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MASTER

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DOUBLE-STEP CROSS SECTION CHANGE IN THE VACUUM CHAMBER

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BROOKHAVEN NATIONAL LABORATORY
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ABSTRACT

The coupling impedance of a double-step cross section change in the vacuum chamber of an accelerator or storage ring is computed in first approximation. The character of the coupling impedance in the various frequency domains, that is low-frequency, above cut-off frequency of the vacuum chamber, and at resonances is discussed.

I. INTRODUCTION

The coupling impedance presented by a single cavity to the circulating beam in a circular accelerator or storage ring is usually analyzed by considering a "developed" periodic structure.^{1,2} However, the typical parameters are such that it appears preferable to treat the cavity as a nonperiodic problem. For example, below the cut-off frequency of the vacuum chamber, the circumference of the machine is very much larger than the cut-off wave length and the circular character of the machine becomes irrelevant. The same reasoning holds above cut-off frequency, if the circumference is also much larger than the attenuation length, typically on the order of few 100 m. Furthermore, since measurements are conveniently performed only on structures of small dimensions the coupling impedance of a single cavity must be understood. These arguments suggest the need for an analytical treatment of nonperiodic cavities. Of course, the coupling impedance of a series of identical cavities as encountered, for example, between the magnets must be analyzed as a periodic structure.

In the present report the simple, yet representative case of a double-step cross section change in a circular vacuum chamber is analyzed (Fig. 1). The case of a modulated beam interacting with a nonperiodic cavity has been previously considered by various authors.^{3,4} The present study was inspired by Hereward who analyzed a single-step cross section change in a very wide rectangular chamber.⁵ His result was that a single-step discontinuity represents a nonresonant inductance below cut-off but that it exhibits also a lossy resistive component above cut-off. The scope of this report is extended by considering a circular geometry and by including the effect of resonances. The mathematical approach taken here employs field expansion in subregions and field matching

1. E. Keil and B. Zotter, *Particle Accelerators*, 3, 11 (1972).
2. M. Month and R.F. Peierls, *Nucl. Instrum. Methods*, 137, 299 (1976).
3. F.J. Kriegler, F.E. Mills, J. van Bladel, *J. Appl. Phys.*, 35, 1721 (1964).
4. J.K. Trickett, *Proc. 8th International Conf. High Energy Accelerators*, CERN, Geneva, p.345 (1971).
5. H.G. Hereward, CERN Report ISR-DI/75-47 (1975).

along common surfaces. A judicious choice of the subregions is important to achieve rapid convergence and transparency of the formulae. Matching of fields on transverse planes⁶ appears particularly suited to the problem on hand. Actually, it is quite adequate for our purposes to retain the dominant space harmonics only. The resulting solution describes all qualitative features of the complete solution⁷: Low-frequency inductance, cavity resonances below cut-off damped only by wall losses, cavity resonances above cut-off damped by radiation into the vacuum chamber. As a matter of fact, the results are believed to be quantitatively correct in the limit of small step sizes.

6. H. Hahn, AEÜ 32, 81 (1978).

7. B. Zotter, Proc. 1976 ISABELLE Workshops, Brookhaven, BNL Report BNL 50611, p.92.

Fig. 1. Geometry of the double-step cross section change in the vacuum chamber.

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II. FIELDS IN A SMOOTH VACUUM CHAMBER

For the purpose of the subsequent analysis we consider a modulated particle beam traveling with velocity $v = \beta c$ on the axis of a smooth vacuum chamber of radius b . The beam is assumed to be stiff, that is, fields generated by the environment do not change the initial motion or charge distribution. The total current I is uniformly distributed over the beam cross section, taken as a circle with radius a . The current density thus becomes

$$\vec{i} = \frac{I}{\pi a^2} e^{-jkz} e^{j\omega t} \quad (1)$$

Wave number k and frequency are related by $\omega = vk$ or $\omega = \beta k$ in natural units ($c = \mu_0 = 1$) which will be used throughout. The charge distribution follows from the continuity condition $\text{div } \vec{i} = -j\omega\rho$ to be

