MIXING CONDITIONS AND LIMIT THEOREMS FOR
MAXIMA OF SOME STATIONARY SEQUENCES

by

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PREFACE

Problems involving the extreme values of stochastic processes have a long history. In ancient Egypt flood control was an important problem and data were recorded on the annual peak level for the Nile river. Recently this author and others (see references [15], [17], and [18]) have been interested in the largest concentration of various air pollutants over fixed time intervals. These are very important practical problems since lack of proper control for extreme observations can have detrimental effects on the population.

Although data have been collected and used for centuries the statistical theory is relatively young. The important early results were obtained by Fréchet in 1927 and Fisher and Tippett in 1928. Gnedenko gave necessary and sufficient conditions for convergence to one of the three extreme value types in 1943. For a more thorough accounting of the history of extreme value theory the reader is referred to Gumbel [15].

In the work mentioned thus far the sequence of random variables are assumed to be independent and identically distributed. These assumptions can be relaxed somewhat. Watson [19] showed that Gnedenko's result holds for m -dependent stationary sequences and in 1965 Loynes [10] generalized this result to certain strictly stationary strong mixing sequences. Leadbetter [8] showed that the assumptions could be weakened still further by introducing the conditions he calls $D(u_n)$ and $D'(u_n)$.

In this thesis some extreme value limit theorems are obtained for specific classes of strictly stationary sequences. In Chapter 1 the results of Loynes and Leadbetter are reviewed. These results, along

with results for associative processes (processes with a specific form of positive dependence) provide the basic tools used to prove the limit theorems in the later chapters.

A class of first order exponential autoregressive processes is studied in Chapter 2. The processes are shown to satisfy the condition of Loynes' theorem. The strong mixing condition which in general would be difficult to check is found to be much easier to verify in this special case due to the Markov structure of the process.

In Chapter 3 a general condition is given which implies strong mixing for Markov and p^{th} order autoregressive processes. This condition is easy to check for Markov processes. Two specific examples are given to illustrate how easy the condition is to verify. For a p^{th} order autoregressive process with small p the condition may be easier to verify than Leadbetter's condition.

In Chapter 4 a class of uniform first order autoregressive processes is studied. These processes are shown to satisfy Leadbetter's $D(u_n)$ condition but they fail to satisfy $D'(u_n)$ and so existing limit theorems cannot be applied. A new limit theorem is obtained for the maximum of such processes and it is seen to be a different result from what would have been obtained if $D'(u_n)$ held. Specifically if M_n^* is the maximum of n independent identically distributed uniform random variables and M_n is the maximum of the first n variables from the uniform first order autoregressive process $\lim_{n \rightarrow \infty} P[M_n^* \leq 1 - \frac{x}{n}] \rightarrow e^{-x} \quad \forall x \geq 0$ but $\lim_{n \rightarrow \infty} P[M_n \leq 1 - \frac{x}{n}] \rightarrow e^{-(1-\rho)x} \quad \forall x \geq 0$ where $\rho = \frac{1}{r}$ and r is an integer with $r \geq 2$. The integer r is a parameter which indexes the family of uniform processes and $\frac{1}{r}$ is the correlation between successive X_n 's.

This result shows that classical extreme value theory cannot be applied to all practical problems. We find that even when we assume stationarity and mixing conditions the limit can differ from the independent case. It also shows that for some first order autoregressive processes the limit distribution can depend on the autocorrelation at lag 1 whereas for processes satisfying Leadbetter's or Loynes' conditions the limit does not depend on the lag 1 autocorrelation. Hopefully this type result will have application to air pollution problems.

In Chapter 5 examples of other processes are studied. The limit theorem of Chapter 2 is extended to a larger class of exponential processes which were studied by Jacobs and Lewis [7]. This class of exponential processes is indexed by two parameters β and ρ with $0 \leq \beta \leq 1$ and $0 \leq \rho < 1$. The case $\rho=1$ is studied in Section 5.4. When $\rho=1$ a limit theorem can be obtained and in this case $D(u_n)$ fails. We find that the limit is not an extreme value type.

For a class of first order autoregressive processes with Cauchy marginal distributions the condition $D'(u_n)$ is shown to fail. A conjectured limit theorem similar to the result for the uniform process appears to be confirmed by the simulation.

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TABLE OF CONTENTS

<u>Chapter</u>		<u>Page</u>
	Preface	iii
	Acknowledgements	vi
1	INTRODUCTION	1
	1.1 Limit Theorems for Stationary Sequences	1
	1.2 Loynes' Conditions	4
	1.3 Leadbetter's Conditions	5
	1.4 Results on Associative Processes	6
2	THE LIMIT THEOREM FOR THE EAR(1) MODEL	10
	2.1 Model Definition	10
	2.2 Existence and Uniqueness for the Distribution of the ϵ_n 's	10
	2.3 The Exact Distributions for M_1 and M_2	11
	2.4 Verifying Strong Mixing	14
	2.5 Verifying Loynes' Sufficient Condition	20
3	SUFFICIENT CONDITIONS FOR STRONG MIXING	25
	3.1 Sufficient Condition for the AR(1) Models and Other Markov Processes	25
	3.2 The Normal Example	27
	3.3 The Cauchy Example	28
	3.4 Sufficient Condition for AR(p) Models	32

TABLE OF CONTENTS (cont.)

<u>Chapter</u>		<u>Page</u>
4	THE LIMIT THEOREM FOR THE UNIFORM AR(1) MODELS	35
4.1	The Class of Models	35
4.2	Proof that $X_n^{(r)} \sim U[0,1]$	35
4.3	The Exact Distribution for M_1 , M_2 , and M_3 when $r=2$	36
4.4	Convergence for Fixed x Along a Subsequence: The Case $r=2$	39
4.5	Verifying $D(u_n)$	42
4.6	Proof that $D'(u_n)$ Does Not Hold	46
4.7	The Limit Theorem	48
4.8	Simulation Results, $x=1$, and $x=1/2$ with $r=2$	57
4.9	Implications on Data Analysis	59
5	OTHER LIMIT THEOREMS	61
5.1	The EARMA(1,1) Model	61
5.2	Special Cases of O'Brien's Examples	65
5.3	The Cauchy Case	68
5.4	A Case Where $D(u_n)$ Fails	71
	APPENDIX 1	75
	APPENDIX 2	87
	REFERENCES	88

1. Introduction

1.1. Limit Theorems for Stationary Sequences. In 1965 R. M. Loynes [10] extended the theorem of Gnedenko to include strictly stationary processes which satisfy a strong mixing condition. He refers to this mixing condition, which will be defined in 1.2, as uniform mixing. In this paper it will be referred to as strong mixing since this usage appears to be more common in the literature. The theorem is as follows:

Theorem 1.1.1. (Loynes). Let $\{X_n\}$ be a strictly stationary stochastic sequence and suppose it satisfies the strong mixing condition. If there exist sequences of real numbers $\{a_n\}$ and $\{b_n\}$ such that $P\{M_n \leq a_n x + b_n\} \rightarrow G(x)$ where $M_n = \max\{X_0, X_1, \dots, X_n\}$ then $G(x)$ must have one of the following forms:

I. $G(x) = \exp(-e^{-x})$.	$-\infty < x < \infty$;
II. $G(x) = 0$	$x \leq 0$,
$\quad = \exp(-x^{-\alpha})$	$x > 0 \quad \alpha > 0 \text{ some constant ;}$
III. $G(x) = \exp(-(-x)^\alpha)$	$x \leq 0 \quad \alpha > 0 \text{ some constant,}$
$\quad = 1$	$x > 0$.

In addition, he proves the following theorem:

Theorem 1.1.2. (Loynes). Let $\{X_n\}$ be a strong mixing strictly stationary stochastic sequence. Let $u_n(x) = a_n x + b_n$ and suppose there exists a function $\tau(x)$ (in Loynes' notation $\xi = \tau(x)$ and $C_n(\xi) = u_n(x)$) such that $P[X_0 > u_n(x)] \leq \frac{\tau(x)}{n} \leq P[X_0 \geq u_n(x)]$. If $P[M_n \leq u_n(x)]$ converges to a nondegenerate distribution function, then it must have the form $e^{-k\tau(x)}$ where $0 < k \leq 1$.

A third theorem of Loynes requires an additional condition which shall be referred to as the sufficient condition and will also be described in 1.2.

Theorem 1.1.3. (Loynes). If a strictly stationary sequence $\{X_n\}$ satisfies the strong mixing condition and the sufficient condition and if $P[M_n^* \leq u_n(x)] \rightarrow G(x)$ where $M_n^* = \max\{X_0^*, X_1^*, \dots, X_n^*\}$ and X_i^* are independent and identically distributed (i.i.d.) and have the same distribution as X_1 , then $P\{M_n \leq u_n(x)\} \rightarrow G(x)$.

Berman [2] proved a theorem similar to 1.1.3 for Gaussian processes.

Theorem 1.1.4. (Berman). Let $\{X_n\}$ be a stationary Gaussian sequence with $EX_n = 0$, $EX_n^2 = 1$, and $r_n = EX_0 X_n$. If $\lim_{n \rightarrow \infty} r_n \ln n = 0$ or $\sum_{n=1}^{\infty} r_n^2 < \infty$, then $P[M_n \leq a_n x + b_n] \rightarrow e^{-e^{-x}}$ as $n \rightarrow \infty$, where $a_n = (2 \ln n)^{-1/2}$ and $b_n = (2 \ln n)^{1/2} - \frac{1}{2}(2 \ln n)^{-1/2} (\ln \ln n + \ln 4\pi)$. We note the similarity of this to 1.1.3 since it is well known that $P[M_n^* \leq a_n x + b_n] \rightarrow e^{-e^{-x}}$ as $n \rightarrow \infty$ for i.i.d. standard normal random variables.

In 1974 Mittal and Ylvisaker [11] showed that if $r_n \rightarrow 0$ and $(r_n \ln n)^{-1}$ is monotone decreasing for large n and tending to zero (i.e. $r_n \ln n \rightarrow \infty$) then $P\{r_n^{-1/2}[M_n - (1-r_n)^{1/2} b_n] \leq x\} \rightarrow \Phi(x)$ where Φ is the standard normal distribution function and b_n is as defined in Theorem 1.1.4. Obviously such processes cannot satisfy the strong mixing condition. These results characterize the possible limit laws for most stationary Gaussian sequences and give rise to many questions about the characterization of limit laws for maxima of other stationary sequences.

In 1974 Leadbetter [8] proved that Theorems 1.1.1 and 1.1.3 hold under weaker conditions. Essentially he gave Loynes' proof of Theorem 1.1.1, but removed from the mixing condition those properties which were not needed in the proof. This condition he called $D(u_n)$. We shall give the definition in 1.3.

Theorem 1.1.5. (Leadbetter). If $\{X_n\}$ is a strictly stationary stochastic sequence satisfying $D(u_n)$ and if $P[M_n \leq a_n x + b_n] \rightarrow G(x)$ for

sequences $\{a_n\}$ and $\{b_n\}$ of real numbers then the conclusion of Theorem 1.1.1 holds.

Leadbetter added a sufficient condition similar to Loynes' condition which he calls $D'(u_n)$. This also will be defined in 1.3.

Theorem 1.1.6. (Leadbetter). Let $\{X_n\}$ be a strictly stationary process satisfying $D(u_n)$ and $D'(u_n)$. If $P[M_n^* \leq u_n(x)] \rightarrow G(x)$ where $u_n(x) = a_n x + b_n$ then $P[M_n \leq u_n(x)] \rightarrow G(x)$ also.

We note that the condition $D'(u_n)$ can be replaced by Loynes' sufficient condition and Theorem 1.1.6 will still hold. In this dissertation, classes of strictly stationary stochastic sequences will be considered. Since they are all autoregressive-moving average models (denoted as in Box and Jenkins [14] ARMA), $r_n \ln n \rightarrow 0$ in each case. For the first order exponential autoregressive process (EAR(1)) and the uniform first order autoregressive process limit theorems for the maximum will be given. The EAR(1) model can be generalized to a mixed autoregressive-moving average model denoted EARMA(1,1) for which a similar limit theorem is obtained. For a Cauchy AR(1) model the strong mixing condition is verified, $D'(u_n)$ is shown to fail, and a limit theorem is conjectured. The conjecture is supported by results from a simulation. Since strong mixing implies $D(u_n)$, we have the option of checking strong mixing or $D(u_n)$. When one of these conditions is shown to hold, it is possible to obtain a limit theorem by verifying either $D'(u_n)$ or Loynes' sufficient condition. The $D'(u_n)$ condition is usually easier to check.

Theorem 1.1.3 holds with $D(u_n)$ replacing strong mixing. This theorem was not given by Leadbetter in his paper, so it shall be given here.

Theorem 1.1.7. Let $\{X_n\}$ be a strictly stationary sequence satisfying $D(u_n)$ and suppose that $P[X_1 > u_n(x)] \approx \frac{\tau(x)}{n}$ as $n \rightarrow \infty$. ($a_n \approx b_n$ means $\frac{a_n}{b_n} \rightarrow 1$ as $n \rightarrow \infty$.) If $P[M_n \leq u_n(x)]$ converges to a nondegenerate distribution function, then it must be that $P[M_n \leq u_n(x)] \rightarrow e^{-k\tau(x)}$ for some $0 < k \leq 1$.

Proof. In [8] Leadbetter showed that $D(u_n)$ implies

$$(1.1) \quad \lim_{m \rightarrow \infty} \{P[M_n \leq u_n(x)] - P^{\ell}[M_m \leq u_n(x)]\} = 0$$

where $n = m\ell$. Loynes in [10] showed that if (1.1) is satisfied for $\ell=2$, then $\lim_{n \rightarrow \infty} P[M_n \leq u_n(x)]$ can only have the form $e^{-k\tau(x)}$ for some $0 < k \leq 1$. He proved this result for strong mixing, but (1.1) with $\ell=2$ is really all he used. Hence the theorem holds under $D(u_n)$.

An interesting quotation from Loynes' paper is, "The question of what values of k can actually arise suggests itself immediately; the author does not know, having been unable to find any process for which the limit exists with $k \neq 1$." In Chapter 5 the class of uniform AR(1) models and the Cauchy AR(1) model satisfy the conditions for 1.1.7 and the limit holds with $k < 1$. O'Brien gives other examples in [12].

1.2. Loynes' Conditions. In this section we define strong mixing and Loynes' sufficient condition. These are the conditions in Theorems 1.1.1, 1.1.2, and 1.1.3.

Definition 1.2.1. A strictly stationary stochastic sequence is said to satisfy the strong mixing condition if for any $n \geq 0$ $|P(A \cap B) - P(A)P(B)| \leq \phi(k)$ for all $A \in \sigma(X_0, X_1, \dots, X_n)$, $B \in \sigma(X_{n+k+1}, X_{n+k+2}, \dots)$ and $\phi(k)$ is a function depending only on k with $\phi(k) \rightarrow 0$ as $k \rightarrow \infty$. $\sigma(X_0, X_1, \dots, X_n)$ denotes the sigma field generated by the random variables $X_0, X_1, \dots, X_n$.

Definition 1.2.2. A strictly stationary sequence which satisfies the strong mixing condition is said to satisfy Loynes' sufficient condition if there exist sequences of integers $\{p_m\}$ and $\{q_m\}$ satisfying $m\phi(q_m) \rightarrow 0$, $q_m/p_m \rightarrow 0$ as $p_{m+1}/p_m \rightarrow 1$ as $m \rightarrow \infty$ and a sequence of real numbers $\{C_n(\xi)\}$ for which

$$\sum_{i=1}^{p_m-1} \left(\frac{p_m-i}{p_m} \right) \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} \rightarrow 0$$

as $m \rightarrow \infty$ and ξ is chosen so that $P[X_0 > C_n(\xi)] \leq \xi/n \leq P[X_0 \geq C_n(\xi)]$. $t_m = m(p_m + q_m)$ and ϕ is defined in definition 1.2.1.

Loynes showed that $P[M_n \leq C_n(\xi)] \rightarrow e^{-\xi}$ when 1.2.1 and 1.2.2 hold. This is theorem 1.1.3 with $\xi = \tau(x)$ and $C_n(\xi) = u_n(x)$.

1.3. Leadbetter's Conditions. We shall now define the conditions $D(u_n)$ and $D'(u_n)$.

Definition 1.3.1. A strictly stationary sequence $\{X_n\}$ is said to satisfy the condition $D(u_n)$ if for any integers $1 \leq i_1 < i_2 < \dots < i_p < j_1 < \dots < j_q \leq n$ with $j_1 - i_p \geq \ell + 1$

$$|F_{i_1 i_2, \dots, i_p j_1, \dots, j_q}^{(u_n)} - F_{i_1 i_2, \dots, i_p}^{(u_n)} F_{j_1, \dots, j_q}^{(u_n)}| \leq \alpha_{n, \ell}$$

where $\lim_{\ell \rightarrow \infty} \lim_{n \rightarrow \infty} \alpha_{n, \ell} = 0$ and u_n is some sequence of real numbers.

$F_{i_1 i_2, \dots, i_p}^{(u_n)}$ is Leadbetter's notation for $P[X_{i_1} \leq u_n, X_{i_2} \leq u_n, \dots, X_{i_p} \leq u_n]$.

We see immediately that strong mixing implies $D(u_n)$ by taking $\alpha_{n, \ell} = \phi(\ell)$ and letting $A = \{X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n\}$ and $B = \{X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n\}$. Note that $A \in \sigma(X_0, \dots, X_{i_p})$ and $B \in \sigma(X_{i_p + \ell + 1}, \dots)$.

Definition 1.3.2. The condition $D'(u_n)$ is said to hold for a strictly stationary sequence $\{X_n\}$ and a sequence of real numbers $\{u_n\}$ if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] = o\left(\frac{1}{k}\right)$ as $k \rightarrow \infty$ (i.e. $\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] = 0$). Notice that definition 1.3.2 is easier to state than 1.2.2. It may also be easier to verify in some examples.

1.4. Results on Associative Processes. For stationary Gaussian processes if $r_n \geq 0 \forall n$ then (1.4.1) $P(M_n \leq t) \geq P(M_n^* \leq t) \forall t$ and $\forall n$. (1.4.1) holds for a large class of stationary processes (the term stationary shall always mean strictly stationary). Most of the processes studied in this thesis satisfy a property called associativity which implies the inequality 1.4.1.

Definition 1.4.1. Random variables $X_1, X_2, \dots, X_n$ are called associated if $\text{Cov}(\Gamma(X_1, X_2, \dots, X_n), \Delta(X_1, X_2, \dots, X_n)) \geq 0$ for all pairs of increasing binary functions¹ Γ and Δ .

We shall say that the sequence $\{X_n\}$ is associated if the random variables $X_0, X_1, X_2, \dots, X_m$ are associated $\forall m$.

There are certain results about associated random variables which are useful, and we list these properties:

- P1 any subset of associated random variables are associated.
- P2 a set consisting of a single random variable is associated.
- P3 increasing functions of associated random variables are associated.

These properties and others are given by Barlow and Proschan [1]. They also give as a theorem the result that independent random variables are associated. Combining P3 with this theorem, we get that increasing functions of independent random variables are associated. This will be a useful result.

¹Increasing means the function does not decrease when any variable is increased with the others held fixed.

For a first order autoregressive process $X_j = \rho X_{j-1} + \varepsilon_j$ with the ε_n 's i.i.d. and $1 > \rho \geq 0$, $X_1 = \rho X_0 + \varepsilon_1$, we can write $X_j = \varepsilon_j + \rho \varepsilon_{j-1} + \dots + \rho^{j-1} \varepsilon_1 + \rho^j X_0 \quad \forall j \geq 1$. This expresses X_j as an increasing function of independent random variables and hence by Theorem 2.2, [1, p. 31] $X_0, X_1, \dots, X_n$ are associated $\forall n$ (i.e. $\{X_n\}$ is an associative process).

The pair (M_{n-1}, X_n) is associated by P3 as M_{n-1} is an increasing function of $X_0, X_1, \dots, X_{n-1}$. Hence $\text{Cov}(f(M_{n-1}), g(X_n)) \geq 0$ for all nondecreasing binary functions f and g . Esary, Proschan, and Walkup [4] give comparisons of concepts of bivariate dependence, and they give applications and examples of associated random variables. In particular, if we take $f(M_{n-1}) = 1_{\{M_{n-1} > s\}}$ and $g(X_n) = 1_{\{X_n > t\}}$ we get

$$(1.4.2) \quad P\{M_{n-1} > s, X_n > t\} \geq P\{M_{n-1} > s\} P\{X_n > t\}$$

$\forall s, t$. It is a simple exercise to show that (1.4.2) implies the following:

$$(1.4.3) \quad P\{M_{n-1} \leq s, X_n \leq t\} \geq P\{M_{n-1} \leq s\} P\{X_n \leq t\}.$$

This property is called positive quadrant dependence. We summarize some of the results presented in this section in the following lemma.

