \underset{\sim}{1}, \underset{\sim}{1^c}) + (\underset{\sim}{1}, \underset{\sim}{n_3}, \underset{\sim}{3^c}) + (\underset{\sim}{1}, \underset{\sim}{\bar{n}_3}, \underset{\sim}{\bar{3}^c})$ $\qquad n_3 > 1$

$SO_n \supset (SO_{n_1} \times SU_{n_3} \times U_1) \times SU_3^c$ $\qquad (n = n_1 + 6n_3)$

Sp_{2n}:

$\underset{\sim}{2n} = (\underset{\sim}{2n_1}, \underset{\sim}{1}, \underset{\sim}{1^c}) + (\underset{\sim}{1}, \underset{\sim}{n_3}, \underset{\sim}{3^c}) + (\underset{\sim}{1}, \underset{\sim}{\bar{n}_3}, \underset{\sim}{\bar{3}^c})$

$Sp_{2n} \supset (Sp_{2n_1} \times SU_{n_3} \times U_1) \times SU_3^c$ $\qquad (n = n_1 + 3n_3)$

Only the fundamental representation satisfies the color restriction.

TABLE 3

Class II $G^{\text{flavor}} = G_{\ell} \times G_q \times G_{\bar{q}} \times U_1 \times U_1$

SU_n:

$$\underset{\sim}{n} = (\underset{\sim}{n_1},\underset{\sim}{1},\underset{\sim}{1},\underset{\sim}{1}^c) + (\underset{\sim}{1},\underset{\sim}{n_3},\underset{\sim}{1},\underset{\sim}{3}^c) + (\underset{\sim}{1},\underset{\sim}{1},\underset{\sim}{n_{\bar{3}}},\underset{\sim}{\bar{3}}^c)$$

$$SU_n \supset (SU_{n_1} \times SU_{n_3} \times SU_{n_{\bar{3}}} \times U_1 \times U_1) \times SU_3^c \qquad (n = n_1 + 3n_3 + 3n_{\bar{3}})$$

(Note that there is a temptation to enlarge the color group to $SU_3 \times SU_3$.)

TABLE 4

Class II' $G^{\text{flavor}} = G_{q+\ell} \times U_1$

(Class I models with $n_3 = 1$)

SU_n:

$$\underset{\sim}{n} = (\underset{\sim}{n-3}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{3}^c)$$

$$SU_n \supset (SU_{n-3} \times U_1) \times SU_3^c$$

Other representations satisfying the color restriction:

$$(\underset{\sim}{n}^k)_A = ((\underset{\sim}{n-3})_A^k, \underset{\sim}{1}^c) + ((\underset{\sim}{n-3})_A^{k-1}, \underset{\sim}{3}^c)$$

$$+ ((\underset{\sim}{n-3})_A^{k-2}, \underset{\sim}{\bar{3}}^c) + ((\underset{\sim}{n-3})_A^{k-3}, \underset{\sim}{1}^c)$$

SO_n:

$$\underset{\sim}{n} = (\underset{\sim}{n-6}, \underset{\sim}{1}^c) + (\underset{\sim}{1}, \underset{\sim}{3}^c) + (\underset{\sim}{1}, \underset{\sim}{\bar{3}}^c)$$

$$SO_n \supset (SO_{n-6} \times U_1) \times SU_3^c$$

Other representations satisfying the color restriction:

$$\underset{\sim}{\sigma} = (\underset{\sim}{\xi}, \underset{\sim}{1}^c) + (\underset{\sim}{\xi}, \underset{\sim}{3}^c) + (\underset{\sim}{\xi}', \underset{\sim}{1}^c) + (\underset{\sim}{\xi}', \underset{\sim}{\bar{3}}^c)$$

$$\underset{\sim}{\sigma}' = (\underset{\sim}{\xi}', \underset{\sim}{1}\ ) + (\underset{\sim}{\xi}', \underset{\sim}{3}^c) + (\underset{\sim}{\xi}, \underset{\sim}{1}^c) + (\underset{\sim}{\xi}, \underset{\sim}{\bar{3}}^c)$$

(ξ and ξ' are spinors of SO_{n-6}; for n odd $\xi = \xi'$.)

TABLE 5

Class IV $G^{\text{flavor}} = G_{q+\ell}$

F_4:

$$\underset{\sim}{26} = (\underset{\sim}{8},\underset{\sim}{1}^c) + (\underset{\sim}{3},\underset{\sim}{3}^c) + (\underset{\sim}{\bar{3}},\underset{\sim}{\bar{3}}^c)$$

$$F_4 \supset SU_3 \times SU_3^c$$

E_6:

$$\underset{\sim}{27} = (\underset{\sim}{3},\underset{\sim}{\bar{3}},\underset{\sim}{1}^c) + (\underset{\sim}{1},\underset{\sim}{3},\underset{\sim}{3}^c) + (\underset{\sim}{\bar{3}},\underset{\sim}{1},\underset{\sim}{\bar{3}}^c)$$

$$E_6 \supset (SU_3 \times SU_3) \times SU_3^c$$

E_7:

$$\underset{\sim}{56} = (\underset{\sim}{20},\underset{\sim}{1}^c) + (\underset{\sim}{6},\underset{\sim}{3}^c) + (\underset{\sim}{\bar{6}},\underset{\sim}{\bar{3}}^c)$$

$$E_7 \supset SU_6 \times SU_3^c$$

Only the fundamental representation satisfies the color restriction.

TABLE 6

Classification of chiral fermion representations

type of representation	$f_R \sim f_L$ (vectorlike)	$f_R \sim \bar{f}_L$ (flavor chiral)
real	possible	identical to vectorlike
complex safe*	possible	possible
complex unsafe*	possible	usually not possible

*) Safe and unsafe from anomalies.