$$\rho = \frac{I}{\pi a^2 \beta} e^{-jkz} \quad (2)$$

with the time dependence $\exp(j\omega t)$ omitted. The current-induced fields must satisfy Maxwell's equations, and in particular the inhomogeneous wave equation

$$\nabla^2 \vec{E} + \omega^2 \vec{E} = j\omega \vec{i} + \nabla\rho$$

In a system with rotational symmetry the wave equation reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{\partial^2 E_z}{\partial r^2} + \omega^2 E_z = \begin{cases} -\frac{j\omega}{(\beta\gamma)^2} \frac{I}{\pi a^2} e^{-jkz} & ; r < a \\ 0 & ; r > a \end{cases}$$

with the relativistic expression for the particle mass $\gamma = (1 - \beta^2)^{-1/2}$.

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The nonvanishing transverse field components follow from the well-known relations

$$\left(\frac{\partial^2}{\partial z^2} + \omega^2 \right) E_r = \frac{\partial^2 E_z}{\partial r \partial z}$$

$$\left(\frac{\partial^2}{\partial z^2} + \omega^2 \right) H_\theta = -j\omega \frac{\partial E_z}{\partial r}.$$

The general expressions for the fields generated by the current within the perfectly conducting smooth vacuum chamber are given here without derivation. Omitting the common factor $\exp j(\omega t - kz)$ one can write the expressions for the field components inside the beam ($r < a$) as

$$E_z = \frac{1}{j\omega} \left\{ c_i I_0(k_r r) - 1 \right\} \frac{I}{\pi a^2}$$

$$E_r = \frac{c_i I_1(k_r r)}{\beta k_r} \frac{I}{\pi a^2}$$

$$H_\theta = \frac{c_i I_1(k_r r)}{k_r} \frac{I}{\pi a^2}$$
(4)

and outside the beam ($a < r < b$) as

$$E_z = \frac{c_o Z_0(k_r r)}{j\omega} \frac{I}{\pi a^2}$$

$$E_r = \frac{c_o Z_1(k_r r)}{\beta k_r} \frac{I}{\pi a^2}$$

$$H_\theta = \frac{c_o Z_1(k_r r)}{k_r} \frac{I}{\pi a^2}$$
(5)

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where $k_r^2 = \omega^2 - k^2$ or $k_r = \gamma^{-1} k$

$$Z_0(k_r r) = I_0(k_r r) K_0(k_r b) - I_0(k_r b) K_0(k_r r)$$

$$Z_1(k_r r) = I_1(k_r r) K_0(k_r b) + I_0(k_r b) K_1(k_r r)$$

and the coefficients

$$c_i = \frac{k_r a Z_1(k_r a)}{I_0(k_r b)}$$

$$c_o = \frac{k_r a I_1(k_r a)}{I_0(k_r b)},$$

with I_0, I_1, K_0, K_1 being modified Bessel functions.

The subsequent discussion will be limited to the case of extreme relativistic particles ($\gamma \rightarrow \infty, \beta \rightarrow 1$ and $\omega = k$), in which the field expressions take the form for $r < a$

$$E_{zB} \cong 0$$

$$E_{rB} \cong \frac{I}{2\pi} \frac{r}{a} \quad (6)$$

$$H_{\theta B} \cong \frac{I}{2\pi} \frac{r}{a}$$

and for $a < r < b$

$$E_{zB} \cong 0$$

$$E_{rB} \cong \frac{I}{2\pi} \frac{1}{r} \quad (7)$$

$$H_{\theta B} \cong \frac{I}{2\pi} \frac{1}{r}$$

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As expected, the fields outside of the beam are only dependent on the total current flow. Note that by retaining the next higher terms one could have obtained the well-known expression for the coupling impedance of the smooth vacuum chamber.⁸ Since the topic of this paper is the additional coupling impedance resulting from a double-step discontinuity in the vacuum chamber, the discussion will proceed without this detour.