Lemma 1.4.1. Let $\{X_n\}$ be a stationary associative process, then we have

- i) $P(X_n \leq t, M_{n-1} \leq s) \geq P(X_n \leq t) P(M_{n-1} \leq s) \quad \forall t, s \text{ and } n$
- ii) $P(X_n > t, M_{n-1} > s) \geq P(X_n > t) P(M_{n-1} > s) \quad \forall t, s \text{ and } n$

- iii) $P(M_n \leq t) \geq P(M_n^* \leq t) \quad \forall t \text{ and } n$
- iv) $P(m_n > t) \geq P(m_n^* > t) \quad \forall t \text{ and } n$ where $m_n = \min(X_0, X_1, \dots, X_n)$ and $m_n^* = \min(X_0^*, X_1^*, \dots, X_n^*)$
- v) $P(X_0 \leq u_0, X_1 \leq u_1, \dots, X_n \leq u_n) \geq \prod_{i=1}^n P(X_i^* \leq u_i) \quad \forall n$ and any choice of u_i 's
- vi) $P(X_0 > u_0, X_1 > u_1, \dots, X_n > u_n) \geq \prod_{i=0}^n P(X_i^* > u_i) \quad \forall n$ and any choice of u_i 's.

Proof. We have already shown i) and indicated that ii) is easy. v) follows from i) by induction and vi) from ii). iii) and iv) are special cases of v) and vi), respectively.

We conclude this section with the following theorem.

Theorem 1.4.2. Let $X_0 = \varepsilon_0$. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$ be a sequence of independent random variables (not necessarily identically distributed so X_n needn't be stationary), $X_n = \rho X_{n-1} + \varepsilon_n$ for $n \geq 1$. We need $0 \leq \rho$. Let $M_n = \max(X_0, X_1, \dots, X_n)$ and $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ where X_i^* has the same distribution as X_i and the sequence $\{X_i^*\}$ is independent, then $P(M_n \leq t) \geq P(M_n^* \leq t)$ and $P(m_n > t) \geq P(m_n^* > t)$.

This theorem is a corollary to Lemma 1.4.1. The theorem holds in general for any associative process. In particular, the class of ARMA models with almost sure representation $X_n = \sum_{r=0}^{\infty} \alpha_r \varepsilon_{n-r}$ with $\alpha_r \geq 0$ $\forall r$ $\sum_{r=0}^{\infty} \alpha_r^2 < \infty$, ε_i 's independent with $\text{Var}(\varepsilon_i) < \infty$ and all finite dimensional distributions continuous are associative processes. This follows since $\{X_n^m\}$ is associative $\forall m$ where $X_n^m = \sum_{r=0}^m \alpha_r \varepsilon_{n-r}$ and hence for every binary increasing Γ, Δ $\text{Cov}(\Gamma(X_1^m, \dots, X_n^m), \Delta(X_1^m, \dots, X_n^m)) \geq 0 \quad \forall m$. Applying part (iii) of Theorem 5.2 of Billingsley [3],

we get $\text{Cov}(\Gamma(X_1^m, \dots, X_n^m), \Delta(X_1^m, \dots, X_n^m)) \rightarrow \text{Cov}(\Gamma(X_1, \dots, X_n), \Delta(X_1, \dots, X_n))$ as $m \rightarrow \infty$. Note that we use here the fact that $X_i^m \xrightarrow{\text{a.s.}} X_i \forall i$ implies the distribution of $(X_1^m, \dots, X_n^m)$ converges to the distribution of $(X_1, \dots, X_n) \forall n$ as $m \rightarrow \infty$. We can conclude from this that $\forall \Gamma, \Delta$ and n $\text{Cov}(\Gamma(X_1, \dots, X_n), \Delta(X_1, \dots, X_n)) \geq 0$ and hence $\{X_n\}$ is an associative process. The continuity of the finite dimensional distributions was needed to apply the theorem in Billingsley since Γ and Δ could have discontinuities at any points.

2. The Limit Theorem for the EAR(1) Model

2.1. Model Definition. We define the EAR(1) process (exponential) autoregressive process of order 1) by the following recursion: $X_n = \rho X_{n-1} + \varepsilon_n$ where $0 \leq \rho < 1$ and the ε_n 's are i.i.d. with ε_n independent of X_{n-1} . We require that X_n has an exponential distribution with parameter 1 (denoted $X_n \sim \exp(1)$) for each $n \geq 0$. The process is initiated with $X_0 \sim \exp(1)$ and independent of ε_n 's. $X_1, X_2, \dots$ are then defined by the recursion.

Requiring $X_n \sim \exp(1)$ leads to the question of whether or not there exists a sequence of random variables ε_n such that the requirement is satisfied for each n . Gaver and Lewis showed that if we let $\varepsilon_n = 0$ with probability ρ and $\varepsilon_n \sim \exp(1)$ with probability $1-\rho$, the X_n 's will be distributed $\exp(1)$. With ρ and the ε_n 's so specified, the process $\{X_n\}_{n=0}^{\infty}$ is a well defined stationary sequence.

We shall see in 2.2 that this is the only way the ε_n 's can be defined for X_n to have the required distribution.

2.2. Existence and Uniqueness for the Distribution of the ε_n 's.

We shall use a characteristic function argument. Suppose $X_{n-1} \sim \exp(1)$. Let $\phi_X(t)$ denote its characteristic function. It is easy to show that $\phi_X(t) = \frac{1}{1-it}$, and letting $\phi_{\rho X}(t)$ denote the characteristic function of ρX_{n-1} we have $\phi_{\rho X}(t) = \phi_X(\rho t) = \frac{1}{1-i\rho t}$. Since ε_n is independent of X_{n-1} , X_n is the convolution of ρX_{n-1} with ε_n . This means the distribution function of X_n is the convolution of the distribution function of ρX_{n-1} with the distribution function of ε_n . Hence the characteristic function for X_n satisfies

$$(2.2.1) \quad \phi_{X_n}(t) = \phi_\epsilon(t) \phi_{\rho X}(t)$$

where $\phi_{X_n}(t)$ is the characteristic function of X_n and $\phi_\epsilon(t)$ is the characteristic function of ϵ_n .

Since we require $\phi_{X_n}(t) = \phi_X(t) \forall n$ by solving 2.2.1 for $\phi_\epsilon(t)$ we get

$$(2.2.2) \quad \phi_\epsilon(t) = \frac{1-i\rho t}{1-it} = \frac{1-\rho + \rho-i\rho t}{1-it} = \rho + (1-\rho) \frac{1}{1-it}.$$

So ϵ_n must have the characteristic function given by 2.2.2. It is easy to check that $\rho + (1-\rho) \frac{1}{1-it}$ is the characteristic function of a random variable which is zero with probability ρ and is distributed $\exp(1)$ with probability $1-\rho$ and hence the existence is shown. Equation 2.2.1 shows that $\rho + (1-\rho) \frac{1}{1-it}$ is the only characteristic function that gives $\phi_{X_n}(t) = \phi_X(t)$ and since the characteristic function uniquely determines the distribution, the ϵ_n 's must have the distribution corresponding to $\phi_\epsilon(t) = \rho + (1-\rho) \frac{1}{1-it}$.

2.3. The Exact Distribution for M_1 and M_2 . We recall that $M_n = \max(X_0, X_1, \dots, X_n)$. We shall determine the exact distribution for $n=1, 2$. By doing that we will see the difficulties involved in obtaining the exact distribution for M_n .

$$\begin{aligned} P(M_{1-} \leq t) &= P(X_{0-} \leq t) \text{ and } X_{1-} \leq t = P(X_{0-} \leq t, \rho X_0 + \epsilon_1 \leq t) \\ &= P(X_{0-} \leq t, 0 < \epsilon_1 \leq t - \rho X_0) + P(X_{0-} \leq t, \epsilon_1 = 0) \\ &= \int_0^t e^{-x} \int_0^{t-\rho x} (1-\rho) e^{-u} du dx + \rho(1-e^{-t}) \\ &= (1-\rho) (1-e^{-t}) - (1-\rho) e^{-t} \left(\frac{1-e^{-(1-\rho)t}}{1-\rho} \right) + \rho(1-e^{-t}). \end{aligned}$$

This simplifies to give

$$(2.3.1) \quad P(M_{1-} \leq t) = 1 - 2e^{-t} + e^{-(2-\rho)t}.$$

Let $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ where X_i^* are i.i.d. and $X_i^* \sim \exp(1)$.

$$P(M_{1-}^* \leq t) = P(X_0^* \leq t) P(X_1^* \leq t) = (1 - e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}.$$

Since $e^{-(2-\rho)t} > e^{-2t} \quad \forall t > 0, \rho > 0$, $P(M_{1-} \leq t) > P(M_{1-}^* \leq t) \quad \forall t, \rho > 0$, and

$$(2.3.2) \quad P(M_{1-} \leq t) \geq P(M_{1-}^* \leq t) \quad \forall t > 0 \text{ and } \rho > 0.$$

Equation 2.3.2 is a consequence of Theorem 1.4.2 with $n=1$, and this calculation shows it explicitly for this special case.

The inequality $P(M_{n-} \leq t) \geq P(M_{n-}^* \leq t)$ of Theorem 1.4.2 is a consequence of the results of Slepian [13] for stationary Gaussian processes. Here since $\rho \geq 0$ the process is associative, and we get the result because Theorem 1.4.2 applies. $\forall n \quad P(M_{n-}^* \leq t) = (1 - e^{-t})^{n+1} \leq P(M_{n-} \leq t)$.

For M_2 ,

$$\begin{aligned} P(M_{2-} \leq t) &= P(X_{0-} \leq t, X_{1-} \leq t, X_{2-} \leq t) = P(X_{0-} \leq t, \rho X_{0-} + \epsilon_1 \leq t, \rho^2 X_{0-} + \rho \epsilon_1 + \epsilon_2 \leq t) \\ &= P(X_{0-} \leq t, 0 < \epsilon_1 \leq t - \rho X_{0-}, \epsilon_2 = 0) \\ &\quad + P(X_{0-} \leq t, \epsilon_1 = 0, 0 < \epsilon_2 \leq t - \rho^2 X_{0-}) + P(X_{0-} \leq t, \epsilon_1 = 0, \epsilon_2 = 0) \\ &\quad + P(X_{0-} \leq t, 0 < \epsilon_1 \leq t - \rho X_{0-}, 0 < \epsilon_2 \leq t - \rho \epsilon_1 - \rho^2 X_{0-}) \\ &= \int_0^t e^{-x} \int_0^{t-\rho x} (1-\rho) e^{-u} \int_0^{t-\rho u-\rho^2 x} (1-\rho) e^{-v} dv du dx \\ &\quad + \rho \int_0^t e^{-x} \int_0^{t-\rho^2 x} (1-\rho) e^{-v} dv dx \\ &\quad + \rho \int_0^t e^{-x} \int_0^{t-\rho x} (1-\rho) e^{-u} du dx + \rho^2 (1 - e^{-t}). \end{aligned}$$

After tedious computations, one gets

$$\begin{aligned}
P(M_{2-} \leq t) &= (1-\rho)^2 (1-e^{-t} (2e^{-(1-\rho)t})) \\
&\quad - (1-\rho)e^{-t} \left[\frac{1-e^{-(1-\rho^2)t}}{(1-\rho^2)} - \frac{e^{-(1-\rho)t} (1-e^{-(1-\rho)t})}{(1-\rho)} \right] \\
&\quad + \rho(1-\rho) \left[1-e^{-t} \left(\frac{2-\rho^2 - e^{-(1-\rho^2)t}}{1-\rho^2} \right) \right] \\
&\quad + \rho(1-\rho) \left[1-e^{-t} \left(\frac{2e^{-(1-\rho)t}}{1-\rho} \right) \right] + \rho^2(1-e^{-t}) .
\end{aligned}$$

After some algebra, this reduces to

$$(2.3.3) \quad P[M_{2-} \leq t] = 1-3e^{-t} + 2e^{-(2-\rho)t} + e^{-(2-\rho^2)t} - e^{-(3-2\rho)t} .$$

We notice that since $P[M_{2-}^* \leq t] = 1-3e^{-t} + 3e^{-2t} - e^{-3t}$,

$$P[M_{2-} \leq t] - P[M_{2-}^* \leq t] = 2e^{-(2-\rho)t} + e^{-(2-\rho^2)t} - 3e^{-2t} + e^{-3t} - e^{-(3-2\rho)t} .$$

A little calculus shows that $P[M_{2-} \leq t] - P[M_{2-}^* \leq t] > 0 \quad \forall t > 0$ if $\rho > 0$.

Having done the computation for M_2 , we see that the computation for general M_n leads to complicated forms that are difficult to simplify.

In principle, it is possible to do the integrations and algebra for any n but even in the case $n=3$ it appears to be difficult to simplify. There may be ways of simplifying the expressions by approximating some of the integrals or by neglecting some of the integrals that would be negligible in the total. Such methods might be useful in determining rates of convergence for the distribution of the maximum to its limit distribution. For rates of convergence, the parameter ρ should play a role, whereas it does not appear in the limit distribution, as we shall see.

In general, we have the following:

$$\begin{aligned}
P(M_{n-} \leq t) &= (1-\rho)^n \int_0^t e^{-x} \int_0^{t-\rho x} e^{-u_1} \int_0^{t-\rho u_1 - \rho^2 x} e^{-u_2} \\
&\quad \dots \int_0^{t-\rho u_{n-1} - \rho^2 u_{n-2} - \dots - \rho^n x} e^{-u_n} du_n du_{n-1} \dots du_1 dx \\
&+ \rho(1-\rho)^{n-1} \int_0^t e^{-x} \int_0^{t-\rho^2 x} e^{-u_2} \int_0^{t-\rho u_2 - \rho^3 x} e^{-u_3} \\
&\quad \dots \int_0^{t-\rho u_{n-1} - \rho^2 u_{n-2} - \dots - \rho^{n-2} u_2 - \rho^n x} e^{-u_n} du_n du_{n-1} \dots du_2 dx \\
&+ \rho(1-\rho)^{n-1} \int_0^t e^{-x} \int_0^{t-\rho x} e^{-u_1} \int_0^{t-\rho u_1 - \rho^3 x} e^{-u_3} \\
&\quad \dots \int_0^{t-\rho u_{n-1} - \rho^2 u_{n-2} - \dots - \rho^{n-3} u_3 - \rho^{n-1} u_1 - \rho^n x} e^{-u_n} du_n \dots du_3 du_1 dx \\
&+ \dots + \rho^n (1-e^{-t}) .
\end{aligned}$$

Note that there are 2^n terms in the sum. Some simplification is possible. For example, one can show that $P(M_{n-} \leq t, \varepsilon_1=0, \varepsilon_i \neq 0 \text{ for } i \neq 1) > P(M_{n-} \leq t, \varepsilon_j=0, \text{ for } i \neq j)$ where $j \neq 1$. This might help to simplify the expressions and get good upper bounds for $P(M_{n-} \leq t)$.

We observe from previous computations that $P(M_{1-} \leq t) - P(M_{1-}^* \leq t) = e^{-2t}(e^{\rho t} - 1)$. This is an increasing function of ρ for each fixed t . Similarly $P(M_{2-} \leq t) - P(M_{2-}^* \leq t)$ is an increasing function of ρ for each fixed t . It seems reasonable to expect $P(M_{n-} \leq t) - P(M_{n-}^* \leq t)$ to be an increasing function of ρ when n and t are fixed. However this may be difficult to prove.

2.4. Verifying Strong Mixing. In order to prove the limit theorem for the EAR(1) model, we will verify the strong mixing condition. This

condition implies that $D(u_n)$ holds and so when $D'(u_n)$ is verified we will have by Theorem 1.1.6 that $P[M_{n-1} \leq u_n(x)] = P[M_{n-1} \leq x + \ln n] \rightarrow e^{-e^{-x}}$.

In Jacobs and Lewis [7] a stronger mixing condition is proved for a class of exponential models which include the EAR(1) model. This result can be applied to conclude that the EAR(1) model satisfies the strong mixing condition.

This author was unaware of the result in [7] and proved that the EAR(1) process satisfied the strong mixing condition independently. The proof is different from the proof in [7], and it has the advantage of having a generalization to a sufficient condition for strong mixing for Markov processes and AR(p) (p^{th} order autoregressive) processes. Such a condition is given in Chapter 3 with examples of its application to AR(1) processes.

Strong mixing is a property which is useful in proving other limit theorems such as central limit theorems for stationary processes. It is therefore worthwhile in specific examples to know whether or not the strong mixing condition is satisfied. In [7] strong mixing was used to obtain several limiting results for the EARMA(1,1) models. This thesis concentrates on limit theorems for maxima, but it is worthwhile to remember that the results in Chapter 3 have other applications.

We shall now proceed with the proof of the results needed to verify the mixing condition. We want to show that if $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, \dots)$ then $|P(A \cap B) - P(A)P(B)| < \phi(j)$ where $\phi(j) \rightarrow 0$ as $j \rightarrow \infty$. We shall use the Markov property, and we need to determine the joint distribution of X_n and X_{n+j+1} .

The following lemma will help us toward the goal:

Lemma 2.4.1. Let $W_j = \sum_{r=1}^j \rho^{j-r} \epsilon_r$ where the ϵ_r 's are as defined for the EAR(1) process.

i) W_j has the following distribution:

$$\begin{aligned} W_j &= 0 \text{ with probability } \rho^j \\ W_j &\sim \exp(1) \text{ with probability } 1-\rho^j. \end{aligned}$$

ii) $W_j \xrightarrow{d} W$ as $j \rightarrow \infty$ where $W \sim \exp(1)$.

Proof. $X_j = \rho^j X_0 + W_j$ and W_j is independent of X_0 . $\phi_{X_j}(t) = \frac{1}{1-it}$

$\forall j$ and

$$\phi_{W_j}(t) = \frac{\phi_{X_j}(t)}{\phi_{\rho^j X_0}(t)} = \frac{\frac{1}{1-it}}{\frac{1}{1-i\rho^j t}} = \frac{1-i\rho^j t}{1-it} = \rho^j + (1-\rho^j) \frac{1}{1-it}.$$

This we recognize as the characteristic function of a random variable which has probability ρ^j of being zero and is distributed $\exp(1)$ with probability $1-\rho^j$. Thus we have shown i).

Now $\lim_{j \rightarrow \infty} \phi_{W_j}(t) = \lim_{j \rightarrow \infty} (1-\rho^j) \frac{1}{1-it} + \rho^j = \frac{1}{1-it}$. Since $\frac{1}{1-it}$ is continuous at $t=0$, we have $P[W_j \leq x] \rightarrow 1-e^{-x}$ as $j \rightarrow \infty \forall x$ (i.e. ii) holds). This completes the proof of the lemma.

At this point it might not be apparent that a function ϕ with the required properties for strong mixing exists. The following lemma shows that $\phi(j) = \rho^j$ works for special sets, and we might expect that a function proportional to ρ^j will work for general sets A and B.

Lemma 2.4.2.

$$\begin{aligned} &|P[X_0 \leq x, X_j \leq y] - P[X_0 \leq x] P[X_j \leq y]| \\ &= |P[X_n \leq x, X_{n+j+1} \leq y] - P[X_n \leq x] P[X_{n+j+1} \leq y]| \leq \rho^j. \end{aligned}$$

Proof. $P(X_0 \leq x, X_j \leq y) \geq P(X_0 \leq x) P(X_j \leq y)$ by Lemma 1.4.1. Also
 $P(X_0 \leq x, X_j \leq y) = P(X_0 \leq x, W_j + \rho^j X_0 \leq y) \leq P(X_0 \leq x, W_j \leq y) = P(X_0 \leq x) P(W_j \leq y).$

Hence

$$(2.4.1) \quad P(X_0 \leq x) (P(W_j \leq y) - P(X_j \leq y)) \geq P(X_0 \leq x, X_j \leq y) - P(X_0 \leq x) P(X_j \leq y) \geq 0.$$

Now

$$\begin{aligned} P(X_0 \leq x) [P(W_j \leq y) - P(X_j \leq y)] &= (1 - e^{-x}) (1 - e^{-y} + \rho^j e^{-y} - (1 - e^{-y})) \\ &= \rho^j e^{-y} (1 - e^{-x}) \leq \rho^j \quad \forall x, y > 0. \end{aligned}$$

$$\therefore |P(X_0 \leq x, X_j \leq y) - P(X_0 \leq x) P(X_j \leq y)| \leq \rho^j.$$

Now let us compute $P(X_0 \leq x, X_j \leq y)$.

$$\begin{aligned} P(X_0 \leq x, X_j \leq y) &= \rho^j \int_0^x e^{-u} du + (1 - \rho^j) \int_0^x \int_0^{y - \rho^j u} e^{-w} e^{-u} dw du \\ &= 1 - e^{-x} - e^{-y} (1 - e^{-(1 - \rho^j)x}) \quad \text{if } y > \rho^j x. \end{aligned}$$

Let $h(x, y)$ denote the joint density of X_n, X_{n+j+1} at $X_n = x, X_{n+j+1} = y$ for $y > \rho^j x$. We note that $h(x, y) = 0$ for $y < \rho^j x$ and since $W_j = 0$ with probability ρ^j , there is probability mass ρ^j concentrated on the line $y = \rho^j x$.

For $y > \rho^j x$

$$h(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} P[X_n \leq x, X_{n+j+1} \leq y] = (1 - \rho^j) e^{-y} e^{-(1 - \rho^j)x}.$$

Let $f(x) = e^{-x}$, the density of X_n . Let $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, X_{n+j+2}, \dots)$

$$|P(A \cap B) - P(A)P(B)| = |E\{P(A \cap B | X_n, X_{n+j+1})\} - E[P(A | X_n)] E[P(B | X_{n+j+1})]|$$

$$= \left| \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) h(u, v) dv du \right.$$

$$\left. + P(A \cap B \cap \{X_{n+j+1} = \rho^j X_n\}) \right.$$

$$\left. - \int_0^\infty P(A | X_n = u) f(u) du \int_0^\infty P(B | X_{n+j+1} = v) f(v) dv \right|$$

$$\leq \rho^j P(A \cap B | X_{n+j+1} = \rho^j X_n)$$

$$+ \left| \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) h(u, v) dv du \right.$$

$$\left. - \int_0^\infty \int_0^\infty P(A | X_n = u) P(B | X_{n+j+1} = v) f(u) f(v) dv du \right|$$

$$\leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty |P(A \cap B | X_n = u, X_{n+j+1} = v) h(u, v)$$

$$- P(A | X_n = u) P(B | X_{n+j+1} = v) f(u) f(v)| dv du + \frac{\rho^j}{1 + \rho^j}.$$

$$P(A \cap B | X_n = u, X_{n+j+1} = v) = P(A | X_n = u, X_{n+j+1} = v) P(B | A, X_n = u, X_{n+j+1} = v)$$

$$= P(A | X_n = u, X_{n+j+1} = v) P(B | X_n = u, X_{n+j+1} = v)$$

since A and B are conditionally independent given X_n and X_{n+j+1} .

$$\forall v > \rho^j u \quad P(A | X_n = u, X_{n+j+1} = v) = P(A | X_n = u) \quad \text{and} \quad \forall u < v/\rho^j \quad P(B | X_n = u,$$

$$X_{n+j+1} = v) = P(B | X_{n+j+1} = v).$$

$$\therefore P(A \cap B | X_n = u, X_{n+j+1} = v) = P(A | X_n = u) P(B | X_{n+j+1} = v).$$

So the inequality reduces to

$$|P(A \cap B) - P(A)P(B)| \leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) |h(u, v) - f(u)f(v)| dv du \\ + \frac{\rho^j}{1+\rho^j} \leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty |h(u, v) - f(u)f(v)| dv du + \frac{\rho^j}{1+\rho^j}.$$

$$\text{Now } |h(u, v) - f(u)f(v)| = |(1-\rho^j) e^{-v} e^{-(1-\rho^j)u} - e^{-u} e^{-v}| \\ = e^{-v} |(1-\rho^j) e^{-(1-\rho^j)u} - e^{-u}| \text{ and}$$

$$|(1-\rho^j) e^{-(1-\rho^j)u} - e^{-u}| = e^{-u} - (1-\rho^j) e^{-(1-\rho^j)u} \text{ for } u < \frac{-\ln(1-\rho^j)}{\rho^j} \\ = (1-\rho^j) e^{-(1-\rho^j)u} - e^{-u} \text{ for } u \geq \frac{-\ln(1-\rho^j)}{\rho^j}.$$

Hence

$$\int_0^\infty \int_{\rho^j u}^\infty |h(u, v) - f(u)f(v)| dv du = \int_0^\infty \int_{\rho^j u}^\infty e^{-v} |(1-\rho^j) e^{-(1-\rho^j)u} - e^{-u}| dv du \\ = \int_0^\infty e^{-\rho^j u} |(1-\rho^j) e^{-(1-\rho^j)u} - e^{-u}| du \\ = \int_0^{\frac{-\ln(1-\rho^j)}{\rho^j}} (e^{-(1+\rho^j)u} - e^{-u}(1-\rho^j)) du \\ + \int_{\frac{-\ln(1-\rho^j)}{\rho^j}}^\infty ((1-\rho^j) e^{-u} - e^{-(1+\rho^j)u}) du \\ = \rho^j \left(\frac{\rho^j}{1+\rho^j} \right) + 2(1-\rho^j)^{(1+\rho^j)/\rho^j} \left[\frac{\rho^j}{1+\rho^j} \right].$$

So for $A \in \sigma(X_0, X_1, \dots, X_n)$, $B \in \sigma(X_{n+j+1}, X_{n+j+2}, \dots)$

$$|P(A \cap B) - P(A)P(B)| < \rho^j + \frac{\rho^j}{1+\rho^j} [\rho^j + 2(1-\rho^j)^{(1+\rho^j)/\rho^j}] + \frac{\rho^j}{1+\rho^j} \\ = \rho^j \left[\frac{2+2\rho^j + 2(1-\rho^j)^{(1+\rho^j)/\rho^j}}{1+\rho^j} \right].$$

$$\frac{2+2\rho^j + 2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} \rightarrow 2 + 2/e \text{ as } j \rightarrow \infty$$

since $(1-\rho^j)^{1/\rho^j} \rightarrow 1/e$ as $j \rightarrow \infty$. Since

$$\frac{2+2\rho^j + 2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} < \frac{6}{1+\rho^j} < 6 \quad \forall j,$$

we can conclude that $|P(A \cap B) - P(A)P(B)| < 6\rho^j \quad \forall j$. Setting $\phi(j) = 6\rho^j$ we see that $\phi(j) \rightarrow 0$ as $j \rightarrow \infty$ and hence the strong mixing condition is verified. Since we were only interested in finding a ϕ with the required properties, we did not take care to get the best upper bound for $|P(A \cap B) - P(A)P(B)|$ for $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, X_{n+j+2}, \dots)$. Since $2 + 2/e < 3$, $3\rho^j$ would be a sufficient bound for large j .

It is interesting to note that in [7] Jacobs and Lewis get the bound $|P(A \cap B) - P(A)P(B)| \leq 5\rho^{j/2}$.

2.5. Verifying Loynes' Sufficient Condition.

$$(2.5.1) \quad \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi)) - (1-\rho^i)C_{t_m}(\xi)}{P(X_1 > C_{t_m}(\xi))} = e$$

where t_m , C_{t_m} , and ξ are defined in Definition 1.2.2.

We shall now verify 2.5.1:

$$\begin{aligned} P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi)) &= P(X_0 > C_{t_m}(\xi), X_i > C_{t_m}(\xi)) \\ &= P(X_0 > C_{t_m}(\xi), W_i + \rho^i X_0 > C_{t_m}(\xi)) \\ &= \int_{C_{t_m}(\xi)}^{\infty} P[W_i > C_{t_m}(\xi) - \rho^i u] e^{-u} du \end{aligned}$$

$$\begin{aligned}
P[W_i > C_{t_m}(\xi) - \rho^i u] &= 1 \quad \text{if } \rho^i u \geq C_{t_m}(\xi) \\
&= (1-\rho^i) e^{-(C_{t_m}(\xi) - \rho^i u)} \quad \text{if } \rho^i u < C_{t_m}(\xi).
\end{aligned}$$

Hence

$$\begin{aligned}
P[X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi)] \\
&= \int_{C_{t_m}(\xi)}^{(C_{t_m}(\xi))/\rho^i} (1-\rho^i) e^{-C_{t_m}(\xi)} e^{\rho^i u} e^{-u} du + \int_{(C_{t_m}(\xi))/\rho^i}^{\infty} e^{-u} du \\
&= e^{-C_{t_m}(\xi)} \left[e^{-(1-\rho^i)C_{t_m}(\xi)} - e^{-(C_{t_m}(\xi)/\rho^i)(1-\rho^i)} \right] + e^{-(C_{t_m}(\xi))/\rho^i} \\
&= e^{-(2-\rho^i)C_{t_m}(\xi)}
\end{aligned}$$

and thus we get 2.5.1 since $P[X_1 > C_{t_m}(\xi)] = e^{-C_{t_m}(\xi)}$. Now

$$\sum_{i=1}^{p_m-1} \left(\frac{p_m-1}{p_m} \right) \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} < e^{-(1-\rho)C_{t_m}(\xi)} \left[\sum_{i=1}^{p_m-1} \left(\frac{p_m-i}{p_m} \right) \right]$$

since $e^{-(1-\rho)C_{t_m}(\xi)} \geq e^{-(1-\rho^i)C_{t_m}(\xi)}$ $\forall i \geq 1$ and $C_{t_m}(\xi) = \ln t_m$
 $-\ln \xi \geq 0$ as $t_m > \xi$.

$$e^{-(1-\rho)C_{t_m}(\xi)} \left[\sum_{i=1}^{p_m-1} \left(\frac{p_m-i}{p_m} \right) \right] = \left(\frac{p_m-1}{2} \right) \xi^{(1-\rho)} e^{-(1-\rho)(\ln m + \ln(p_m+q_m))}.$$

Note that $t_m = m(p_m+q_m)$.

For each fixed ξ we get

$$(2.5.2) \quad \sum_{i=1}^{p_m-1} \left(\frac{p_m-1}{p_m} \right) \frac{P(X_1 > c_{t_m}(\xi), X_{i+1} > c_{t_m}(\xi))}{P(X_1 > c_{t_m}(\xi))} \\ < \left(\frac{p_m-1}{2} \right) \frac{\xi^{(1-\rho)}}{m^{(1-\rho)}} e^{-(1-\rho) \ln(p_m+q_m)} < \frac{p_m \xi^{(1-\rho)}}{m^{(1-\rho)}}.$$

The last inequality above follows since $(p_m-1)/2 < p_m$ and

$$-(1-\rho) \ln(p_m+q_m) < 0 \text{ implying } e^{-(1-\rho) \ln(p_m+q_m)} < 1.$$

Equation 2.5.2 holds for any choice of p_m and q_m . Let $p_m = [m^{(1-\rho)/2}]$ where $[x]$ is the greatest integer less than or equal to x .

$$\frac{p_m \xi^{(1-\rho)}}{m^{(1-\rho)}} < m^{(1-\rho)/2} \frac{\xi^{(1-\rho)}}{m^{(1-\rho)}} = \frac{\xi^{(1-\rho)}}{m^{(1-\rho)/2}} \text{ and } \lim_{m \rightarrow \infty} \frac{\xi^{(1-\rho)}}{m^{(1-\rho)/2}} = 0 \quad \forall \xi > 0$$

and $0 < \rho < 1$. So the condition

$$(2.5.3) \quad \sum_{i=1}^{p_m-1} \left(\frac{p_m-1}{p_m} \right) \frac{P(X_1 > c_{t_m}(\xi), X_{i+1} > c_{t_m}(\xi))}{P(X_1 > c_{t_m}(\xi))} \rightarrow 0 \text{ as } m \rightarrow \infty$$

holds for this choice of $\{p_m\}$.

It remains for us to check that

$$(2.5.4) \quad \frac{q_m}{p_m} \rightarrow 0, \quad \frac{p_{m+1}}{p_m} \rightarrow 1, \text{ and } m\phi(q_m) \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Equation 2.5.3 holds regardless of the choice of q_m . We choose $q_m = [m^{(1-\rho)/4}]$ in order to get the conditions in 2.5.4 to hold.

$$0 < \frac{q_m}{p_m} < \frac{m^{(1-\rho)/4}}{m^{(1-\rho)/2} - 1} \quad \text{and} \quad \frac{m^{(1-\rho)/4}}{m^{(1-\rho)/2} - 1} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Hence $\frac{q_m}{p_m} \rightarrow 0$ as $m \rightarrow \infty$.

$$\frac{(m+1)^{(1-\rho)/2} - 1}{m^{(1-\rho)/2}} < \frac{p_{m+1}}{p_m} < \frac{(m+1)^{(1-\rho)/2}}{(m^{(1-\rho)/2} - 1)}$$

and both

$$\frac{(m+1)^{(1-\rho)/2}}{(m^{(1-\rho)/2} - 1)} \quad \text{and} \quad \frac{(m+1)^{(1-\rho)/2} - 1}{m^{(1-\rho)/2}} \rightarrow 1$$

as $m \rightarrow \infty$. So $1 \leq \liminf_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} \leq \limsup_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} \leq 1$ implying $\lim_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} = 1$. We

must check that $m\phi(q_m) \rightarrow 0$ as $m \rightarrow \infty$ in order to verify 2.5.3 and complete the verification of Loynes' conditions.

$$m\phi(q_m) = 6m \rho^{[m^{(1-\rho)/4}]} < 6m \rho^{(m^{(1-\rho)/4} - 1)} = 6m \frac{\rho^{m^{(1-\rho)/4}}}{\rho},$$

$$m \rho^{m^{(1-\rho)/4}} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

We see this by letting $u = m^{(1-\rho)/4}$ and hence $m = u^{4/(1-\rho)}$. Let k be an integer such that $k > 4/(1-\rho)$ and hence $u^k > u^{4/(1-\rho)}$. We can do this since $\rho < 1$.

Now $u^{4/(1-\rho)} \rho^u < u^k \rho^u$ and $u^k \rho^u \rightarrow 0$ as $u \rightarrow \infty$. $u \rightarrow \infty$ as $m \rightarrow \infty$ so $6m \rho^{m^{(1-\rho)/4}} \rightarrow 0$ as $m \rightarrow \infty$. So $6m \rho^{m^{(1-\rho)/4}} / \rho \rightarrow 0$ as $m \rightarrow \infty$ and

hence $m\phi(q_m) \rightarrow 0$ as $m \rightarrow \infty$. This completes the requirements for Loynes' sufficient condition and so Theorem 1.1.3 applies to the EAR(1) model.

We state the theorem for this case.

Theorem 2.5.1. For the EAR(1) model $P\{M_n \leq x + \ln n\} \rightarrow e^{-e^{-x}}$ as $n \rightarrow \infty$ for each $-\infty < x < \infty$.

We have seen here a particular case where the limit law for the maximum parallels the result for Gaussian processes. In Chapter 4 we shall see an example where this is not the case.

3. Sufficient Conditions for Strong Mixing

3.1. The Sufficient Condition for the AR(1) Models and Other Markov Processes. We generalize the proof of the sufficient condition in the exponential case to other Markov processes. In 3.2 and 3.3 we will illustrate the usefulness of the condition by exhibiting AR(1) processes which satisfy the condition in addition to the EAR(1) process.

Suppose $\{X_n\}$ is a strictly stationary Markov process. Consider for $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, \dots)$ $|P(A \cap B) - P(A)P(B)|$.

Using the same conditioning argument, we get

$$|P(A \cap B) - P(A)P(B)| = |E\{P(A \cap B | X_n, X_{n+j+1})\} - E\{P(A | X_n)\} E\{P(B | X_{n+j+1})\}|$$

$$P(A \cap B | X_n, X_{n+j+1}) = P(A | X_n, X_{n+j+1}) P(B | X_n, X_{n+j+1}) = P(A | X_n) P(B | X_{n+j+1}).$$

Hence

$$|P(A \cap B) - P(A)P(B)| = |E\{P(A | X_n) P(B | X_{n+j+1})\} - E\{P(A | X_n)\} E\{P(B | X_{n+j+1})\}|$$

$$= \left| \iint P(A | x_n) P(B | x_{n+j+1}) dG_j(x_n, x_{n+j+1}) \right.$$

$$\quad \left. - \int P(A | x_n) dF(x_n) \int P(B | x_{n+j+1}) dF(x_{n+j+1}) \right|$$

$$= \left| \iint P(A | x_n) P(B | x_{n+j+1}) \{dG(x_n, x_{n+j+1}) \right.$$

$$\quad \left. - dF(x_n) dF(x_{n+j+1})\} \right|$$

where $G_j(x_n, x_{n+j+1})$ is the joint distribution function for X_n, X_{n+j+1} and $F(x_n)$ is the cumulative distribution function for X_n , and $P(A | x_n)$ is short for $P(A | X_n = x_n)$.

$$\begin{aligned}
& \left| \iint P(A|x_n) P(B|x_{n+j+1}) \{dG_j(x_n, x_{n+j+1}) - dF(x_n) dF(x_{n+j+1})\} \right| \\
& \leq \iint |P(A|x_n) P(B|x_{n+j+1}) \{dG_j(x_n, x_{n+j+1}) - dF(x_n) dF(x_{n+j+1})\}| \\
& \leq \iint |dG_j(x_n, x_{n+j+1}) - dF(x_n) dF(x_{n+j+1})|.
\end{aligned}$$

So if $\iint |dG_j(x_n, x_{n+j+1}) - dF(x_n) dF(x_{n+j+1})| \rightarrow 0$ as j tends to infinity, we can take $\phi(j) = \iint |dG_j(x_n, x_{n+j+1}) - dF(x_n) dF(x_{n+j+1})|$. This is precisely what was done in the exponential example. If G_j and F have densities g_j and f , respectively, the condition can be written as

$$(3.1.1) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g_j(x_n, x_{n+j+1}) - f(x_n) f(x_{n+j+1})| dx_n dx_{n+j+1} \rightarrow 0.$$

Condition (3.1.1) can be more conveniently expressed as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{g_j(x_n, x_{n+j+1})}{f(x_n) f(x_{n+j+1})} - 1 \right| f(x_n) f(x_{n+j+1}) dx_n dx_{n+j+1} \rightarrow 0.$$

Sometimes $g_j(x_n, x_{n+j+1})/(f(x_n) f(x_{n+j+1}))$ is a bounded function. When this is the case, 3.1.1 can often be shown to hold by application of the bounded convergence theorem. Note that strong mixing implies $g_j(x_n, x_{n+j+1})/(f(x_n) f(x_{n+j+1})) \rightarrow 1$ as $j \rightarrow \infty$ for any possible x_n and x_{n+j+1} , so we expect $g_j(x_n, x_{n+j+1})/(f(x_n) f(x_{n+j+1})) \rightarrow 1$ as $j \rightarrow \infty$ for examples where we expect the strong mixing condition to hold. We will exploit these ideas in the examples given in the next two sections. Let us now state the condition as a lemma.

Lemma 3.1.1. Let $\{X_n\}$ be a strictly stationary Markov sequence. A sufficient condition for $\{X_n\}$ to satisfy the strong mixing property is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |dG_j(x_0, x_1) - dF(x_0) dF(x_1)| \rightarrow 0 \text{ as } j \rightarrow \infty$$

where $G_j(x_0, x_1) = P[X_0 \leq x_0, X_j \leq x_1]$ and $F(x_0) = P[X_0 \leq x_0] = P[X_n \leq x_0] \quad \forall n$.

3.2. The Normal Example. Let $X_n = \rho X_{n-1} + \varepsilon_n$, $n \geq 1$, $-1 < \rho < 1$, ε_n independent of X_{n-1} , $\varepsilon_n \sim N(0, 1-\rho^2)$ for $n \geq 1$ and $X_0 \sim N(0, 1)$. X_0 is independent of the ε_n 's. We have that $\{X_n\}$ is a strictly stationary Markov AR(1) process and $X_n \sim N(0, 1) \quad \forall n$. Let $W_n = \sum_{r=1}^n \rho^{n-r} \varepsilon_r$. We see that $W_n \sim N(0, 1-\rho^{2n})$. (X_0, X_j) has a bivariate normal distribution. We now compute the covariance

$$\text{Cov}(X_0, X_j) = E(X_0 X_j) = E(X_0 (\rho^j X_0 + W_j)) = \rho^j E X_0^2 = \rho^j.$$

Hence X_0, X_j have joint density

$$g_j(x_0, x_1) = \frac{1}{2\pi\sqrt{1-\rho^{2j}}} e^{-\frac{(x_0^2 - 2\rho^j x_0 x_1 + x_1^2)}{2(1-\rho^{2j})}},$$

$$f(x_0) f(x_1) = \frac{1}{2\pi} e^{-\frac{(x_0^2 + x_1^2)}{2}},$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g_j(x_0, x_1) - f(x_0) f(x_1)| dx_0 dx_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(x_0^2 + x_1^2)}{2}} \left| \frac{g_j(x_0, x_1)}{f(x_0) f(x_1)} - 1 \right| dx_0 dx_1, \end{aligned}$$

and

$$\frac{g_j(x_0, x_1)}{f(x_0) f(x_1)} = \frac{1}{\sqrt{1-\rho^{2j}}} e^{-\left(\rho^{2j}(x_0^2+x_1^2)-2\rho^j x_0 x_1\right)/(2(1-\rho^{2j}))}$$

We observe that $\rho^{2j}(x_0^2+x_1^2) - 2\rho^j x_0 x_1 \rightarrow \infty$ as $x_0 \rightarrow \pm\infty$ or $x_1 \rightarrow \pm\infty$.