8. A.M. Sessler, IEEE Trans., NS-18, No. 3, p.1039 (1971).

III. DEFINITIONS OF THE LONGITUDINAL COUPLING IMPEDANCE

The exchange of energy between an electromagnetic field and moving charges is fully described by the integral $\int \vec{E} \cdot \vec{i}^* dV$. Assuming time-harmonic fields and the current density given by Eq. (1) one finds for the complex power flow from the beam into the fields

$$P_{\text{beam}} = - \frac{I}{2\pi a^2} \int E_z e^{jkz} dV$$

with the volume integral extended over the full beam region. In analogy to circuit theory one can define a longitudinal coupling impedance by $P_{\text{beam}} = \frac{1}{2} Z I^2$ leading to

$$Z = - (\pi a^2 I)^{-1} \int E_z e^{jkz} dV \quad (8)$$

It is sometimes useful to use an equivalent voltage, $U = ZI$, defined by

$$U = \frac{-1}{\pi a^2} \int E_z e^{jkz} dV \quad (9)$$

which, of course, is related to the power flow by $P_{\text{beam}} = \frac{1}{2} Z^{-1} U U^*$.

Using Poynting's theorem, one can find an alternate expression for P_{beam} (and thus the coupling impedance)

$$P_{\text{beam}} = \frac{1}{2} j\omega \int (\vec{H} \cdot \vec{H}^* - \vec{E} \cdot \vec{E}^*) dV + \frac{1}{2} \int \vec{E} \times \vec{H}^* \cdot d\vec{S}.$$

Considering, for example, the situation of Fig. 1 one must evaluate the volume integral over the entire cavity region, including the beam region and the smooth vacuum chamber, whereas the surface integral can be limited to the contribution from the walls. The above expression is particularly relevant at the cavity resonances where it reduces to

$$P_{\text{beam}} = P_{\text{wall}} = \frac{1}{2} r_s \int \vec{H} \cdot \vec{H}^* dS$$

with r_s the surface impedance of the wall. Together with the general expression for the quality factor of a cavity

$$Q r_s = \frac{\omega \int \vec{H} \cdot \vec{H}^* dV}{\int \vec{H} \cdot \vec{H}^* dS}$$

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one obtains the formula for the coupling impedance

$$\frac{Z}{Q} = \frac{UU^*}{\omega \int \vec{H} \cdot \vec{H}^* dV} \quad (10)$$

to be used at the cavity resonances. It is seen that at a resonance the coupling impedance is equal to the shunt impedance of a cavity.⁸

An approximate expression for the coupling impedance which remains valid at all frequencies and in presence of low-Q, i.e. overlapping resonances can be obtained by starting with a normal mode expansion of the fields. One finds

$$Z \cong j \sum_v \left\{ \frac{\omega_v}{\omega} - \frac{\omega}{\omega_v} + \frac{(1+j)r_s}{(Qr_s)_v} \right\}^{-1} \left(\frac{Z}{Q} \right)_v$$

with the sum to be taken over all normal modes.

Computations of the coupling impedance in this paper assume low losses and high-Q resonances and thus will be based on the definitions of Eq. (10) at resonance, and Eq. (8) for all other frequencies. Furthermore, to simplify the expressions $I = 1$ and $a \rightarrow 0$, that is a filamentary current of unit strength will be assumed. The volume integral in Eq. (8) then reduces to a simple line integral

$$Z = - \int_{-\infty}^{\infty} (E_z)_{r=0} e^{jkz} dz \quad (11)$$

A corresponding simplification of Eq. (10) will result from the above assumptions.

The usual stability criterion¹ involves the coupling impedance Z divided by the mode number $n = kR$, with R the average radius, thus

$$\frac{Z}{n} = \frac{Z}{kR} \quad (12)$$

IV. THE OFF-RESONANCE SOLUTION

1. General Expressions

The general expressions for the fields in the presence of the cavity are represented as the sum of (1) the smooth-wall beam fields, and (2) the additional fields required to satisfy the boundary conditions on the cavity walls, $\vec{E} = \vec{E}_B + \vec{E}_{TM}$ and $\vec{H} = \vec{H}_B + \vec{H}_{TM}$. The beam fields are given by Eqs. (6) and (7), whereas the induced fields are found from the homogenous wave equation