Hence $\rho^{2j}(x_0^2+x_1^2) - 2\rho^j x_0 x_1$ attains an absolute minimum in $-\infty < x_0 < \infty$, $-\infty < x_1 < \infty$, and

$$e^{-\left(\rho^{2j}(x_0^2+x_1^2)-2\rho^j x_0 x_1\right)/(2(1-\rho^{2j}))}$$

attains its maximum at the point where $\rho^{2j}(x_0^2+x_1^2) - 2\rho^j x_0 x_1$ attains its minimum. So $g_j(x_0, x_1)/[f(x_0)f(x_1)] \leq M \quad \forall -\infty < x_0 < \infty, -\infty < x_1 < \infty$ where M is some constant.

So by bounded convergence

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x_0^2+x_1^2)/2} \left| \frac{g_j(x_0, x_1)}{f(x_0) f(x_1)} - 1 \right| dx_0 dx_1 \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x_0^2+x_1^2)/2} \left[\lim_{j \rightarrow \infty} \left| \frac{g_j(x_0, x_1)}{f(x_0) f(x_1)} - 1 \right| \right] dx_0 dx_1 = 0 \end{aligned}$$

since $\lim_{j \rightarrow \infty} g_j(x_0, x_1) = f(x_0)f(x_1)$.

We have by Lemma 3.1.1 that $\{X_n\}$ satisfies the strong mixing condition. We could obtain a limit theorem for the maximum since $D'(u_n)$ holds, but this is unnecessary since Berman's theorem applies in this case.

3.3. The Cauchy Example. We shall use the notation Cauchy $(0, \beta)$

for the distribution with density function $f(x) = \frac{1}{\beta\pi} \frac{1}{[1 + (\frac{x}{\beta})^2]}$.

Let $X_n = \rho X_{n-1} + \varepsilon_n$, $n \geq 1$, $-1 < \rho < 1$ where $\varepsilon_n \sim \text{Cauchy}(0, 1-|\rho|)$,

$\{\varepsilon_n\}$ independent and $\forall i \ \varepsilon_i$ is independent of X_{i-1} . As a result of this, we get that $X_n \sim \text{Cauchy}(0,1)$. We can write $X_n = \rho^n X_0 + W_n$ where $W_n \sim \text{Cauchy}(0, 1-|\rho|^n)$ and from this fact we get that

$$g_j(x_0, x_1) = \frac{1}{(1-|\rho|^j)\pi} \frac{1}{\left(1 + \left(\frac{x_1 - \rho^j x_0}{1-|\rho|^j}\right)^2\right)} \frac{1}{\pi} \frac{1}{1+x_0^2}.$$

Again we shall consider

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0) f(x_1) \left| \frac{g_j(x_0, x_1)}{f(x_0) f(x_1)} - 1 \right| dx_1 dx_0 \quad \text{where} \quad f(x_0) = \frac{1}{\pi(1+x_0^2)}.$$

where $f(x_0) = 1/\pi(1+x_0^2)$.

We can determine when $g_j(x_0, x_1) \geq f(x_0) f(x_1)$.

$$g_j(x_0, x_1) \geq f(x_0) f(x_1) \Leftrightarrow \frac{1}{(1-|\rho|^j)} \frac{1}{\left[1 + \left(\frac{x_1 - \rho^j x_0}{1-|\rho|^j}\right)^2\right]} \geq \frac{1}{1+x_1^2}$$

$$\Leftrightarrow -x_1^2 \left(\frac{|\rho|^j}{1-|\rho|^j} \right) + \frac{2\rho^j}{(1-|\rho|^j)} x_0 x_1 + |\rho|^j \frac{(1-|\rho|^j) - \rho^{2j} x_0^2}{(1-|\rho|^j)} \geq 0$$

$$\Leftrightarrow |\rho|^j x_1^2 - 2\rho^j x_0 x_1 - |\rho|^j (1-|\rho|^j) - \rho^{2j} x_0^2 \leq 0.$$

Consider the following equation:

$$(3.3.1) \quad |\rho|^j x_1^2 - 2\rho^j x_0 x_1 - |\rho|^j (1-|\rho|^j) - \rho^{2j} x_0^2 = 0.$$

Solving (3.3.1) gives $x_1 = \frac{\rho}{|\rho|} \pm \sqrt{2+|\rho|^j(x_0^2-1)}$. When

$$x_1 \geq \frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)} \quad \text{or} \quad x_1 \leq \frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)}, \quad f(x_0) f(x_1)$$

$$\geq g_j(x_0, x_1) \quad \text{and hence} \quad |g_j(x_0, x_1) - f(x_0) f(x_1)| = f(x_0) f(x_1) - g_j(x_0, x_1)$$

for these pairs (x_0, x_1) . Otherwise $g_j(x_0, x_1) \geq f(x_0) f(x_1)$ and $|g_j(x_0, x_1) - f(x_0) f(x_1)| = g_j(x_0, x_1) - f(x_0) f(x_1)$. From this we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g_j(x_0, x_1) - f(x_0) f(x_1)| dx_0 dx_1 \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi^2 (1+x_0^2)} \left\{ \frac{\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)}}{\frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)}} \left[\frac{1}{(1-|\rho|^j) \left[1 + \frac{(x_1 - \rho^j x_0)^2}{(1-|\rho|^j)^2} \right]} \right. \right. \\
&\quad \left. \left. - \frac{1}{1+x_1^2} \right] dx_1 dx_0 \right. \\
&+ \int_{-\infty}^{\infty} \frac{1}{\pi^2 (1+x_0^2)} \left[\frac{\frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)}}{\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)}} \left[\frac{1}{1+x_1^2} - \frac{1}{(1-|\rho|^j) \left(1 + \frac{(x_1 - \rho^j x_0)^2}{(1-|\rho|^j)^2} \right)} \right] dx_1 dx_0 \right. \\
&+ \left. \int_{-\infty}^{\infty} \frac{1}{\pi^2 (1+x_0^2)} \left[\frac{\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)}}{\frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)}} \left[\frac{1}{1+x_1^2} - \frac{1}{(1-|\rho|^j) \left(1 + \frac{(x_1 - \rho^j x_0)^2}{(1-|\rho|^j)^2} \right)} \right] dx_1 dx_0 \right.
\end{aligned}$$

After some computations one gets

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g_j(x_0, x_1) - f(x_0) f(x_1)| dx_0 dx_1 \\
&= 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \left[\tan^{-1} \left(\frac{\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)} + \rho^j x_0}{1-|\rho|^j} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& - \tan^{-1} \left(\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)} \right) \Big] dx_0 \\
& - 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \left[\tan^{-1} \left(\frac{\frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)} + \rho^j x_0}{1-|\rho|^j} \right) \right. \\
& \left. - \tan^{-1} \left(\frac{\rho}{|\rho|} - \sqrt{2+|\rho|^j(x_0^2-1)} \right) \right] dx_0 .
\end{aligned}$$

We observe that for each u $|\tan^{-1} u| \leq \pi/2$ and hence we can apply the bounded convergence theorem to each of the integrals. So, for example

$$\begin{aligned}
& \lim_{j \rightarrow \infty} 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)} + \rho^j x_0}{1-|\rho|^j} \right) dx_0 \\
& = 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\rho}{|\rho|} + \sqrt{2} \right) dx_0
\end{aligned}$$

and similarly

$$\begin{aligned}
& \lim_{j \rightarrow \infty} 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\rho}{|\rho|} + \sqrt{2+|\rho|^j(x_0^2-1)} \right) dx_0 \\
& = 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\rho}{|\rho|} + \sqrt{2} \right) dx_0
\end{aligned}$$

also. We get cancellation. For the other two terms we also get cancellation since they both converge to

$$(3.3.2) \quad 2 \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\rho}{|\rho|} - \sqrt{2} \right) dx_0 .$$

We can therefore apply Lemma 3.1.1 to conclude that the Cauchy AR(1) process satisfies the strong mixing condition. Notice that we did not have to evaluate explicitly any integral such as the one given in (3.3.2).

3.4. Sufficient Condition for AR(p) Models. A slightly more complicated sufficient condition for strong mixing can be given for p^{th} order autoregressive processes.

Again $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, \dots)$. $\{X_n\}$ is a strictly stationary AR(p) process:

$$\begin{aligned} |P(A \cap B) - P(A)P(B)| &= |E\{P(A \cap B | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, X_{n+j+2}, \\ &\quad \dots, X_{n+j+p})\} - E\{P(A | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p})\} \\ &\quad E\{P(B | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p})\}|. \end{aligned}$$

Since the process is p^{th} order autoregressive, we get the following

$$\begin{aligned} (3.4.1) \quad P(A | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p}) \\ = P(A | X_{n+1-p}, \dots, X_n) \end{aligned}$$

$$\begin{aligned} (3.4.2) \quad P(B | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p}) \\ = P(B | X_{n+j+1}, \dots, X_{n+j+p}) \end{aligned}$$

and

$$\begin{aligned} (3.4.3) \quad P(A \cap B | X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p}) \\ = P(A | X_{n+1-p}, X_{n+2-p}, \dots, X_n) P(B | X_{n+j+1}, \dots, X_{n+j+p}). \end{aligned}$$

Combining these equations we get

$$\begin{aligned}
|P(A \cap B) - P(A)P(B)| &= |E\{P(A|X_{n+1-p}, \dots, X_n) P(B|X_{n+j+1}, \dots, X_{n+j+p})\} \\
&\quad - E[P(A|X_{n+1-p}, \dots, X_n)] E[P(B|X_{n+j+1}, \dots, X_{n+j+p})]| \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |dG_j(x_{n+1-p}, \dots, x_n, x_{n+j+1}, \dots, x_{n+j+p}) \\
&\quad \text{2p fold integral} \\
&\quad - dF(x_{n+1-p}, \dots, x_n) dF(x_{n+j+1}, \dots, x_{n+j+p})|
\end{aligned}$$

where G_j is the joint distribution of $X_{n+1-p}, X_{n+2-p}, \dots, X_n, X_{n+j+1}, \dots, X_{n+j+p}$ and F is the joint distribution of p consecutive X_i 's. When $p=1$, F is the marginal distribution of X_1 and the expression reduces to the one in Section 3.1.

$$\begin{aligned}
(3.4.4) \quad \phi(j) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |dG_j(x_{n+1-p}, \dots, x_n, x_{n+j+1}, \dots, x_{n+j+p}) \\
&\quad - dF(x_{n+1-p}, \dots, x_n) dF(x_{n+j+1}, \dots, x_{n+j+p})|
\end{aligned}$$

$\phi(j) \rightarrow 0$ as $j \rightarrow \infty$ is a sufficient condition for strong mixing. For large p this may not be a very useful condition. If one is interested in a limit theorem for the maximum of a stationary $AR(p)$ process, for small p it may be easier to check this condition than Leadbetter's $D(u_n)$ condition. When G_j and F have densities g_j and f , respectively, we can write

$$\begin{aligned}
\phi(j) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |g_j(x_1, x_2, \dots, x_p, x_{p+j+1}, \dots, x_{2p+j}) \\
&\quad - f(x_1, x_2, \dots, x_p) f(x_{p+j+1}, \dots, x_{2p+j})| dx_1 dx_2 \dots dx_p dx_{p+j+1} \dots dx_{2p+j}.
\end{aligned}$$

The sufficient condition is actually a sufficient condition in a slightly more general context. Any strictly stationary sequence $\{X_n\}$ satisfying (3.4.1), (3.4.2), and (3.4.3) satisfies the strong mixing condition if $\phi(j) \rightarrow 0$ as $j \rightarrow \infty$ where ϕ is as defined in (3.4.4).

4. The Limit Theorem for the Uniform AR(1) Models

4.1. The Class of Models. We define the uniform AR(1) model as follows:

Let $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \varepsilon_n$, $n \geq 1$ for r an integer with $r \geq 2$ and the ε_n 's i.i.d. with $X_{n-1}^{(r)}$ independent of ε_n and the ε_n 's have their distribution given by $P\{\varepsilon_n = k/r\} = 1/r$ for $k=0, 1, 2, \dots, r-1$ and $n \geq 1$. Let $X_0^{(r)} \sim U[0,1]$ and independent of the ε_n 's. In 4.2 we shall see that $\forall n \geq 1$ and $r \geq 2$ $X_n^{(r)} \sim U[0,1]$ and so the above definition gives a family of strictly stationary autoregressive processes of first order. We note that the correlation ρ between X_0 and X_1 equals $\frac{1}{r}$.

4.2. Proof that $X_n^{(r)} \sim U[0,1]$. Let $\phi_\varepsilon(t) = E(e^{it\varepsilon_n})$ and $\phi_{X_n^{(r)}}(t) = E(e^{itX_n^{(r)}})$. These are the characteristic functions of ε_n and $X_n^{(r)}$, respectively. Define $\phi_U(t) = E(e^{itU})$ where $U \sim U[0,1]$. It is easy to show that $\phi_U(t) = (e^{it}-1)/it$ and $\phi_\varepsilon(t) = \sum_{k=0}^{r-1} (e^{itk/r})/r$. We have $\phi_{X_0^{(r)}}(t) = \phi_U(t) = (e^{it}-1)/it$. The proof goes by induction. Suppose $\phi_{X_{n-1}^{(r)}}(t) = (e^{it}-1)/it$. We must show that this implies

$$(4.2.1) \quad \phi_{X_n^{(r)}}(t) = \frac{e^{it} - 1}{it}.$$

$\phi_{X_n^{(r)}}(t) = \phi_{X_{n-1}^{(r)}}(t/r) \phi_\varepsilon(t)$ since $X_{n-1}^{(r)}$ and ε_n are independent.

$$\begin{aligned}\phi_{X_n(r)}(t) &= \frac{(e^{it/r} - 1)}{(it/r)} \cdot \frac{1}{r} \sum_{k=0}^{r-1} e^{itk/r} = \frac{(e^{it/r} - 1)}{it} \sum_{k=0}^{r-1} e^{itk/r} \\ &= \sum_{k=0}^{r-1} \left(\frac{e^{it(k+1)/r} - e^{itk/r}}{it} \right).\end{aligned}$$

This sum telescopes, so after cancellation only $e^{it} - 1$ remains. Hence (4.2.1) is satisfied and the proof is complete.

4.3. The Exact Distribution for M_1, M_2, M_3 When $r=2$. We define $M_n = \max(X_0, X_1, \dots, X_n)$ where X_i is used to denote $X_i^{(2)}$. $P(M_{1-} \leq 1+y) = P(X_{0-} \leq 1+y, X_{1-} \leq 1+y)$ for $-1 \leq y \leq 0$. We recall that $X_1 = (X_0/2) + \epsilon_1$. By conditioning on ϵ_1 we get $P(M_{1-} \leq 1+y) = \frac{1}{2} P(X_{0-} \leq 1+y) + \frac{1}{2} P(X_{0-} \leq 1+2y)$ and we observe that $P(X_{0-} \leq 1+2y) = 0$ if $y \leq -1/2$. Hence

$$P[M_{1-} \leq 1+y] = \begin{cases} 1 + 3y/2 & \text{for } -1/2 \leq y \leq 0 \text{ and} \\ 1/2 + y/2 & \text{for } -1 \leq y \leq -1/2. \end{cases}$$

Now let us consider the case $n=2$. The analysis is similar.

$$P(M_2 \leq 1 + \frac{y}{2}) = P(X_0 \leq 1 + \frac{y}{2}, \frac{X_0}{2} + \epsilon_1 \leq 1 + \frac{y}{2}, \frac{X_0}{4} + \frac{\epsilon_1}{2} + \epsilon_2 \leq 1 + \frac{y}{2})$$

since $X_1 = (X_0/2) + \epsilon_1$ and $X_2 = (X_0/4) + (\epsilon_1/2) + \epsilon_2$. We now condition on ϵ_1 and ϵ_2 and arrive at the following result:

$$\begin{aligned}(4.3.1) \quad P(M_2 \leq 1 + \frac{y}{2}) &= \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}) + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 2+2y) \\ &\quad + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 1+y) + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 1+y, X_0 \leq 1+2y)\end{aligned}$$

and this is defined for all $-2 \leq y \leq 0$. Using some algebra and taking care to notice at which values of y terms in (4.3.1) become zero, we get the following:

$$\begin{aligned}
 P(M_2 \leq 1 + \frac{y}{2}) &= 1+y & -\frac{1}{2} \leq y \leq 0 \\
 &= \frac{3}{4} + \frac{y}{2} & -\frac{2}{3} \leq y \leq -\frac{1}{2} \\
 &= 1 + \frac{7}{8} y & -1 \leq y \leq -\frac{2}{3} \\
 &= \frac{1}{4} + \frac{y}{8} & -2 \leq y \leq -1 .
 \end{aligned}$$

Similarly

$$\begin{aligned}
 P(M_3 \leq 1 + \frac{y}{3}) &= P(X_0 \leq 1 + \frac{y}{3}, \frac{X_0}{2} + \epsilon_1 \leq 1 + \frac{y}{3}, \frac{X_0}{4} + \frac{\epsilon_1}{2} + \epsilon_2 \leq 1 + \frac{y}{3}, \\
 &\quad \frac{X_0}{8} + \frac{\epsilon_1}{4} + \frac{\epsilon_2}{2} + \epsilon_3 \leq 1 + \frac{y}{3})
 \end{aligned}$$

and when we condition on ϵ_1 , ϵ_2 , and ϵ_3 we get eight terms to evaluate with the final result as follows:

$$\begin{aligned}
 P(M_3 \leq 1 + \frac{y}{3}) &= 1 + \frac{5}{6} y & -\frac{3}{8} \leq y \leq 0 \\
 &= \frac{7}{8} + \frac{y}{2} & -\frac{3}{7} \leq y \leq -\frac{3}{8} \\
 &= 1 + \frac{19}{24} y & -\frac{3}{4} \leq y \leq -\frac{3}{7} \\
 &= \frac{5}{8} + \frac{7}{24} y & -1 \leq y \leq -\frac{3}{4} \\
 &= 1 + \frac{2}{3} y & -\frac{9}{8} \leq y \leq -1 \\
 &= \frac{5}{8} + \frac{y}{3} & -\frac{9}{7} \leq y \leq -\frac{9}{8} \\
 &= 1 + \frac{5}{8} y & -\frac{3}{2} \leq y \leq -\frac{9}{7} \\
 &= \frac{1}{8} + \frac{y}{24} & -3 \leq y \leq -\frac{3}{2} .
 \end{aligned}$$

The important thing to observe from these calculations is that for every n $P(M_n \leq 1 + \frac{Y}{n})$ is a piecewise linear function on $[-n, 0]$ increasing from 0 to 1.

Plotting the curve for $n=3$ already shows us that the curve approximates the exponential shape. It is well known and easy to verify that if $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ and X_i^* are i.i.d. $U[0,1]$ then $P[M_n^* \leq 1 + \frac{Y}{n}] \rightarrow e^Y$ as $n \rightarrow \infty$ for $-\infty < Y < 0$. The limit distribution is an extreme value type II distribution with $\alpha=1$. We shall see in 4.7 that

$$(4.3.2) \quad P(M_n \leq 1 + \frac{Y}{n}) \rightarrow e^{Y/2} \text{ as } n \rightarrow \infty.$$

If we could calculate the exact distribution of M_n for "sufficiently large" n , we could see that (4.3.2) holds by plotting the curves. It may be feasible to do this with the aid of the computer. Feasibility depends on how large "sufficiently large" turns out to be.

It is apparent that $n=3$ is not large enough, as can be seen by comparing e^Y , $e^{Y/2}$, and the values for $P(M_1 \leq 1+Y)$, $P(M_2 \leq 1 + \frac{Y}{2})$, and $P(M_3 \leq 1 + \frac{Y}{3})$. It may be that $P(M_3 \leq 1 + \frac{Y}{4})$ gives a better approximation to the limiting distribution. Calculations are not very convincing as $n=3$ is too small to get good approximations to the limit distribution. We see in the following table that when comparing $P(M_3 \leq 1 + \frac{Y}{3})$ to e^Y and $e^{Y/2}$, we have a closer approximation to e^Y than to $e^{Y/2}$.

TABLE 1

y	$P(M_1 \leq 1+y)$	$P(M_2 \leq 1 + \frac{y}{2})$	$P(M_3 \leq 1 + \frac{y}{3})$	e^y	$e^{y/2}$
- 3	0	0	0	.0498	.2231
- 2	0	0	.0417	.1353	.3679
$-\frac{3}{2}$	0	.0625	.0625	.2231	.4724
$-\frac{9}{7}$	0	.0893	.1964	.2765	.5258
$-\frac{9}{8}$	0	.1094	.25	.3247	.5698
- 1	0	.125	.3333	.3679	.6065
$-\frac{3}{4}$	.125	.3438	.4063	.4724	.6873
$-\frac{2}{3}$	.1667	.4167	.4722	.5134	.7165
$-\frac{1}{2}$	.25	.5	.6042	.6065	.7788
$-\frac{3}{7}$	.3571	.5714	.6607	.6514	.8071
$-\frac{3}{8}$	.4375	.625	.6875	.6873	.8290
0	1.0	1.0	1.0	1.0	1.0

4.4. Convergence for Fixed x Along a Subsequence: The Case $r=2$.

We define $n_k = [2^k x]$ where $[z]$ denotes the greatest integer less than or equal to z . M_n, X_n are as defined in 4.3, and x is taken to be positive. The variable y in the previous section can be thought of as $-x$.

Lemma 4.4.1. a) $P[M_{n_k} \leq 1 - \frac{1}{2^k}] \rightarrow e^{-x/2}$ as $k \rightarrow \infty$, and

b) $P[M_{n_k} \leq 1 - \frac{x}{n_k}] \rightarrow e^{-x/2}$ as $k \rightarrow \infty$.

Proof. $P\{M_{n_k} \leq 1 - \frac{1}{2^k}\} = 1 - P\{M_{n_k} > 1 - \frac{1}{2^k}\} = 1 - P\{X_i > 1 - \frac{1}{2^k}$

for some $i \geq 0\}$, $X_i = \frac{X_1}{2^{i-1}} + \sum_{j=0}^{i-2} \frac{\epsilon_{i-j}}{2^j}$ for $i \geq 2$.

Suppose k consecutive ε_i 's = $1/2$. Call them $\varepsilon_\ell, \varepsilon_{\ell+1}, \dots, \varepsilon_{\ell+k-1}$.

In this event

$$X_{k+\ell-1} = \frac{X_{\ell-1}}{2^k} + \sum_{j=0}^{k-1} \frac{\varepsilon_{k+\ell-1-j}}{2^j} = 1 - \frac{1}{2^k} + \frac{X_{\ell-1}}{2^k} > 1 - \frac{1}{2^k}$$

since $X_{\ell-1} > 0$ with probability 1. Hence $M_{n_k} > 1 - \frac{1}{2^k}$. We can conclude $\{k \text{ consecutive } \varepsilon_i \text{'s} = \frac{1}{2} \text{ in } n_k \text{ trials}\} \subseteq \{M_{n_k} > 1 - \frac{1}{2^k}\}$ and by taking complements we get $\{\text{no run of length } k \text{ in } n_k \text{ trials}\} \supseteq \{M_{n_k} \leq 1 - \frac{1}{2^k}\}$. Consequently, $P\{\text{no run of length } k \text{ in } n_k \text{ trials}\} \geq P[M_{n_k} \leq 1 - \frac{1}{2^k}]$.

Now $P\{\text{no run of length } k \text{ in } n_k \text{ trials}\} \rightarrow e^{-x/2}$ as $k, n_k \rightarrow \infty$.

This result is a consequence of a Poisson limit law first proved by von Mises. Feller [5] gives this result on p. 341. We identify $\frac{x}{2}$ with the parameter λ since

$$\frac{n_k}{2} \cdot \frac{1}{2^k} = \frac{n_k}{2^{k+1}} \rightarrow \frac{x}{2}$$

by the definition of n_k . We can now conclude that $\lim_{k \rightarrow \infty} P[M_{n_k} \leq 1 - \frac{1}{2^k}] \leq e^{-x/2}$.

Let $B_k = \{\text{no run of length } k \text{ in } n_k \text{ trials}\}$ and $A_k = \{M_{n_k} \leq 1 - \frac{1}{2^k}\}$. $B_k \supseteq A_k$ and $P(B_k) - P(A_k) = P(B_k \cap A_k^C)$.

$$(4.4.1) \quad P(B_k \cap A_k^C) \leq P\{X_0 > 1 - \frac{1}{2^k} \cup (1 - \frac{1}{2^j} < X_0 \leq 1 - \frac{1}{2^{j+1}})$$

for some $1 \leq j \leq k-1 \cap \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-j} \neq 0\}$ $\stackrel{\text{def.}}{=} P\{C_k\}$.

The inequality (4.4.1) holds because if $\varepsilon_\ell = 0$ for some $1 \leq \ell \leq k-j$ and $1 - \frac{1}{2^j} < X_0 \leq 1 - \frac{1}{2^{j+1}}$ it will take a run of at least k ε 's = $1/2$

for X_i to be greater than $1 - \frac{1}{2^k}$ for some $i \leq n_k$.

$$P\{C_k\} = P[X_0 > 1 - \frac{1}{2^k}] + \sum_{j=1}^{k-1} P[1 - \frac{1}{2^j} < X_0 < 1 - \frac{1}{2^{j+1}}]$$

$$P\{\epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{1}{2}, \dots, \epsilon_{k-j} = \frac{1}{2}\} = \frac{1}{2^k} + \sum_{j=1}^{k-1} \frac{1}{2^{j+1}} \frac{1}{2^{k-j}} = \frac{k+1}{2^{k+1}}.$$

So $P(C_k) = (k+1)/(2^{k+1}) \rightarrow 0$ as $k \rightarrow \infty \Rightarrow P(B_k) - P(A_k) \rightarrow 0 \Rightarrow P[M_{n_k} \leq 1 - \frac{1}{2^k}] \rightarrow e^{-x/2}$.

To show part b) of the lemma, we have

$$\begin{aligned} |P[M_{n_k} \leq 1 - \frac{1}{2^k}] - P[M_{n_k} \leq 1 - \frac{x}{n_k}]| &= P[m < M_{n_k} \leq M] \leq \sum_{i=0}^{n_k} P[m \leq X_i \leq M] \\ &= (n_k+1) P[m \leq X_0 \leq M] \end{aligned}$$

where $M = \max(1 - \frac{1}{2^k}, 1 - \frac{x}{n_k})$ and $m = \min(1 - \frac{1}{2^k}, 1 - \frac{x}{n_k})$.

Now

$$\begin{aligned} (n_k+1) P[m \leq X_0 \leq M] &= (n_k+1) \left| \frac{x}{n_k} - \frac{1}{2^k} \right| \\ &= \left| x \left(\frac{n_k+1}{n_k} \right) - \frac{n_k+1}{2^k} \right| \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ since $(n_k+1)/2^k \rightarrow x$ as $k \rightarrow \infty$ and $(n_k+1)/n_k \rightarrow 1$ as $k \rightarrow \infty$.

We conclude that $P[M_{n_k} \leq 1 - \frac{x}{n_k}] \rightarrow e^{-x/2}$ and this completes the proof of the lemma.

The author acknowledges that Professor Faith conjectured the lemma after recalling the Poisson limit theorem. Professor Faith also provided the proof of part b) of the lemma. Lemma 4.4.1 will not be used

to prove the limit theorem, but it is an important result as it tells us that we cannot expect $P(M_n \leq 1-x/n)$ to converge to e^{-x} as we originally might have expected.

4.5. Verifying $D(u_n)$. In this section we will verify the condition $D(u_n)$ for the class of uniform AR(1) processes. The proof will then be generalized to show $D(u_n)$ holds for a slightly more general class of AR(1) processes.

Theorem 4.5.1. The uniform AR(1) process defined by $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \epsilon_n$, ϵ_n i.i.d. with $P\{\epsilon_n = \frac{k}{r}\} = \frac{1}{r}$ for $k=0, 1, 2, \dots, r-1$ and $X_0 \sim U[0,1]$ with X_0 independent of $\{\epsilon_n\}$ satisfies

$$(4.5.1) \quad |F_{i_1, i_2, \dots, i_p, j_1, \dots, j_q}^{(u_n)} - F_{i_1, i_2, \dots, i_p}^{(u_n)} F_{j_1, \dots, j_q}^{(u_n)}| \leq \alpha_{n, \ell} = \phi(\ell)$$

where $\ell = j_1 - i_p$, $p+q \leq n$, $\phi(\ell) = (\frac{1}{r})^\ell$ and $i_1 < i_2 < \dots < i_p < j_1 < \dots < j_q$.

Proof. For convenience we shall suppress the superscript r . It shall be understood that $X_n = X_n^{(r)}$ and $M_n = M_n^{(r)}$. $\frac{1}{r} > 0$, so by the results in Chapter 1 $\{X_n\}_{n=0}^\infty$ is associative, and by lemma 1.4.1

$$0 \leq F_{i_1, \dots, i_p, j_1, \dots, j_q}^{(u_n)} - F_{i_1, \dots, i_p}^{(u_n)} F_{j_1, \dots, j_q}^{(u_n)}. \text{ Now}$$

$$\begin{aligned} & F_{i_1, \dots, i_p, j_1, \dots, j_q}^{(u_n)} - F_{i_1, \dots, i_p}^{(u_n)} F_{j_1, \dots, j_q}^{(u_n)} \\ &= F_{i_1, \dots, i_p}^{(u_n)} \{P[X_{j_1 - \frac{u_n}{n}} < u, \dots, X_{j_q - \frac{u_n}{n}} < u | X_{i_1 - \frac{u_n}{n}} < u, \dots, X_{i_p - \frac{u_n}{n}} < u] - F_{j_1, \dots, j_q}^{(u_n)}\} \\ &\leq P[X_{j_1 - \frac{u_n}{n}} < u, \dots, X_{j_q - \frac{u_n}{n}} < u | X_{i_1 - \frac{u_n}{n}} < u, \dots, X_{i_p - \frac{u_n}{n}} < u] - F_{j_1, \dots, j_q}^{(u_n)}. \end{aligned}$$

$$P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{i_1} \leq u, \dots, X_{i_p} \leq u]$$

$$= \int_{v \leq u_n} P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{i_p} = v] dG_{i_p}(v)$$

where $G_{i_p}(v) = P[X_{i_1} \leq v | X_{i_1} \leq u, \dots, X_{i_p} \leq u]$. Let $\rho = \frac{1}{r}$ and define

$$W_{j_t} = X_{j_t} - \rho^{(j_t - i_p)} X_{i_p} \quad \text{for } t=1, 2, \dots, q. \quad \text{We consider}$$

$$\int_{v \leq u_n} P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{i_p} = v] dG_{i_p}(v).$$

Now

$$P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{i_p} = v] = \int_0^u P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{j_1} = u, X_{i_p} = v] dH_v(u)$$

where $H_v(u) = P[X_{j_1} \leq u | X_{i_p} = v]$. For each v $P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{j_1} = u, X_{i_p} = v]$
 $= P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{j_1} = u].$

Define $g(u) = P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{j_1} = u]$ for $0 \leq u \leq u_n$ and let
 $g(u) = 1$ for $u < 0$. We have

$$P[X_{j_1} \leq u, \dots, X_{j_q} \leq u | X_{i_p} = v] = \int_0^u g(u) dH_v(u).$$

It is important to note that $g(u)$ is a nonincreasing function of u .

We now consider the measure $H_v(u)$. $H_v(u) = P[X_{j_1} \leq u | X_{i_p} = v]$
 $= P[W_{j_1} + \rho^{\ell} v \leq u] \quad \text{and} \quad \forall 1 \leq t \leq q, X_{j_t} > W_{j_t} = X_{j_t} - \rho^{(j_t - i_p)} X_{i_p} > X_{j_t} - \rho^{\ell}.$
Hence $P[W_{j_1} + \rho^{\ell} v \leq u] \leq P[X_{j_1} - \rho^{\ell} + \rho^{\ell} v \leq u] \leq P[X_{j_1} \leq u + \rho^{\ell}] \quad \forall 0 \leq u, v \leq 1.$

Since g is a nonincreasing function of u

$$\int_0^u g(u) dH_v(u) \leq \int_0^u g(u) dP[X_{j_1} - \rho^{\ell} \leq u] = \int_{-\rho^{\ell}}^{u - \rho^{\ell}} g(w + \rho^{\ell}) dw \leq \int_{-\rho^{\ell}}^{u - \rho^{\ell}} g(w) dw$$

$$= \rho^\ell + \int_0^{u_n - \rho^\ell} g(w) dw \leq \rho^\ell + \int_0^{u_n} g(w) dw$$

$$= \rho^\ell + P[X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n].$$

So

$$\begin{aligned} & \int_{v \leq u_n} P[X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_p} = v] dG_{i_p}(v) \\ & \int_{v \leq u_n} \{\rho^\ell + P[X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n]\} dG_{i_p}(v) \\ & = \rho^\ell + P[X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n] \end{aligned}$$

and from this we conclude that (4.5.1) holds with $\phi(\ell) = \rho^\ell = (\frac{1}{r})^\ell$.

We now generalize the theorem.

Theorem 4.5.2. Suppose $\{X_n\}$ is a strictly stationary first order autoregressive process with $1 > \rho > 0$. Let F be the c.d.f. of X_n and assume $X_0 \sim F$ and is independent of the ε_n 's. If $\exists y \ni F(x) = 1 \forall x \geq y$ and $F(x) < 1 \forall x < y$, then $D(u_n)$ is satisfied. We assume F does not have mass concentrated at y .

Proof. We lose no generality by assuming the distribution is concentrated on $[0, y]$ for if F is concentrated on $[-a, y-a]$ where $a, y > 0$ we simply define $X'_n = X_n + a$ and $u'_n = u_n + a$. The distribution F_1 of the X'_n 's is then concentrated on $[0, y]$ and if $D(u'_n)$ holds for $\{X'_n\}$ then $D(u_n)$ holds for $\{X_n\}$. We shall now show that $D(u_n)$ holds when F is concentrated on $[0, y]$.

Since $\rho > 0$ we again have $0 \leq F_{i_1, \dots, i_p, j_1, \dots, j_q}^{(u_n)} - F_{i_1, \dots, i_p}^{(u_n)} F_{j_1, \dots, j_q}^{(u_n)}$.

As in the previous theorem, $X_{j_t} > W_{j_t} > X_{i_t} - \rho^{(j_t - i_t)} X_{i_t} > X_{j_t} - \rho^{(j_t - i_t)} y > X_{j_t} - \rho^\ell y$ where $\ell = j_t - i_t$. The arguments and inequalities go through as in theorem 4.5.1 and we get

$$\begin{aligned} \int_0^{u_n} g(u) dH_v(u) &= \int_0^{u_n} g(u) dP[X_{j_1} - \rho^\ell y \leq u] \\ &= \int_{-\rho^\ell y}^{u_n - \rho^\ell y} g(w + \rho^\ell y) dF(w + \rho^\ell y) \leq \int_{-\rho^\ell y}^{u_n - \rho^\ell y} g(w) dF(w + \rho^\ell y) \\ &= \int_{-\rho^\ell y}^0 dF(w + \rho^\ell y) + \int_0^{u_n - \rho^\ell y} g(w) dF(w + \rho^\ell y) \\ &\leq \int_0^{\rho^\ell y} dF(w) + P[X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n] \end{aligned}$$

and hence $F_{i_1, \dots, i_p, j_1, \dots, j_q}(u_n) - F_{i_1, \dots, i_p}(u_n) F_{j_1, \dots, j_q}(u_n) \leq F(\rho^\ell y) - F(0)$ and by right continuity of F , $F(\rho^\ell y) \rightarrow F(0)$ as $\ell \rightarrow \infty$.

So $D(u_n)$ holds by taking $\alpha_{n, \ell} = F(\rho^\ell y) - F(0)$. We should point out that the assumptions about F imply $M_n \rightarrow y$ almost surely as $n \rightarrow \infty$ and $\forall x \ u_n = u_n(x) \uparrow y$ as $n \rightarrow \infty$.

In the case where $F'(x)$ exists (i.e. F has a density) we can be more explicit in describing $\alpha_{n, \ell}$. Let $f(x) = F'(x)$. $F(0) = 0$ and $F(\rho^\ell y) \leq \rho^\ell y f(c)$ where $f(c)$ is an absolute maximum for f . In this case we can let $\alpha_{n, \ell} = \rho^\ell y f(c)$. We see that in the case of a uniform distribution F on $[0, y]$, $f(c) = \frac{1}{y}$ and $\alpha_{n, \ell} = \rho^\ell$ just as in the case where F is uniform on $[0, 1]$.

4.6. Proof That $D'(u_n)$ Does Not Hold. We consider $n \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}]$. Associativity implies $\forall j \geq 2 \quad P[X_1 > u_{nk}, X_j > u_{nk}] \geq P[X_1 > u_{nk}] P[X_j > u_{nk}] = P^2[X_1 > u_{nk}] = x^2/(nk)^2$. So

$$n \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] \geq n \sum_{j=2}^n \left(\frac{x}{nk}\right)^2 = \frac{(n-1)}{n} \frac{x^2}{k^2}.$$

Also $X_j = W_j + \rho^{j-1}X_1$ where $W_j = \varepsilon_j + \rho\varepsilon_{j+1} + \dots + \rho^{j-2}\varepsilon_2$, $0 \leq X_1 \leq 1 \Rightarrow X_j \leq W_j + \rho^{j-1}$. So $P[X_j > u_{nk}] < P[W_j > u_{nk} - \rho^{j-1}]$. W_j is independent of X_1 , so

$$\begin{aligned} n \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] &\leq n \sum_{j=2}^n P[X_1 > u_{nk}, W_j > u_{nk} - \rho^{j-1}] \\ &= n \sum_{j=2}^n P[X_1 > u_{nk}] P[W_j > u_{nk} - \rho^{j-1}] = nP[X_1 > u_{nk}] \left\{ \sum_{j=2}^n P[W_j > u_{nk} - \rho^{j-1}] \right\} \\ &= \frac{x}{k} \sum_{j=2}^n P[W_j > u_{nk} - \rho^{j-1}] < \frac{x}{k} \sum_{j=2}^n P[X_j > u_{nk} - \rho^{j-1}] \\ &= \frac{n-1}{n} \frac{x^2}{k^2} + \frac{x}{k} \sum_{j=2}^n \rho^{j-1} < \frac{x^2}{k^2} + \frac{x}{k}. \end{aligned}$$

We see that $\overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}]$ is no larger than $0(\frac{1}{k})$. To see that the order is $0(\frac{1}{k})$ and not $o(\frac{1}{k})$ we must work a little harder. The following lemma shows this when $r=2$.

Lemma 4.6.1. In the case $r=2$, $\overline{\lim}_{n \rightarrow \infty} nk \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] \geq x > 0$ and hence $D'(u_n)$ fails to hold.

Proof. For each j ,

$$P[X_1 > u_{nk}, X_j > u_{nk}] = \int_{u_{nk}}^1 P[W_j > u_{nk} - v/2^{j-1}] dv.$$

$$P[W_j > u_{nk} - v/2^{j-1}] = 1 - \frac{(\ell_j + 1)}{2^{j-1}} \quad \text{if } u_{nk} - v/2^{j-1} \geq \frac{\ell_j}{2^{j-1}}$$

$$= 1 - \frac{\ell_j}{2^{j-1}} \quad \text{otherwise ,}$$

where ℓ_j is defined by $\ell_j/2^{j-1} \leq u_{nk} < (\ell_j + 1)/2^{j-1}$ and ℓ_j is an integer.

Suppose $u_{nk} > (2^{j-1} - 1)/2^{j-1}$. Then $\ell_j/2^{j-1} < u_{nk}$ since the following inequality holds.

$$(4.6.1) \quad \frac{\ell_j}{2^{j-1}} \leq \frac{2^{j-1} - 1}{2^{j-1}} .$$

(4.6.1) holds because if $\ell_j/2^{j-1} > (2^{j-1} - 1)/2^{j-1}$ then $\ell_j/2^{j-1} = 1$ and $u_{nk} < 1$, but by definition $\ell_j/2^{j-1} \leq u_{nk} < 1$, a contradiction.