- in region I

$$\begin{aligned} E_{zTM}^I &= -j(c_e^I - jc_o^I) J_0(k_r^I r) \exp k_z^I (g + z) \\ E_{rTM}^I &= j(c_e^I - jc_o^I) \frac{k_z^I}{k_r^I} J_1(k_r^I r) \exp k_z^I (g + z) \\ H_{\theta TM}^I &= (c_e^I - jc_o^I) \frac{k}{k_r^I} J_1(k_r^I r) \exp k_z^I (g + z) \end{aligned} \quad (13)$$

with $k_r^{Ib} = j_{01}$ and $k_z^{I2} = k_r^{I2} - k^2$

- in region III

$$\begin{aligned} E_{zTM}^{III} &= -j(c_e^I + jc_o^I) J_0(k_r^I r) \exp k_z^I (g - z) \\ E_{rTM}^{III} &= -j(c_e^I + jc_o^I) \frac{k_z^I}{k_r^I} J_1(k_r^I r) \exp k_z^I (g - z) \\ H_{\theta TM}^{III} &= (c_e^I + jc_o^I) \frac{k}{k_r^I} J_1(k_r^I r) \exp k_z^I (g - z) \end{aligned} \quad (14)$$

- in region II

$$\begin{aligned}
 E_{zTM}^{II} &= \left\{ u^{II} \frac{\sinh k_z^{II} z}{\sinh k_z^{II} g} - j i^{II} \frac{\cosh k_z^{II} z}{\cosh k_z^{II} g} \right\} J_0(k_r^{II} r) \\
 E_{rTM}^{II} &= \left\{ - u^{II} \frac{\cosh k_z^{II} z}{\sinh k_z^{II} g} + j i^{II} \frac{\sinh k_z^{II} z}{\cosh k_z^{II} g} \right\} \frac{k_z^{II}}{k_r^{II}} J_1(k_r^{II} r) \\
 H_{\theta TM}^{II} &= \left\{ j u^{II} \frac{\sinh k_z^{II} z}{\sinh k_z^{II} g} + i^{II} \frac{\cosh k_z^{II} z}{\cosh k_z^{II} g} \right\} \frac{k}{k_r^{II}} J_1(k_r^{II} r)
 \end{aligned} \quad (15)$$

with $k_r^{II} d = j_{01}$ and $k_z^{II2} = k_r^{II2} - k^2$.

The solution is found by the usual modal field matching.⁹ Treating the electric field in regions I and III as the excitation for the fields in region II and taking into account the boundary condition leads to the relation

$$\int_0^d E_r^{II} J_1(k_r^{II} r) r dr = \int_0^b E_r^I J_1(k_r^{II} r) r dr,$$

which is evaluated on the interface between I and II. Combining this relation with the corresponding one for the interface between II and III one obtains

$$\begin{aligned}
 u^{II} \frac{k_z^{II}}{k_r^{II}} \operatorname{ctgh} k_z^{II} g \langle II, II \rangle + c_o^I \frac{k_z^I}{k_r^I} \langle I, II \rangle &= \frac{\cos kg}{2\pi} \langle II \rangle \\
 i^{II} \frac{k_z^{II}}{k_r^{II}} \operatorname{tgh} k_z^{II} g \langle II, II \rangle + c_e^I \frac{k_z^I}{k_r^I} \langle I, II \rangle &= \frac{\sin kg}{2\pi} \langle II \rangle.
 \end{aligned} \quad (16)$$

Treating the magnetic field in II as the excitation for the fields in I and III leads to

$$\int_0^b H_{\theta}^I J_1(k_r^I r) r dr = \int_0^b H_{\theta}^{II} J_1(k_r^I r) r dr$$