So

$$u_{nk} = 1 - \frac{x}{nk} > \frac{2^{j-1} - 1}{2^{j-1}} \Leftrightarrow 2^{j-1} (1 - \frac{x}{nk}) > 2^{j-1} - 1$$

$$\Leftrightarrow 2^{j-1} < \frac{nk}{x} \Leftrightarrow j < \log_2 n + \log_2 k - \log_2 x + 1.$$

When $j \leq [\log_2 n + \log_2 k - \log_2 x] + 1$ we have $u_{nk} > (2^{j-1} - 1)/2^{j-1}$ and $P[W_j > u_{nk} - v/2^{j-1}] = 1 - \ell_j/2^{j-1} \quad \forall u_{nk} \leq v \leq 1$. Define $f(n) = [\log_2 n + \log_2 k - \log_2 x] + 1$. Then

$$n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) = n \left\{ \sum_{j=2}^{f(n)} (1 - u_{nk}) \left(1 - \frac{\ell_j}{2^{j-1}} \right) \right\}$$

$$+ n \left\{ \sum_{f(n)+1}^n \int_{u_{nk}}^1 P[W_j > u_{nk} - v/2^{j-1}] dv \right\} \geq n \sum_{j=2}^{f(n)} \frac{x}{nk} \left(1 - \frac{\ell_j}{2^{j-1}} \right) .$$

$$n \sum_{j=2}^{f(n)} \frac{x}{nk} \left(1 - \frac{\ell_j}{2^{j-1}}\right) = \frac{x}{k} \sum_{j=2}^{f(n)} \frac{1}{2^{j-1}}$$

since $\ell_j = 2^{j-1} - 1$ for $j \leq f(n)$ and it follows that $\lim_{n \rightarrow \infty} n \sum_{j=2}^n$
 $P[X_1 > u_{nk}, X_j > u_{nk}] \geq \frac{x}{k}$ as $\frac{x}{k} \sum_{j=2}^{f(n)} 1/2^{j-1} \nearrow \frac{x}{k}$ as $n \rightarrow \infty$.

$$\therefore \lim_{n \rightarrow \infty} nk \sum_{j=2}^n P[X_1 > u_{nk}, X_j > u_{nk}] \geq x > 0$$

and hence $D'(u_n)$ fails. This completes the proof of the lemma.

A similar argument could probably be given to show $D'(u_n)$ fails for general r . Loynes' condition also fails, and this indicates the possibility that $P[M_n \leq u_n]$ does not converge to e^{-x} . We shall see in the next section that $P[M_n^{(r)} \leq u_n] \rightarrow e^{-((r-1)/r)x}$ as $n \rightarrow \infty \quad \forall r \geq 2$ and this limit is of course not equal to e^{-x} .

4.7. The Limit Theorem. We shall prove the theorem for all $r \geq 2$, but for simplicity of notation we will continue to suppress the superscript r . We first need to do some calculations and obtain a lemma.

Let $M_0 = X_0$. We have $P[M_0 \leq 1 - \frac{x}{n}] = 1 - \frac{x}{n}$.

$$\begin{aligned} P[M_1 \leq 1 - \frac{x}{n}] &= P[X_0 \leq 1 - \frac{x}{n}, X_1 \leq 1 - \frac{x}{n}] \\ &= P[X_0 \leq 1 - \frac{x}{n}, \frac{X_0}{r} + \varepsilon_1 \leq 1 - \frac{x}{n}] \end{aligned}$$

and by conditioning on ε_1 we get

$$(4.7.1) \quad P[M_1 \leq 1 - \frac{x}{n}] = \sum_{k=0}^{r-1} \frac{1}{r} P[X_0 \leq 1 - \frac{x}{n}, X_0 \leq r-k - \frac{rx}{n}]$$

Now $r-k - \frac{rx}{n} \geq 1 - \frac{x}{n}$ if $r-k-1 \geq \frac{(r-1)x}{n}$. For n sufficiently large $\frac{(r-1)x}{n} < 1 \leq r-k-1$ for $k \leq r-2$. So for n large we have

$$P[M_1 \leq 1 - \frac{x}{n}] = \frac{1}{r} \left(\sum_{k=0}^{r-1} P[X_0 \leq 1 - \frac{x}{n}] \right) + \frac{1}{r} P[X_0 \leq 1 - \frac{rx}{n}] = 1 - \frac{(2r-1)x}{rn}.$$

In general if r is small enough relative to n (i.e. $\frac{(r-1)x}{n} < 1$)

then $\forall m$

$$(4.7.2) \quad P[M_m \leq 1 - \frac{x}{n}] = \frac{r-1}{r} P[M_{m-1} \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_{m-1} \leq 1 - \frac{x}{n}, X_{m-1} \leq 1 - \frac{rx}{n}]$$

and

$$(4.7.3) \quad P[M_m \leq 1 - \frac{x}{n}, X_m \leq 1 - \frac{r^i x}{n}] = \frac{r-1}{r} P[M_{m-1} \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_{m-1} \leq 1 - \frac{x}{n}, X_{m-1} \leq 1 - \frac{r^{i+1} x}{n}]$$

where $i \leq j-1$ and j will be defined shortly.

These recursive type formulae allow the computation of

$P[M_m \leq 1 - \frac{x}{n}]$ when m is small relative to n .

We give the computation for $P[M_2 \leq 1 - \frac{x}{n}]$ in order to indicate how the general formula is calculated.

$$\begin{aligned} P[M_2 \leq 1 - \frac{x}{n}] &= \frac{r-1}{r} P[M_1 \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_1 \leq 1 - \frac{x}{n}, X_1 \leq 1 - \frac{rx}{n}] \\ &= \frac{r-1}{r} P[M_1 \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_1 \leq 1 - \frac{x}{n}, X_1 \leq 1 - \frac{rx}{n}]. \end{aligned}$$

We continue to substitute using the recursive relations, and we get that

$$\begin{aligned} P[M_2 \leq 1 - \frac{x}{n}] &= \left(\frac{r-1}{r} \right)^2 \left(1 - \frac{x}{n} \right) + \frac{(r-1)}{r^2} \left[1 - \frac{rx}{n} \right] \\ &\quad + \frac{1}{r^2} \sum_{k=0}^{r-1} P[X_0 \leq 1 - \frac{x}{n}, X_0 \leq r-k - \frac{r^2 x}{n}]. \end{aligned}$$

Now $r-k - \frac{r^2 x}{n} \geq 1 - \frac{x}{n}$ if $r-k-1 \geq \frac{(r^2-1)x}{n}$ and for n sufficiently large $\frac{(r^2-1)x}{n} < \frac{r^2 x}{n} < 1 \leq r-k-1$ for $k \leq r-2$. So

$$P[M_2 \leq 1 - \frac{x}{n}] = \left(\frac{r-1}{r}\right)^2 \left(1 - \frac{x}{n}\right) + \frac{r-1}{r^2} \left(1 - \frac{rx}{n}\right) + \frac{r-1}{r^2} \left(1 - \frac{x}{n}\right) + \frac{1}{r^2} \left(1 - \frac{r^2 x}{n}\right).$$

After some algebra we get $P[M_2 \leq 1 - \frac{x}{n}] = 1 - \frac{(3r-2)x}{rn}$.

We now define j to be an integer for which $1 - \frac{r^{j-1}x}{n} > 0$
 $\geq 1 - \frac{r^j x}{n}$. We can use (4.7.2) and (4.7.3) or other methods to show
 $\forall k \leq j-1$

$$(4.7.4) \quad P[M_k \leq 1 - \frac{x}{n}] = 1 - \frac{\{(k+1)r-k\}x}{rn}$$

and in particular we have

$$(4.7.5) \quad P[M_{j-1} \leq 1 - \frac{x}{n}] = 1 - \frac{jx}{n} + \frac{(j-1)x}{rn}.$$

Using (4.7.5) and associativity we can show $\lim_{n \rightarrow \infty} P[M_n \leq 1 - \frac{x}{n}] \geq e^{-\frac{(r-1)x}{r}}$. Hence $P[M_n \leq 1 - \frac{x}{n}] \not\rightarrow e^{-x}$ for any finite r .

Lemma 4.7.1. Let $k = \lfloor \frac{n}{j} \rfloor$ where $\lfloor z \rfloor$ is again the greatest integer less than or equal to z . Define

$$\begin{aligned} I_1 &= \{0, 1, 2, \dots, j-1-m\}, & I_1^* &= \{j-m, \dots, j-1\} \\ I_2 &= \{j, j+1, \dots, 2j-1-m\}, & I_2^* &= \{2j-m, \dots, 2j-1\} \\ &\vdots & & \\ I_k &= \{(k-1)j, \dots, kj-1-m\}, & I_k^* &= \{kj-m, \dots, kj-1\} \end{aligned}$$

and then $|P[M_n \leq 1 - \frac{x}{n}] - P^k[M_{j-1} \leq 1 - \frac{x}{n}]| \rightarrow 0$ as m, n , and $j \rightarrow \infty$ as long as $\frac{m}{j} \rightarrow 0$.

The proof of this lemma is somewhat like the proof of Lemma 2.4 in Leadbetter [8], pp. 293-294. We denote by $M(I_i)$ the maximum of the X_j 's for $j \in I_i$.

$$\begin{aligned} |P[M_n \leq 1 - \frac{x}{n}] - P^k[M_{j-1} \leq 1 - \frac{x}{n}]| &\leq |P[M_n \leq 1 - \frac{x}{n}] - P[M_{jk} \leq 1 - \frac{x}{n}]| \\ &+ |P\left(\bigcap_{i=1}^k \left(M(I_i) \leq 1 - \frac{x}{n}\right)\right) - P[M_{jk} \leq 1 - \frac{x}{n}]| \\ &+ |P\left(\bigcap_{i=1}^k \left(M(I_i) \leq 1 - \frac{x}{n}\right)\right) - P^k\left(M(I_1) \leq 1 - \frac{x}{n}\right)| \\ &+ |P^k\left(M(I_1) \leq 1 - \frac{x}{n}\right) - P^k\left(M_{j-1} \leq 1 - \frac{x}{n}\right)|. \end{aligned}$$

We must show that all four terms tend to zero as j, n , and $m \rightarrow \infty$ with $\frac{m}{j} \rightarrow 0$.

$$\begin{aligned} \text{First } |P^k(M(I_1) \leq 1 - \frac{x}{n}) - P^k(M_{j-1} \leq 1 - \frac{x}{n})| &\rightarrow 0 \text{ since} \\ P^k(M_{j-1} \leq 1 - \frac{x}{n}) &\rightarrow e^{-\left(\frac{r-1}{r}\right)x} \text{ as } j, n \rightarrow \infty \text{ and } P[M(I_1) \leq 1 - \frac{x}{n}] \\ &= P[M_{j-1-m} \leq 1 - \frac{x}{n}] = 1 - (j-1-m) \left(\frac{r-1}{r}\right) \frac{x}{n} - \frac{x}{n}. \text{ If } \frac{m}{j} \rightarrow 0 \\ [1 - \frac{(j-1-m)x}{n} \left(\frac{r-1}{r}\right) - \frac{x}{n}]^k &\rightarrow e^{-\left(\frac{r-1}{r}\right)x}. \end{aligned}$$

$$\begin{aligned} 0 &\leq P\left[\bigcap_{i=1}^k \left(M(I_i) \leq 1 - \frac{x}{n}\right)\right] - P(M_{jk} \leq 1 - \frac{x}{n}) \\ &\leq kP[M(I_1) \leq 1 - \frac{x}{n} \leq M(I_1^*)]. \end{aligned}$$

This follows just as in Lemma 2.4 of Leadbetter [8]. $kP[M(I_1) \leq 1 - \frac{x}{n} \leq M(I_1^*)] \leq kP[1 - \frac{x}{n} \leq M(I_1^*)] < \frac{kx}{n} + \frac{kmx}{n} \left(\frac{r-1}{r}\right)$. Now $\frac{kx}{n} < \frac{x}{j}$ and $\frac{kmx}{n} \left(\frac{r-1}{r}\right) < \frac{mx}{j} \left(\frac{r-1}{r}\right)$. So $\frac{kx}{n} + \frac{kmx}{n} \left(\frac{r-1}{r}\right) \rightarrow 0$ as m, n and $j \rightarrow \infty$ since $\frac{m}{j} \rightarrow 0$ as m and $j \rightarrow \infty$.

It remains for us to show that $|P[M_n \leq 1 - \frac{x}{n}] - P[M_{jk} \leq 1 - \frac{x}{n}]| \rightarrow 0$ and $|P(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})) - P^k(M(I_1) \leq 1 - \frac{x}{n})| \rightarrow 0$. $n \geq jk$ and hence $P[M_n \leq 1 - \frac{x}{n}] \leq P[M_{jk} \leq 1 - \frac{x}{n}]$. When $n > jk$, $P[M_n \leq 1 - \frac{x}{n}] = P[M_{jk} \leq 1 - \frac{x}{n}, X_{jk+1} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \geq P[M_{jk} \leq 1 - \frac{x}{n}] P[X_{jk+1} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}]$ by associativity. As $n \rightarrow \infty$ the number $n-jk$ always remains less than j and hence $\forall n$ $P[X_{jk+1} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \geq P[M_{j-1} \leq 1 - \frac{x}{n}]$. So $P[M_{jk} \leq 1 - \frac{x}{n}] \geq P[M_n \leq 1 - \frac{x}{n}] \geq P[M_{jk} \leq 1 - \frac{x}{n}] P[M_{j-1} \leq 1 - \frac{x}{n}]$. Since $P[M_{j-1} \leq 1 - \frac{x}{n}] \rightarrow 1$ as n and $j \rightarrow \infty$, $P[M_{jk} \leq 1 - \frac{x}{n}] - P[M_n \leq 1 - \frac{x}{n}] \rightarrow 0$.

In Lemma 2.3, Leadbetter [8] shows that $|P(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})) - P^k(M(I_1) \leq 1 - \frac{x}{n})| < k \alpha_{n,m}$ where $\alpha_{n,m}$ is the bound in the definition of $D(u_n)$. We showed that ρ^m could be used for $\alpha_{n,m}$ in the verification of $D(u_n)$. Unfortunately, $k\rho^m \not\rightarrow 0$ as n, j and $m \rightarrow \infty$. Since $\frac{m}{j} \rightarrow 0$, ρ^m does not go to zero fast enough for $k\rho^m$ to go to zero.

Let $A_i = \{M(I_i) \leq 1 - \frac{x}{n}\}$ for $i=1, 2, \dots, k$. We then have that

$$\begin{aligned} & |P\left(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})\right) - P^k(M(I_1) \leq 1 - \frac{x}{n})| \\ &= |P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i)| \end{aligned}$$

and we shall see that this can be bounded by $2k \frac{x}{n}$. Since $2k \frac{x}{n} < 2 \frac{x}{j}$ and $2 \frac{x}{j} \rightarrow 0$ as $j \rightarrow \infty$, we get the result of the lemma when this bound is verified.

$$\begin{aligned}
\left| P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \right| &= P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \\
&= P\left(A_k \mid \bigcap_{i=1}^{k-1} A_i\right) P\left(\bigcap_{i=1}^{k-1} A_i\right) - P(A_k) P\left(\bigcap_{i=1}^{k-1} A_i\right) \\
&\quad + P(A_k) P\left(\bigcap_{i=1}^{k-2} A_i\right) P\left(A_{k-1} \mid \bigcap_{i=1}^{k-2} A_i\right) - P(A_k) P(A_{k-1}) P\left(\bigcap_{i=1}^{k-2} A_i\right) \\
&\quad + \dots + \left[\prod_{j=3}^k P(A_j) \right] P(A_2 \mid A_1) - \prod_{j=1}^k P(A_j) \\
&\leq P\left(A_k \mid \bigcap_{i=1}^{k-1} A_i\right) - P(A_k) + P\left(A_{k-1} \mid \bigcap_{i=1}^{k-2} A_i\right) - P(A_{k-1}) \\
&\quad + \dots + P(A_2 \mid A_1) - P(A_2) .
\end{aligned}$$

A typical term is $P(A_S \mid \bigcap_{i=1}^{S-1} A_i) - P(A_S)$. We shall show that

$$0 \leq P\left(A_S \mid \bigcap_{i=1}^{S-1} A_i\right) - P(A_S) < 2 \frac{x}{n}$$

for any $2 \leq S \leq k$ and hence

$$\left| P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \right| \leq 2k \frac{x}{n} = \frac{2x}{j} .$$

First we will show this for $P(A_2 \mid A_1) - P(A_2)$. Let $q = j-1-m$

$$P(A_2 \mid A_1) - P(A_2) = P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n} \mid A_1]$$

$$- P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n}] .$$

$$P(A_2) = 1 - \frac{qx}{n} + \frac{(q-1)x}{rn} .$$

$$P(A_2 | A_1) = \int_0^{1-x/n} P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n} | X_q = v] dF_{xq}(v)$$

where $F_{xq}(v) = P[X_q \leq v | X_0 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}]$. For each ℓ , $X_{q+\ell} = \rho^\ell X_q + W_\ell$ where W_ℓ is a discrete uniform random variable with $P[W_\ell = \frac{h}{r} \rho^\ell] = \frac{1}{r} \rho^\ell$ for $h=0, 1, \dots, \frac{r^\ell - 1}{\rho^\ell}$.

$$\begin{aligned} P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n} | X_q = v] \\ = P[W_{m+1} \leq 1 - \frac{x}{n} - \rho^{m+1} v, \dots, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v]. \end{aligned}$$

We observe that $P[W_\ell \leq 1 - \frac{x}{n} - \rho^\ell v] = 1$ if $v \leq 1 - r^\ell \frac{x}{n}$ and $P[W_\ell \leq 1 - \frac{x}{n} - \rho^\ell v] = 1 - \rho^\ell$ if $v > 1 - r^\ell \frac{x}{n}$. This follows since $v \leq 1 - r^\ell \frac{x}{n} \Rightarrow 1 - \frac{x}{n} - \rho^\ell v \geq 1 - \frac{x}{n} - \rho^\ell + \frac{x}{n} = 1 - \rho^\ell$ and $P[W_\ell \leq 1 - \rho^\ell] = 1$, and $v > 1 - r^\ell \frac{x}{n} \Rightarrow 1 - \rho^\ell > 1 - \frac{x}{n} - \rho^\ell v > 1 - \frac{x}{n} - \rho^\ell > 1 - 2\rho^\ell$ and $P[W_\ell \leq 1 - 2\rho^\ell] = \frac{r^\ell - 1}{r}$. (Note that $1 - 2\rho^\ell < 1 - \frac{x}{n} - \rho^\ell \Leftrightarrow \frac{x}{n} < \rho^\ell \Leftrightarrow \ell \leq j-1$, and in this case $\ell \leq m+q = j-1$.)

From this we see that

$$\begin{aligned} P(A_2 | A_1) &= \int_0^{1-r^{j-1} \frac{x}{n}} dF_{xq}(v) + \int_{1-r^{j-1} \frac{x}{n}}^{1-r^{j-2} \frac{x}{n}} P[W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{xq}(v) \\ &\quad + \int_{1-r^{j-2} \frac{x}{n}}^{1-r^{j-3} \frac{x}{n}} P[W_{m+q-1} \leq 1 - \frac{x}{n} - \rho^{m+q-1} v, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{xq}(v) \\ &\quad + \dots + \int_{1-r^{m+1} \frac{x}{n}}^{1 - \frac{x}{n}} P[W_{m+1} \leq 1 - \frac{x}{n} - \rho^{m+1} v, \dots, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{xq}(v). \end{aligned}$$

The integrands $P[W_\ell \leq 1 - \frac{x}{n} - \rho^\ell v, \dots, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v]$ are seen to be constant over the interval of integration with the constant being $1 - \rho^\ell$ and the integrals simplify as follows:

$$(4.7.6) \quad P(A_2 | A_1) = F_{xq} \left(1 - r^{j-1} \frac{x}{n} \right) + \sum_{\ell=1}^{q-1} \left(1 - \frac{1}{r^{j-\ell}} \right) \left[F_{xq} \left(1 - r^{j-\ell-1} \frac{x}{n} \right) - F_{xq} \left(1 - r^{j-\ell} \frac{x}{n} \right) \right] + \left[1 - F_{xq} \left(1 - r^{m+1} \frac{x}{n} \right) \right] \left(1 - \frac{1}{r^{m+1}} \right).$$

Claim. $\forall 0 \leq v \leq 1, F_{xq}(v) \leq v + \rho^q.$

Proof. $F_{xq}(v) = P[X_q \leq v | X_0 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}] \leq P[X_q \leq v | X_0 = 0, X_1 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}] = P[X_q \leq v | X_0 = 0] = P[W_q \leq v].$

Now $W_q = X_q - \rho^q X_0 \geq X_q - \rho^q$ with probability 1 and hence $P[W_q \leq v] \leq P[X_q - \rho^q \leq v] = P[X_q \leq v + \rho^q] = v + \rho^q.$

After some algebra (4.7.6) simplifies to

$$(4.7.7) \quad P(A_2 | A_1) = 1 - \sum_{\ell=1}^{q-1} \left\{ \frac{1}{r^{j-\ell}} \left[F_{xq} \left(1 - r^{j-\ell-1} \frac{x}{n} \right) - F_{xq} \left(1 - r^{j-\ell} \frac{x}{n} \right) \right] \right\} + \frac{1}{r^{m+1}} F_{xq} \left(1 - r^{m+1} \frac{x}{n} \right) - \frac{1}{r^{m+1}}.$$

$F_{xq} \left(1 - r^{j-\ell-1} \frac{x}{n} \right) - F_{xq} \left(1 - r^{j-\ell} \frac{x}{n} \right) \geq 1 - r^{j-\ell-1} \frac{x}{n} - (1 - r^{j-\ell} \frac{x}{n} + \rho^q)$ and $F_{xq} \left(1 - r^{m+1} \frac{x}{n} \right) \leq 1 - r^{m+1} \frac{x}{n} + \rho^q.$ Using these two inequalities with (4.7.7) we get

$$\begin{aligned} P(A_2 | A_1) &\leq 1 - \frac{qx}{n} + \frac{(q-1)x}{rn} + \frac{\rho^q}{r^{m+1}} + \rho^q \sum_{\ell=1}^{q-1} \frac{1}{r^{j-\ell}} \\ &= 1 - \frac{qx}{n} + \frac{(q-1)x}{rn} + \frac{1}{r^j} + \frac{1}{r^j} \left(1 - \frac{1}{r^{j-m-2}} \right). \end{aligned}$$

We get

$$P(A_2|A_1) - P(A_2) \leq \frac{1}{r^j} + \frac{1}{r^j} \left(1 - \frac{1}{r^{j-m-2}}\right) = \frac{1}{r^j} \left(2 - \frac{1}{r^{j-m-2}}\right) < \frac{2}{r^j} \leq \frac{2x}{n}.$$

Now consider $P(A_S | \bigcap_{i=1}^{S-1} A_i) = P(A_S | A_{S-1} \cap B_{S-2})$ for $S \geq 3$ where
 $B_{S-2} = \bigcap_{i=1}^{S-2} A_i$. $A_{S-1} = \{X_{(S-2)j} \leq 1 - \frac{x}{n}, \dots, X_{(S-1)j-1-m} \leq 1 - \frac{x}{n}\}$.

$$P(A_S | A_{S-1} \cap B_{S-2}) = \int_0^{1 - \frac{x}{n}} P[X_{(S-1)j} \leq 1 - \frac{x}{n},$$

$$\dots, X_{Sj-1-m} \leq 1 - \frac{x}{n} | X_{(S-1)j-1-m} \leq 1 - \frac{x}{n}] dF_S(v)$$

where $F_S(v) = P[X_{(S-1)j-1-m} \leq v | A_{S-1} \cap B_{S-2}]$. $F_S(v) \leq P[X_{(S-1)j-1-m} \leq v |$

$X_{(S-2)j} = 0, X_{(S-2)j+1} \leq 1 - \frac{x}{n}, \dots, X_{(S-1)j-1-m} \leq 1 - \frac{x}{n} \cap B_{S-2})$

$= P[X_{(S-1)j-1-m} \leq v | X_{(S-2)j} = 0] \leq v + \rho^q$. Applying stationarity we see

$P(A_S | A_{S-1} \cap B_{S-2})$ has the same upper bound as $P(A_2 | A_1)$ since the

integrands are equal and F_S has the same upper bound as F_{xq} . Hence

$$P(A_S | A_{S-1} \cap B_{S-2}) - P(A_S) \leq \frac{2}{r^j} < \frac{2x}{n}.$$

This gives us that

$$\left| P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \right| \leq k \left(\frac{2x}{n}\right) = \frac{2x}{j} \rightarrow 0$$

as $j \rightarrow \infty$ and the lemma is proved.

As a corollary of this lemma we get the following:

Theorem 4.7.2. For the processes $\{X_n^{(r)}\}_{n=0}^{\infty}$ $r \geq 2$ defined in

Section 4.1

$$P\{M_n^{(r)} \leq 1 - \frac{x}{n}\} \rightarrow e^{-\left(\frac{r-1}{r}\right)x}$$

for all $x \geq 0$ or equivalently

$$P\{M_n^{(r)} \leq 1 + \frac{y}{n}\} \rightarrow e^{-\left(\frac{r-1}{r}\right)y}$$

for all $y \leq 0$.

4.8. Simulation Results, $x=1$ and $x = 1/2$ with $r=2$. A simulation of the process was carried out for the case $r=2$. The results are presented here to give independent confirmation that $P\{M_n \leq 1 - \frac{x}{n}\}$ appears to be converging to $e^{-x/2}$.

The simulation was done on the LOTS system using the library random number generator. The following table summarizes the results. n refers to the number of X_i 's over which the maximum is taken and m is the number of replications. $nP(1)$ is the number of X_i 's $\leq 1 - \frac{1}{n}$ and $nP(1/2)$ is the number of X_i 's $\leq 1 - \frac{1}{2n}$.

Since the limit value for $x=1$ is $e^{-1/2}$ and for $x = 1/2$ is $e^{-1/4}$, $P(1)$ and $P(1/2)$ should be compared to the values .6065 and .7788 respectively. We see that the Monte Carlo approximation is good to within $\pm .02$ for n ranging from 50 to 725 and $x=1$. When $x = 1/2$ the approximation is even better with differences only in the third decimal place when $n \geq 500$. It is apparent that $P(1)$ is not converging to $e^{-1} \approx .3679$ and $P(1/2)$ is not converging to $e^{-1/2} \approx .6065$.

TABLE 2

m = 500			m = 1000		
<u>n</u>	<u>P(1)</u>	<u>P(1/2)</u>	<u>n</u>	<u>P(1)</u>	<u>P(1/2)</u>
50	.606	.790	50	.583	.765
75	.550	.744	75	.586	.765
100	.600	.756	100	.616	.782
125	.612	.774	125	.577	.775
150	.582	.780	150	.608	.767
175	.600	.796	175	.584	.753
200	.584	.752	200	.583	.773
			225	.607	.795
			250	.592	.771
			275	.591	.762
			300	.593	.768
			325	.602	.779
			350	.610	.777
			375	.593	.778
			400	.622	.792
			425	.590	.781
			450	.602	.780
			475	.604	.775
			500	.613	.770
			525	.621	.776
			550	.593	.775
			575	.609	.792
			600	.593	.787
			625	.608	.783
			650	.608	.782
			675	.605	.794
			700	.588	.778
			725	.603	.771

Table 3 is exhibited to give some indication of the variability due to the random sampling. M_{500} was generated 1000 times and $P(1)$ and $P(1/2)$ were computed. This was replicated five times using a different sequence of random numbers each time.

TABLE 3

m = 1000	n = 500	
<u>replication</u>	<u>P(1)</u>	<u>P(1/2)</u>
1	.625	.789
2	.585	.778
3	.588	.774
4	.614	.780
5	.626	.773

We see that nearly all the variation between $P(1)$ and $e^{-1/2}$, $P(1/2)$ and $e^{-1/4}$ can be explained by sampling variation. This means that although the results are useful in confirming the convergence to $e^{-x/2}$ there is too much sampling variation to say anything meaningful about the rate of convergence.

4.9. Implications on Data Analysis. The results of Berman [2], Loynes [10], and Leadbetter [8] indicate that in many cases stationary stochastic processes have maxima behaving asymptotically as for independent identically distributed random variables. The uniform example in this thesis shows that even when mixing conditions are satisfied the correlation structure can have an effect on the limiting distribution. It shows that the sufficient conditions like those posed in [10] and [8] are needed to obtain the limit theorems.

The uniform example is not intended to be a model which is applicable to a particular data analysis problem, but rather it is intended

to illustrate a behavior that may also be present in other stationary models which do have applicability. It shows that Theorems 1.1.3 and 1.1.6 can fail to hold when the sufficient conditions are not satisfied. Lewis in [9] has developed ARMA models with gamma marginal distributions. These models may have applications in queueing problems or in the analysis of air pollution data. Since these models are generalizations of the exponential model studied in Chapter 2, we might suspect that Theorem 1.1.3 holds, but this requires checking the mixing condition and the sufficient condition.

We have seen that if a stationary $AR(1)$ process has a marginal distribution concentrated on a closed interval $D(u_n)$ is satisfied. Such models may have applicability in a variety of problems as we often know that the X_n 's will never exceed a known value M and never go below a known lower bound m .

5. Other Limit Theorems

5.1. The EARMA(1,1) Model. The EARMA(1,1) model is a generalization of the EAR(1) process of Chapter 2. Jacobs and Lewis [7] describe the process and obtain some properties including verification of the strong mixing condition. We shall prove the following theorem:

Theorem 5.1. Let $M_n = \max\{X_1, X_2, \dots, X_n\}$ where $\{X_n\}$ is an EARMA(1,1) process. $P[M_n \leq \frac{x + \ln n}{\lambda}] \rightarrow e^{-e^{-x}}$ as $n \rightarrow \infty$. The proof of the theorem will involve verifying $D'(u_n)$. Earlier we showed that the EAR(1) process satisfied Loynes' sufficient condition.

First we need to give the definition of the process.

Definition. $\{X_n\}_{n=1}^{\infty}$ is an EARMA(1,1) process if $X_n = \beta \epsilon_n + U_n A_{n-1}$, $A_n = \rho A_{n-1} + V_n \epsilon_n$ for $n \geq 1$. ϵ_n 's are i.i.d., $\epsilon_n \sim \exp(\lambda)$. $0 \leq \beta \leq 1$ and $0 \leq \rho < 1$. $A_0 \sim \exp(\lambda)$ is independent of the ϵ_n 's.

$$\begin{aligned} U_n &= 0 \text{ with probability } \beta, \\ &= 1 \text{ with probability } 1-\beta; \\ V_n &= 0 \text{ with probability } \rho, \\ &= 1 \text{ with probability } 1-\rho. \end{aligned}$$

The V_n 's and U_n 's are all independent and are independent of the ϵ_n 's and A_n 's. Under these conditions $\{X_n\}_{n=1}^{\infty}$ is strictly stationary and $X_1 \sim \exp(\lambda)$.

In order to verify $D'(u_n)$ we need to consider probabilities of the form $P[X_1 > z, X_j > z]$. We have

$$\begin{aligned} P[X_1 > z, X_j > z] &= \beta^2 P[\beta \epsilon_1 > z, \beta \epsilon_j > z] + \beta(1-\beta) P[\beta \epsilon_1 > z, \beta \epsilon_j + A_{j-1} > z] \\ &\quad + \beta(1-\beta) P[\beta \epsilon_1 + A_0 > z, \beta \epsilon_j > z] + (1-\beta)^2 P[\beta \epsilon_1 + A_0 > z, \beta \epsilon_j + A_{j-1} > z] \\ &\quad < P[\beta \epsilon_1 + A_0 > z, \beta \epsilon_j + A_{j-1} > z]. \end{aligned}$$

To verify $D'(u_n)$ it suffices to show that

$$\lim_{n \rightarrow \infty} n \sum_{j=2}^n P[\beta \epsilon_1 + A_0 > u_{nk}(x), \beta \epsilon_j + A_{j-1} > u_{nk}(x)] = 0 \left(\frac{1}{k}\right).$$

$$\begin{aligned} (5.1.1) \quad P[\beta \epsilon_1 + A_0 > z, \beta \epsilon_j + A_{j-1} > z] &= \int_{\frac{z}{\beta}}^{\infty} \int_{\frac{z}{\beta}}^{\infty} \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv \\ &+ \int_{\frac{z}{\beta}}^{\infty} \int_0^{\frac{z}{\beta}} P[A_0 > z - \beta u] \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv \\ &+ \int_0^{\frac{z}{\beta}} \int_{\frac{z}{\beta}}^{\infty} P[A_{j-1} > z - \beta v | \epsilon_1 = u] \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv \\ &+ \int_0^{\frac{z}{\beta}} \int_0^{\frac{z}{\beta}} P[A_0 > z - \beta u, A_{j-1} > z - \beta v | \epsilon_1 = u] \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv. \end{aligned}$$

Computations show that the first integral is $e^{-2\lambda(z/\beta)}$ and the second $\frac{1}{(1-\beta)} [e^{-\lambda(1+\beta)(z/\beta)} - e^{-2\lambda(z/\beta)}]$.

$$n \sum_{j=2}^n e^{-2\lambda u_{nk}(x)/\beta} = n \sum_{j=2}^n \frac{e^{-2x/\beta}}{(nk)^{2/\beta}}$$

$$\text{since } u_{nk}(x) = \frac{x + \ln nk}{\lambda}.$$

Now

$$n \sum_{j=2}^n \frac{e^{-2x/\beta}}{(nk)^{2/\beta}} < \frac{1}{k} \sum_{j=2}^n \frac{1}{\frac{2-\beta}{\beta}} = \frac{n-1}{k(nk)^{\frac{2-\beta}{\beta}}}$$

which tends to zero as $n \rightarrow \infty$.

Similarly the second integral is bounded by $\frac{1}{(1-\beta)} e^{-\lambda(1+\beta) z/\beta}$

and

$$n \sum_{j=2}^n \frac{e^{-(1+\beta) x/\beta}}{(1-\beta)(nk)^{\frac{1+\beta}{\beta}}} \leq \frac{1}{k} \sum_{j=2}^n \frac{1}{(1-\beta)(nk)^{\frac{1}{\beta}}} = \frac{n-1}{(1-\beta)k(nk)^{\frac{1}{\beta}}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We are assuming here that $\beta > 0$. For $\beta=0$ we simply get

$$P[X_1 > z, X_j > z] = P[A_0 > z, A_{j-1} > z] = e^{-(2-\rho^{j-1})z} \text{ and}$$

$$n \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] = \frac{1}{k} \sum_{j=2}^n \frac{e^{-(2-\rho^{j-1})x}}{(nk)^{1-\rho^{j-1}}} < \frac{1}{k^{2-\rho}} \sum_{j=2}^n \frac{1}{n^{1-\rho^{j-1}}}.$$

It can be shown as was shown to the author by Professor Faith that

if $\rho < 1$

$$\lim_{n \rightarrow \infty} \sum_{j=2}^n \frac{1}{n^{1-\rho^{j-1}}} \leq 2.$$

So $\lim_{n \rightarrow \infty} n \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] = o(\frac{1}{k})$ when $\beta=0$. This verifies $D'(u_n)$ for the EAR(1) process.

To show that $D'(u_n)$ is satisfied for the EARMA(1,1) model we must show that the third and fourth integrals in (5.1.1) are appropriately small.

$A_{j-1} = \rho^{j-1} A_0 + W_{j-1}^* + \rho^{j-2} v_1 \epsilon_1$ where $W_{j-1}^* = \sum_{r=1}^{j-2} \rho^{j-1-(r+1)} v_{r+1} \epsilon_{r+1}$ and as has been shown before $W_{j-1}^* = 0$ with probability ρ^{j-2} and $W_{j-1}^* \sim \exp(\lambda)$ with probability $1-\rho^{j-2}$. The third integral becomes

$$\begin{aligned}
& (1-\rho) \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P(\rho^{j-1} A_0 + W_{j-1}^* > z - \beta v - \rho^{j-2} u) \lambda e^{-\lambda u} du dv \\
& + \rho \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P(\rho^{j-1} A_0 + W_{j-1}^* > z - \beta v) \lambda e^{-\lambda u} du dv \\
& < \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P[\rho^{j-1} A_0 + W_{j-1}^* > z - \beta v - \rho^{j-2} u] \lambda e^{-\lambda u} du dv \\
& = \rho^{j-2} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} \lambda e^{-\lambda u} P(\rho^{j-1} A_0 > z - \beta v - \rho^{j-2} u) du dv \\
& + (1-\rho^{j-2}) \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P(\rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u) \lambda e^{-\lambda u} du dv \\
& < \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P(\rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u) \lambda e^{-\lambda u} du dv
\end{aligned}$$

where Z_{j-1}^* denotes the random variable W_{j-1}^* conditional on $W_{j-1}^* \neq 0$.

The fourth integral is bounded by

$$\begin{aligned}
& \rho^{j-2} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda u} P[A_0 > z - \beta u, \rho^{j-1} A_0 > z - \beta v - \rho^{j-2} u] du dv \\
& + (1-\rho^{j-2}) \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda u} P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] du dv \\
& < \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda u} P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] du dv .
\end{aligned}$$

It suffices to show that

$$(5.1.2) \quad \overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n \int_0^{u_{nk}(x)/\beta} \lambda e^{-\lambda v} \int_{u_{nk}(x)/\beta}^{\infty} P[\rho^{j-1} A_0 + Z_{j-1}^* > u_{nk}(x) - \beta v - \rho^{j-2} u] \lambda e^{-\lambda u} du dv = o\left(\frac{1}{k}\right)$$

and

$$(5.1.3) \quad \overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n \int_0^{u_{nk}(x)/\beta} \lambda e^{-\lambda v} \int_0^{u_{nk}(x)/\beta} P[A_0 > u_{nk}(x) - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > u_{nk}(x) - \beta v - \rho^{j-2} u] \lambda e^{-\lambda u} du dv = o\left(\frac{1}{k}\right).$$

Tedious computations show that (5.1.2) and (5.1.3) do indeed hold and hence $D'(u_n)$ is satisfied and the limit theorem holds. Detailed computations are given in Appendix 1 for the verification of (5.1.2) and (5.1.3).

5.2. Special Cases of O'Brien's Examples. O'Brien [12] gave the following example:

Let Y_n be i.i.d. random variables with cumulative distribution function F . We define a strictly stationary sequence $\{X_n\}$ as follows: For each $n \geq 1$, $X_n = Y_n$ if $J_n = 0$ and $X_n = X_{n-1}$ if $J_n = 1$. $\{J_n\}_{n=1}^{\infty}$ are i.i.d. with $P[J_n = 0] = \alpha = 1 - P[J_n = 1]$, $X_0 = Y_0$ and $\{J_n\}$ is independent of $\{Y_n\}$.

O'Brien shows that the sequence $\{X_n\}_{n=0}^{\infty}$ is strong mixing with $|P(A \cap B) - P(A)P(B)| \leq g(k) = (1-\alpha)^k \quad \forall A \in \sigma(X_0, X_1, \dots, X_m) \text{ and } B \in \sigma(X_{m+k+1}, \dots) \text{ for any } m \geq 1.$

If as usual we let $M_n^* = \max(Y_0, Y_1, \dots, Y_n)$ and $M_n = \max(X_0, X_1, \dots, X_n)$ and suppose $P[M_n^* \leq a_n x + b_n] \rightarrow G(x)$ as $n \rightarrow \infty$ then $P[M_n \leq a_n x + b_n] \rightarrow G^\alpha(x)$. Heuristically we can argue that this result holds because in m observations of the process $\{X_n\}$ we expect to see αm different Y_n 's. Since the limit for M_n is not the same as for M_n^* when $\alpha \neq 1$, $D'(u_n)$ must fail.

We consider three special cases.

- a) F is $\exp(1)$.
- b) F is $U(0,1)$.
- c) F is Cauchy $(0,1)$.

The corresponding $u_{nk}(x)$'s are as follows:

- a) $x + \ln nk$,
- b) $1 + \frac{x}{nk}$ for $x \leq 0$, and
- c) $\frac{nkx}{\pi}$ for $x \geq 0$.

The corresponding $G(x)$'s are:

- a) $e^{-e^{-x}}$,
- b) e^x for $x \leq 0$, and
- c) $e^{-x^{-1}}$ for $x \geq 0$.

For c) we refer the reader to Feller [6], p. 278.

The limit theorems are then:

- a) $P[M_n \leq x + \ln n] \rightarrow e^{-\alpha e^{-x}}$,
- b) $P[M_n \leq 1 + \frac{x}{n}] \rightarrow e^{\alpha x}$ for $x \leq 0$, and
- c) $P[M_n \leq \frac{nx}{\pi}] \rightarrow e^{-\alpha/x}$ for $x \geq 0$.