9. W.C. Hahn, J. Appl. Phys. 12, p.62 (1941).

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and finally to

$$\begin{aligned} c_e^I &= \frac{\langle I, II \rangle}{x \langle I, I \rangle} i^{II} \\ c_o^I &= \frac{\langle I, II \rangle}{x \langle I, I \rangle} u^{II} \end{aligned} \quad (17)$$

where the ratio $x = b/d = k_r^{II}/k_r^I$ is introduced. In the above equations we have used the shorthand symbols for the integrals involved,

$$\begin{aligned} \langle I, I \rangle &= \int_0^b J_1^2(k_r^I r) r dr = \frac{1}{2} \frac{j_{01}^2}{k_r^{I2}} J_1^2(j_{01}) \\ \langle II, II \rangle &= \int_0^d J_1^2(k_r^{II} r) r dr = \frac{1}{2} \frac{j_{01}^2}{x^2 k_r^{II2}} J_1^2(j_{01}) \\ \langle I, II \rangle &= \int_0^b J_1(k_r^I r) J_1(k_r^{II} r) r dr = \frac{x j_{01}}{k_r^{I2} (1 - x^2)} J_1(j_{01}) J_0(x j_{01}) \\ \langle II \rangle &= \int_b^d J_1(k_r^{II} r) dr = \frac{1}{x k_r^I} J_0(x j_{01}) \end{aligned}$$

The longitudinal coupling impedance is now obtained from Eq. (11).

Integration leads to

$$\begin{aligned} Z &= j2i^{II} \left\{ \left(\frac{k_z^{II}}{k_r^{II2}} \tanh k_z^{II} g + B \frac{k_z^I}{k_r^{I2}} \right) \cos kg + \left(\frac{k}{k_r^{II2}} - B \frac{k}{k_r^{I2}} \right) \sin kg \right\} \\ &\quad - j2u^{II} \left\{ \left(\frac{k_z^{II}}{k_r^{II2}} \coth k_z^{II} g + B \frac{k_z^I}{k_r^{I2}} \right) \sin kg - \left(\frac{k}{k_r^{II2}} - B \frac{k}{k_r^{I2}} \right) \cos kg \right\} \end{aligned} \quad (18)$$

$$\text{with } B = \frac{2}{(1 - x^2)} \frac{J_0(x j_{01})}{j_{01} J_1(j_{01})}$$

and

$$i^{II} = \text{sinkg} \frac{k_r^{II} J_0(xj_{01})}{\pi j_{01}^2 J_1^2(j_{01})} \psi_e^{-1}$$

$$u^{II} = \text{coskg} \frac{k_r^{II} J_0(xj_{01})}{\pi j_{01}^2 J_1^2(j_{01})} \psi_o^{-1}$$

where

$$\psi_e = \frac{k_z^{II}}{k_r^{II}} \text{tgh} k_z^{II} g + \frac{k_z^I}{k_r^I} \frac{4x^3}{(1-x^2)^2} \frac{J_0^2(xj_{01})}{j_{01}^2 J_1^2(j_{01})}$$

$$\psi_o = \frac{k_z^{II}}{k_r^{II}} \text{ctgh} k_z^{II} g + \frac{k_z^I}{k_r^I} \frac{4x^3}{(1-x^2)^2} \frac{J_0^2(xj_{01})}{j_{01}^2 J_1^2(j_{01})} \quad (19)$$

The expression for the coupling impedance given by Eq. (18) is valid for extreme relativistic particles. Although derived for frequencies below cut-off ($k < k_r^I$) it remains valid above cut-off. At resonances below cut-off, Z tends to infinity because wall losses were not taken into account in deriving Eq. (18). The relevant quantity here is Z/Q which must be derived directly using Eq. (10). In the subsequent sections we will discuss the general characteristics of Z in the various frequency domains.

A simplification of Eq. (18) is possible in the case of a small step, where we find

$$Z \approx j \frac{(1-x)^2}{\pi j_{01}^2 J_1^2(j_{01})} \frac{\sin 2kg}{1 + \text{tgh} k_z^{II} g} \times$$

$$\times \left\{ \left(2 - \frac{k_z^I}{k_z^{II}} \right) \left(1 - \text{tgh} k_z^{II} g \right) + \frac{k}{k_z^{II}} \left(\text{tgkg} + \text{ctgkg} \text{tgh} k_z^{II} g \right) \right\}.$$

2. The Low-Frequency Expression

In the limit $k \rightarrow 0$ the coupling impedance tends to zero, however, Z/n retains a finite value which is given by the expression

$$\begin{aligned} \frac{Z_o}{n} = j 2 \frac{g}{R} i' \left\{ \operatorname{tgh} k_r^{II} g + Bx \right\} \\ - j 2 \frac{g}{R} u' \left\{ \operatorname{ctgh} k_r^{II} g - \frac{1}{k_r^{II} g} + \frac{Bx^2}{k_r^{II} g} + Bx \right\} \end{aligned} \quad (20)$$

with

$$\begin{aligned} i' &= \frac{J_0(xj_{01})}{\pi j_{01}^2 J_1^2(j_{01})} \left\{ \operatorname{tgh} k_r^{II} g + B^2 x^3 \right\}^{-1} \\ u' &= \frac{J_0(xj_{01})}{\pi j_{01}^2 J_1^2(j_{01})} \left\{ \operatorname{ctgh} k_r^{II} g + B^2 x^3 \right\}^{-1} \end{aligned}$$

Again, a simplification is possible in the case of a small step, that is $x \rightarrow 1$, where one finds

$$\left(\frac{Z_o}{n} \right)_{x \rightarrow 1} = j \frac{g}{R} \frac{2(1-x)^2}{\pi j_{01} J_1(j_{01})} \frac{(1 - \operatorname{tgh} k_r^{II} g + (k_r^{II} g)^{-1} \operatorname{tgh} k_r^{II} g)}{(1 + \operatorname{tgh} k_r^{II} g)} \quad (21)$$

The range of parameters over which the small-step approximation is useful can be assessed by plotting the form factor F relating

$$\frac{Z_o}{n} = F \left(\frac{Z_o}{n} \right)_{x \rightarrow 1}$$

One can see in Fig. 2 that F is approximately independent of the ratio g/d in the range $0.6 < x < 1$.

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The low frequency Z/n equals an equivalent step inductance divided by the machine radius. For long cavities, the inductance equals twice the value of a single discontinuity and is independent of the cavity length

$$\left(\frac{Z_o}{n} \right)_{x \rightarrow 1} \approx j \frac{d}{R} \frac{1}{\pi j_{01}^2 J_1(j_{01})} (1-x)^2 \quad \text{if } k_r^{II} g \gg 1$$

This result is comparable to Eq. (2.13) of Keil and Zotter¹, who give

$$\left(\frac{Z_o}{n} \right)_{x \rightarrow 1} \approx j 0.17 \frac{d}{R} (1-x)^2$$

which differs only in the numerical factor 0.17 from the 0.11 in our solution.

For short cavities, the inductance is proportional to the cavity length

$$\left(\frac{Z_o}{n} \right)_{x \rightarrow 1} \approx j \frac{g}{R} \frac{4}{\pi j_{01}^2 J_1(j_{01})} (1-x)^2 \quad \text{if } k_r^{II} g \ll 1$$

Note, however, that in the case of very short cavities ($g \ll d - b$) the above formulation is inadequate since a linear dependence on the step size would be correct (Appendix),

$$\left(\frac{Z_o}{n} \right)_{g \rightarrow 0} \approx j \frac{g}{\pi R} (1-x) \quad \text{if } k_r^{II} g \ll 1-x$$

This result can be obtained directly from

$$\frac{Z_o}{n} \approx j \frac{1}{R} H_{\theta B}^2 \Delta V$$

where $H_{\theta B} \approx (2\pi b)^{-1}$ and $\Delta V \approx 4\pi g b(d-b)$ because it is justified to assume that the electric field does not penetrate into very short cavities.

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Fig. 2. Form factor in the small step approximation for the coupling impedance at low frequencies.

3. Expressions Above Cut-Off

The expressions for the coupling impedance given by Eq. (18) remain valid above the cut-off frequency of the smooth vacuum chamber. It is, however, convenient to make the substitutions $k_z^I = j|k_z^I|$ and $k_z^{II} = j|k_z^{II}|$ respectively. Note that one has the identities

$$\left. \begin{aligned} \frac{k_z^{II}}{k_r^{II2}} \operatorname{tgh} k_z^{II} g &= - \frac{|k_z^{II}|}{k_r^{II2}} \operatorname{tg} |k_z^{II}| g \\ \frac{k_z^{II}}{k_r^{II2}} \operatorname{ctgh} k_z^{II} g &= \frac{|k_z^{II}|}{k_r^{II2}} \operatorname{ctg} |k_z^{II}| g \end{aligned} \right\} \text{ for } k > k_r^{II}.$$

It is clear that the coefficients i^{II} and u^{II} in Eq. (18) assume complex values which reflects the damping of the cavity resonances by power radiated into the beam tubes above cut-off. As illustration we have plotted in Figs. 3 and 4 the frequency dependence (normalized to k_r^I) of the absolute value of $|Z/n|$ (normalized to the dc value Z_0/n) for two cavities of different length.

Fig. 3/4. Coupling impedance (normalized to the low-frequency value) versus frequency (normalized to the cut-off frequency of the beam tube) above cut-off.

V. THE SOLUTION AT RESONANCES

The expression in Eq. (18) for the coupling impedance below cut-off shows poles at the resonant frequencies of the cavity; Figs. 5 and 6 illustrate the frequency dependence of the coupling impedance. The TM_{0ln} resonances occur at frequencies given by

$$\psi_e(k_{0ln}) = 0 ; \quad n = \text{even}$$

and

$$\psi_o(k_{0ln}) = 0 ; \quad n = \text{odd} .$$

The dependence of the resonant frequencies on the geometrical parameters is shown in Fig. 7. It is seen that in long cavities the resonant frequency is essentially determined by the cavity radius $k_{0ln} d \approx j_{01}$.

It was mentioned above that at resonance the field configuration is dominated by the homogenous solution and that the relevant expression for the coupling impedance is obtained by applying Eq. (10) to the TM_{0ln} fields only. After some manipulations, the terms in

$$\frac{Z}{nQ} = \frac{1}{kR} \frac{UU^*}{k \int \vec{H} \cdot \vec{H}^* dV}$$

can be written as (with $k = k_{0ln}$):

- even modes

$$U = j \frac{2}{k_{II}^2} \left\{ \left(\frac{k_{II}^2}{k_r^2} \tanh k_z^2 g + Bx \frac{k_z^I}{k_r^I} \right) \cos kg + \left(\frac{k}{k_{II}^2} - Bx \frac{k}{k_r^I} \right) \sin kg \right\}$$

$$\int \vec{H} \cdot \vec{H}^* dV = \pi j_{01}^2 J_1^2(j_{01}) \frac{1}{k_{II}^3} \left\{ \frac{k}{k_z^2} \tanh k_z^2 g + \frac{kg}{\cosh^2 k_z^2 g} + B^2 x^4 \frac{k}{k_z^I} \right\}$$

Fig. 5/6. Coupling impedance (normalized to the low-frequency value) versus frequency (normalized to the cut-off frequency of the beam tube) below cut-off.

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Fig. 7. Cavity resonant frequencies (normalized to the cut-off frequency of the beam tube) as function of geometry.

- odd modes

$$U = -j \frac{2}{k_{II}^r} \left\{ \left(\frac{k_z^{II}}{k_r^{II}} \operatorname{ctgh} k_z^{II} g + Bx \frac{k_z^I}{k_r^I} \right) \operatorname{sink} g - \left(\frac{k}{k_r^{II}} - Bx \frac{k}{k_r^I} \right) \operatorname{cos} k g \right\}$$

$$\int \vec{H} \cdot \vec{H}^* dV = \pi j_{01}^2 J_1^2(j_{01}) \frac{1}{k_{II3}^r} \left\{ \frac{k}{k_z^{II}} \operatorname{ctgh} k_z^{II} g - \frac{kg}{\sinh^2 k_z^{II} g} + B^2 x^4 \frac{k}{k_z^I} \right\}.$$

The value of Z/nQ (normalized to the dc value of Z_0/n) for the TM_{010} mode is plotted versus the gap length (g/d) in Fig. 8, with the ratio $x = 0.9, 0.7, 0.5$ as parameters. The dips in the coupling impedance reflect the influence of a transit time factor. A worst-case discussion must be based on the envelope of these curves. The envelopes for the normalized coupling impedances of the TM_{0ln} modes are shown in Figs. 9, 10 and 11. One can deduce that the resonance with k_z^{II} closest to k entails the highest coupling impedance. Typically, damping of the critical resonances to achieve $Q \leq 10$ is required.

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Fig. 8. Coupling impedance divided by Q (and normalized to the low frequency value) of the TM_{010} mode as function of gap length.

Figs. 9, 10 and 11. Envelope for normalized coupling impedance/ Q as function of gap length.

APPENDIX

It was noted above that the field expansion used is ill-suited, in first approximation, to the analysis of very short cavities. Here it is preferable to further subdivide region II by a cylindrical cut at $r = b$. The central region $|z| < g$ and $r < b$ generated by this cut will be denoted by the superscript V. The field expansion in region I and III remains unchanged, however one can impose $c_0^I = 0$, retaining only the even solution with the expansion coefficient c_e^I . The expansion in region II no longer separates the fields into homogenous and inhomogenous solution, and is taken as ($r > b$)

$$E_z^{II} = -j c^{II} \frac{B_0(kr)}{B_1(kb)}$$

$$E_r^{II} = 0$$

$$H_\theta^{II} = c^{II} \frac{B_1(kr)}{B_1(kb)}$$

with $B_0(kr) = J_0(kr) N_0(kd) - J_0(kd) N_0(kr)$

$$B_1(kr) = J_1(kr) N_0(kd) - J_0(kd) N_1(kr) .$$

The total field in region V can now be written as⁶

$$E_z^V = j i^V J_0(k_r^I r) \frac{\cosh k_z^I z}{\cosh k_z^I g} - j c^V J_0(kr)$$

$$E_r^V = -j i^V \frac{k_z^I}{k_r^I} J_1(k_r^I r) \frac{\sinh k_z^I z}{\cosh k_z^I g} + \frac{1}{2\pi r} e^{-jkz}$$

$$H_\theta^V = -i^V \frac{k}{k_r^I} J_1(k_r^I r) \frac{\cosh k_z^I z}{\cosh k_z^I g} + c^V J_1(kr) + \frac{1}{2\pi r} e^{-jkz} .$$

Field matching of the electric and magnetic field along the interfaces leads, after some manipulations, to the expressions for the expansion coefficients ($I = 1$)

$$c^{II} = \frac{1}{2} \frac{B_1(kb)}{B_0(kb)} \psi^{-1}$$

$$c^V = \frac{1}{2} \frac{1}{J_0(kb)} \psi^{-1}$$

$$c^I = \frac{1}{j_{01} J_1(j_{01})} \frac{k_r^{I2}}{k_r^{I2} - k^2} \frac{\tanh k_z^I g}{1 + \tanh k_z^I g} \psi^{-1}$$

$$i^V = \frac{1}{j_{01} J_1(j_{01})} \frac{k_r^{I2}}{k_r^{I2} - k^2} \frac{1}{1 + \tanh k_z^I g} \psi^{-1}$$

with

$$\psi = \pi b \frac{kg}{\sin kg} \left\{ \frac{2 J_0(kd)}{\pi kb B_0(kb) J_0(kb)} + \frac{2}{j_{01}} \frac{k k_r^I}{k_r^{I2} - k^2} \frac{\tanh k_z^I g}{k_z^I g (1 + \tanh k_z^I g)} \right\}$$

Integration of Eq. (11) yields the coupling impedance

$$\begin{aligned} Z &= 2jc^I \left(\frac{k_z^I}{k_r^{I2}} \cos kg - \frac{k}{k_r^{I2}} \sin kg \right) \\ &\quad - 2ji^V \left(\frac{k_z^I}{k_r^{I2}} \tanh k_z^I g \cos kg + \frac{k}{k_r^{I2}} \sin kg \right) \\ &\quad + 2jc^V \frac{\sin kg}{k} . \end{aligned}$$

Of particular interest is the low-frequency limit, $k \rightarrow 0$, where one finds $\psi^{-1} = \pi^{-1} k \ln(d/b)$ leading to

$$\left(\frac{Z_0}{n} \right)_{g \rightarrow 0} = \frac{g}{\pi R} \ln \frac{d}{b}$$

in full agreement with Eq. (2.8) of Keil and Zotter.¹