We now shall consider $\overline{\lim}_{n \rightarrow \infty} \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)]$.

$$\begin{aligned} nk \cdot \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] &= nk \cdot \sum_{j=2}^n \{ [1 - (1-\alpha)^{j-1}] P^2[X_1 > u_{nk}(x)] \\ &\quad + (1-\alpha)^{j-1} P[X_1 > u_{nk}(x)] \} . \end{aligned}$$

This follows since $X_1 = X_j$ with probability $(1-\alpha)^{j-1}$ (i.e. $j-1$ consecutive J_n 's are equal to one). Hence

$$nk \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] > nk P[X_1 > u_{nk}(x)] \left(\sum_{j=2}^n (1-\alpha)^{j-1} \right) .$$

Now we shall compute $nk P[X_1 > u_{nk}(x)]$ for all the three cases:

$$a) \quad nk P[X_1 > u_{nk}(x)] = e^{-x} > 0 \quad \forall n, k ,$$

$$b) \quad nk P[X_1 > u_{nk}(x)] = -x > 0 \quad \forall n, k , \text{ and}$$

$$c) \quad nk P[X_1 > u_{nk}(x)] = nk \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi(1+y^2)} dy .$$

In case c)

$$\int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi(1+y^2)} dy \approx \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi y^2} dy = \frac{1}{nkx} .$$

By " $\approx$ " we mean the two integrals are approximately equal for large n in the sense that their ratio tends to one as $n \rightarrow \infty$. $\overline{\lim}_{n \rightarrow \infty} nk P[X_1 > u_{nk}(x)] = \frac{1}{x} > 0 \quad \forall k$ in case c).

We see in all three cases that $0 < \lim_{k \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} nk \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] < \infty$ and so just as in the uniform case $D'(u_n)$ just

barely fails. We have seen that $D'(u_n)$ holds in the examples where the limit theorem is the same as for the i.i.d. case and it barely fails in the cases where the limits are different. This gives evidence that $D'(u_n)$ is a good sufficient condition for the limit theorem for strong mixing sequences.

5.3. The Cauchy Case. We shall now consider the Cauchy AR(1) process given in Section 3.3. We will compute $\sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)]$ and show that $D'(u_n)$ fails. Here $u_{nk}(x) = \frac{nkx}{\pi}$ for $x \geq 0$.

$$P[X_1 > u_{nk}(x), X_j > u_{nk}(x)]$$

$$= \int_{\frac{nkx}{\pi}}^{\infty} \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2(1+x_0^2)} \frac{1}{(1-|\rho|^{j-1}) \left[1 + \left(\frac{x_j - \rho^{j-1}x_0}{1-|\rho|^{j-1}} \right)^2 \right]} dx_j dx_0.$$

Let

$$z = \frac{x_j - \rho^{j-1}x_0}{(1-|\rho|^{j-1})}.$$

Applying this transformation we get

$$\begin{aligned} P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] &= \int_{\frac{nkx}{\pi}}^{\infty} \int_{\frac{(\frac{nkx}{\pi} - \rho^{j-1}x_0)}{(1-|\rho|^{j-1})}}^{\infty} \frac{1}{\pi^2(1+x_0^2)(1+z^2)} dz dx_0 \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{nkx}{\pi} \right) \right] - \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2(1+x_0^2)} \tan^{-1} \left(\frac{\frac{nkx}{\pi} - \rho^{j-1}x_0}{1-|\rho|^{j-1}} \right) dx_0. \end{aligned}$$

Simplifying we get

$$(5.3.1) \quad P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] = \frac{1}{4} - \frac{1}{2\pi} \tan^{-1} \left(\frac{nkx}{\pi} \right) \\ - \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2 (1+x_0^2)} \tan^{-1} \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{1 - |\rho|^{j-1}} \right) dx_0, \quad j \geq 2.$$

As mentioned in Section 5.2, $P[M_n^* \leq \frac{nx}{\pi}] \rightarrow e^{-x-1}$ for $x > 0$. We shall see that a conjectured result $P[M_n \leq \frac{nx}{\pi}] \rightarrow e^{-(1-\rho)/x}$ appears to be confirmed by simulation results which are given in Appendix 2.

Equation (5.3.1) holds for $-1 < \rho < 1$. In the following we restrict attention to $0 < \rho < 1$.

Using integration by parts we get

$$\int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2} \frac{1}{(1+x_0^2)} \tan^{-1} \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{1 - \rho^{j-1}} \right) dx_0 = -\frac{1}{4} - \frac{1}{\pi^2} [\tan^{-1} \left(\frac{nkx}{\pi} \right)]^2 \\ + \frac{\rho^{j-1}}{(1-\rho^{j-1})} \int_{\frac{nkx}{\pi}}^{\infty} \frac{\tan^{-1} x_0}{\pi^2 \left[1 + \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{1 - \rho^{j-1}} \right)^2 \right]} dx_0.$$

For large x_0 , $\tan^{-1} x_0 \approx \frac{\pi}{2} - \frac{1}{x_0}$. We get

$$P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] \approx \frac{1}{2} + \frac{1}{\pi^2} [\tan^{-1} \left(\frac{nkx}{\pi} \right)]^2 - \frac{1}{2\pi} \tan^{-1} \left(\frac{nkx}{\pi} \right) \\ - \frac{\rho^{j-1}}{(1-\rho^{j-1})} \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2} \frac{\left(\frac{\pi}{2} - \frac{1}{x_0} \right)}{\left(1 + \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{(1-\rho^{j-1})} \right)^2 \right)} dx_0$$

$$= \frac{1}{4} + \frac{1}{\pi^2} (\tan^{-1} (\frac{nkx}{\pi}))^2 - \frac{1}{\pi} \tan^{-1} (\frac{nkx}{\pi})$$

$$+ \frac{\rho^{j-1}}{(1-\rho^{j-1})} \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2} \frac{1}{x_0 \left(1 + \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{1-\rho^{j-1}} \right)^2 \right)} dx_0.$$

$\tan^{-1} (\frac{nkx}{\pi}) \approx \frac{\pi}{2} - \frac{\pi}{nkx}$ and we shall use this asymptotic approximation throughout.

$$(5.3.2) \quad P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] \approx \frac{1}{n^2 k^2 x^2}$$

$$+ \frac{\rho^{j-1}}{(1-\rho^{j-1})} \int_{\frac{nkx}{\pi}}^{\infty} \frac{1}{\pi^2} \frac{1}{x_0 \left[1 + \left(\frac{\frac{nkx}{\pi} - \rho^{j-1} x_0}{1-\rho^{j-1}} \right)^2 \right]} dx_0$$

$$= \frac{1}{n^2 k^2 x^2} - \frac{\rho^{j-1}}{(1-\rho^{j-1})} \int_{-\infty}^{\frac{nkx}{\pi}} \frac{1}{\pi^2 (1+z^2)} \frac{1}{\left(z - \frac{nkx}{\pi(1-\rho^{j-1})} \right)} dz$$

where

$$z = \frac{\rho^{j-1} x_0 - \frac{nkx}{\pi}}{\rho^{j-1} - 1}.$$

Using partial fractions to do the integration we get

$$\int_{-\infty}^{\frac{nkx}{\pi}} \frac{1}{\pi^2 (1+z^2)} \frac{1}{\left(z - \frac{nkx}{\pi(1-\rho^{j-1})} \right)} dz = \frac{1}{\pi^2} \left[\frac{1}{1 + \left(\frac{nkx}{\pi(1-\rho^{j-1})} \right)^2} \right]$$

$$\left\{ \lim_{m \rightarrow \infty} \int_{-m}^{\frac{nkx}{\pi}} \frac{1}{\left(z - \frac{nkx}{\pi(1-\rho^{j-1})} \right)} dz - \int_{-m}^{\frac{nkx}{\pi}} \frac{1}{(1+z^2)} dz \right.$$

$$\left. - \frac{nkx}{(1-\rho^{j-1})} \int_{-\infty}^{\frac{nkx}{\pi}} \frac{1}{(1+z^2)} dz \right\}.$$

After making changes of variable and letting $m \rightarrow \infty$ we arrive at

$$\frac{1}{\pi^2 \left(1 + \left[\frac{nkx}{\pi(1-\rho^{j-1})}\right]^2\right)} \left\{ \ln \left(\frac{\rho^{j-1} nkx}{(1-\rho^{j-1})\pi} \right) - \frac{nkx}{\pi(1-\rho^{j-1})} \left[\frac{\pi}{2} + \tan^{-1} \left(\frac{nkx}{\pi} \right) \right] \right\}.$$

We now substitute into (5.3.2) and use $\tan^{-1} \left(\frac{nkx}{\pi} \right) \approx \frac{\pi}{2} - \frac{\pi}{nkx}$ and

$$1 + \left(\frac{nkx}{\pi(1-\rho^{j-1})} \right)^2 \approx \left(\frac{nkx}{\pi(1-\rho^{j-1})} \right)^2$$

to get

$$\begin{aligned} P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] &\approx \frac{C_1}{n^2 k^2 x^2} + \frac{(j-1) C_2 \rho^{j-1}}{n^2 k^2 x^2} + \frac{C_3 \rho^{j-1}}{n^2 k^2 x^2} + \frac{\rho^{j-1}}{nkx} \\ &\quad - \frac{\rho^{j-1}}{(1-\rho^{j-1})} \frac{\ln nkx}{n^2 k^2 x^2} + \frac{\rho^{j-1}(1-\rho^{j-1})}{n^2 k^2 x^2} \frac{\ln(1-\rho^{j-1})\pi}{n^2 k^2 x^2} \end{aligned}$$

where C_1, C_2, C_3 are positive constants. We see that

$$\begin{aligned} n \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] &\approx \frac{C_1}{k^2 x^2} + \frac{C_4}{nk^2 x^2} + \frac{\rho}{(1-\rho)kx} \\ &\quad + O\left(\frac{\ln n}{n}\right) + \frac{C_2}{nk^2 x^2} \sum_{j=2}^n (j-1)\rho^{j-1} \end{aligned}$$

where $C_4 = C_3 \sum_{j=2}^{\infty} \rho^{j-1}$. We see that $n \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] \rightarrow \infty$ as $n \rightarrow \infty$. The condition $D'(u_n)$ fails just as in the uniform example or O'Brien's example.

5.4. A Case Where $D(u_n)$ Fails. If we consider the EARMA(1,1)

model and allow $\rho=1$ we find that $D(u_n)$ fails when $\rho=1$. In the special case where $\beta=0$ we see that $X_n = A_0 \quad \forall n$ and hence $P[M_n \leq x] = P[A_0 \leq x] = 1 - e^{-\lambda x}$. So $P[M_n \leq x] \rightarrow 1 - e^{-\lambda x}$ as $n \rightarrow \infty$ and this is not an extreme value distribution. For $1 > \beta > 0$ the result is that $D(u_n)$ fails and the limit distribution is not an extreme value type. For $\beta=1$

$X_n = \epsilon_n$ regardless of ρ and the result for i.i.d. random variables applies. We shall derive the limit distribution for $1 > \beta > 0$ and shall see that a limiting argument for β gives us the i.i.d. result when $\rho=1$.

Theorem 5.4. Let $\{X_n\}$ be an EARMA(1,1) process with $1 > \beta > 0$ and $\rho=1$. So $X_n = \beta\epsilon_n + U_n A_0$. We then have

$$\lim_{n \rightarrow \infty} P[M_n \leq \frac{\beta(x + \ln n)}{\lambda}] = \int_0^\infty e^{-e^{-x}(\beta+(1-\beta)e^{\lambda a/\beta})} \lambda e^{-\lambda a} da \stackrel{\text{def}}{=} G_\beta(x).$$

Proof. We observe that the sequence is exchangeable and that $\text{Cov}(X_i, X_j)$ is the same $\forall i \neq j$. So the condition $D(u_n)$ could not possibly be satisfied.

We now consider $P[M_n \leq \frac{\beta(x + \ln n)}{\lambda}]$. Define $M'_n = \max\{X_i \text{ for which } U_i = 0\}$ and $M_n^2 = \max\{X_i \text{ for which } U_i = 1\}$. So $M_n = \max\{M'_n, M_n^2\}$. Now $P[M_n \leq \frac{\beta(x + \ln n)}{\lambda}] = P[M'_n \leq \frac{\beta(x + \ln n)}{\lambda}, M_n^2 \leq \frac{\beta(x + \ln n)}{\lambda}]$. We let K_n be the random variable which counts the number of U_i 's = 0.

We have $P[M_n \leq \frac{\beta(x + \ln n)}{\lambda}] = E\{P[M'_n \leq \frac{\beta(x + \ln n)}{\lambda}, M_n^2 \leq \frac{\beta(x + \ln n)}{\lambda} | K_n = k, A_0 = a]\}$. Conditional on K_n, M'_n and M_n^2 are independent and hence we get

$$(5.4.1) \quad P[M_n \leq \frac{\beta(x + \ln n)}{\lambda}] = E\{P[M'_n \leq \frac{\beta(x + \ln n)}{\lambda} | K_n = k, A_0 = a]\}$$

$$P[M_n^2 \leq \frac{\beta(x + \ln n)}{\lambda} | K_n = k, A_0 = a]\}.$$

Now

$$P[M'_n \leq \frac{\beta(x + \ell_n n)}{\lambda} | K_n = k, A_0 = a] = P[M'_n \leq \frac{\beta(x + \ell_n n)}{\lambda} | K_n = k] = \left(1 - \frac{e^{-x}}{n}\right)^k$$

and

$$P[M_n^2 \leq \frac{\beta(x + \ell_n n)}{\lambda} | K_n = k, A_0 = a] = \left(1 - \frac{e^{-x}}{n} e^{\lambda a / \beta}\right)^{n-k}.$$

Now K_n has a binomial distribution with parameters n and β and we recall that $A_0 \sim \exp(\lambda)$. We therefore obtain

$$(5.4.2) \quad P[M_n \leq \frac{\beta(x + \ell_n n)}{\lambda}] = \int_0^{\frac{\beta(x + \ell_n n)}{\lambda}} \sum_{k=0}^n \binom{n}{k} \beta^k (1-\beta)^{n-k} \left(1 - \frac{e^{-x}}{n}\right)^k \left(1 - e^{-x} \frac{e^{\lambda a / \beta}}{n}\right)^{n-k} \lambda e^{-\lambda a} da.$$

sider

$$\sum_{k=0}^n \binom{n}{k} \beta^k (1-\beta)^{n-k} \left(1 - \frac{e^{-x}}{n}\right)^k \left(1 - \frac{e^{-x} e^{\lambda a / \beta}}{n}\right)^{n-k}.$$

$$a_n = \beta \left(1 - \frac{e^{-x}}{n}\right)$$

id

$$b_n = (1-\beta) \left(1 - \frac{e^{-x} e^{\lambda a / \beta}}{n}\right).$$

hen the above expression is equal to

$$\sum_{k=0}^n \binom{n}{k} a_n^k b_n^{n-k} = (a_n + b_n)^n = \left(1 - \frac{e^{-x}}{n} (\beta + (1-\beta) e^{\lambda a/\beta})\right)^n.$$

Now

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^{\frac{\beta(x + \ln n)}{\lambda}} \sum_{k=0}^n \binom{n}{k} a_n^k b_n^{n-k} \lambda e^{-\lambda a} da \\ = \int_0^{\infty} \lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} a_n^k b_n^{n-k} \lambda e^{-\lambda a} da \\ = \int_0^{\infty} \lim_{n \rightarrow \infty} (a_n + b_n)^n \lambda e^{-\lambda a} da \\ = \int_0^{\infty} \lim_{n \rightarrow \infty} \left(1 - \frac{e^{-x}}{n} (\beta + (1-\beta) e^{\lambda a/\beta})\right)^n \lambda e^{-\lambda a} da \\ = \int_0^{\infty} e^{-(\beta + (1-\beta) e^{\lambda a/\beta}) e^{-x}} \lambda e^{-\lambda a} da \end{aligned}$$

and this completes the proof.

It is interesting to note that $\lim_{\beta \rightarrow 1} e^{-(\beta + (1-\beta) e^{\lambda a/\beta}) e^{-x}} = e^{-e^{-x}}$ and since

$$\int_0^{\infty} e^{-e^{-x}} \lambda e^{-\lambda a} da = e^{-e^{-x}} \int_0^{\infty} \lambda e^{-\lambda a} da = e^{-e^{-x}}$$

we have $\lim_{\beta \rightarrow 1} G_{\beta}(x) = e^{-e^{-x}}$ and this agrees with the result for i.i.d. random variables. We note that

$$G_{\beta}(x) = e^{-\beta e^{-x}} \int_0^{\infty} e^{-(1-\beta) e^{\lambda a/\beta} e^{-x}} \lambda e^{-\lambda a} da.$$

So $G_{\beta}(x) < e^{-e^{-x}}$ since $e^{\lambda a/\beta} > 1 \quad \forall a > 0.$

Appendix 1

Computation of Integrals from Theorem 5.1

We need to show that equations (5.1.2) and (5.1.3) hold. We first consider $P[\rho^{j-1}A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2}u]$. We have

$$(A.1.1) \quad P[\rho^{j-1}A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2}u] \\ = \int_0^\infty P[Z_{j-1}^* > z - \beta v - \rho^{j-2}u - \rho^{j-1}w] \lambda e^{-\lambda w} dw.$$

We observe that

$$P[Z_{j-1}^* > z - \beta v - \rho^{j-2}u - \rho^{j-1}w] = 1 \quad \text{if } w > \frac{z - \beta v - \rho^{j-2}u}{\rho^{j-1}} \\ = e^{-\lambda(z - \beta v - \rho^{j-2}u - \rho^{j-1}w)} \quad \text{otherwise.}$$

Evaluating the right hand side of equation (A.1.1) we get

$$(A.1.2) \quad P[\rho^{j-1}A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2}u] = 1 \quad \text{if } u > \frac{z - \beta v}{\rho^{j-2}} \\ = e^{-\lambda \frac{(z - \beta v - \rho^{j-2}u)}{\rho^{j-1}}} + \frac{1}{(1 - \rho^{j-1})} e^{-\lambda(z - \beta v - \rho^{j-2}u)} \\ \left[1 - e^{-\lambda \frac{(1 - \rho^{j-1})}{\rho^{j-1}} (z - \beta v - \rho^{j-2}u)} \right] \quad \text{otherwise.}$$

When $v \geq (\beta - \rho^{j-2}) \frac{z}{\beta}$, $\frac{z - \beta v}{\rho^{j-2}} \leq \frac{z}{\beta}$. So in this case $u \geq \frac{(z - \beta v)}{\rho^{j-2}}$.

This will be true for all v if $\beta \leq \rho^{j-2}$ (i.e. $j-2 \leq \frac{\log \beta}{\log \rho}$).

Otherwise we have

$$\begin{aligned}
(A.1.3) \quad & \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} P[\rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] \lambda e^{-\lambda u} du dv \\
&= \int_0^{\frac{(\beta - \rho^{j-2})z}{\beta^2}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\frac{z - \beta v}{\rho^{j-2}}} \left\{ e^{-\lambda \frac{(z - \beta v - \rho^{j-2} u)}{\rho^{j-1}}} + \frac{e^{-\lambda(z - \beta v - \rho^{j-2} u)}}{(1 - \rho^{j-1})} \right. \\
&\quad \left. \left(1 - e^{-\lambda \left(\frac{1 - \rho^{j-1}}{\rho^{j-1}} \right) (z - \beta v - \rho^{j-2} u)} \right) \right\} \lambda e^{-\lambda u} du dv \\
&+ \int_0^{\frac{(\beta - \rho^{j-2})z}{\beta^2}} \lambda e^{-\lambda v} \int_{\frac{z - \beta v}{\rho^{j-2}}}^{\infty} \lambda e^{-\lambda u} du dv \\
&+ \int_{\frac{(\beta - \rho^{j-2})z}{\beta^2}}^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} \lambda e^{-\lambda u} du dv .
\end{aligned}$$

For $0 \leq j-2 \leq \frac{\log \beta}{\log \rho}$ we get

$$\int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} \lambda e^{-\lambda u} du dv = e^{-\lambda z / \beta} (1 - e^{-\lambda z / \beta}) < e^{-\lambda z / \beta} .$$

The contribution to (5.1.2) is then less than

$$n \sum_{j=2}^{\ell+2} e^{-\lambda u_{nk}(x)/\beta} = n \sum_{j=2}^{\ell+2} \frac{e^{-x/\beta}}{(nk)^{1/\beta}}$$

where $\ell \leq \frac{\log \beta}{\log \rho}$ and $\ell+1 > \frac{\log \beta}{\log \rho}$. Now

$$n \sum_{j=2}^{l+2} \frac{e^{-x/\beta}}{(nk)^{1/\beta}} < \frac{\log \beta}{\log \rho} \frac{e^{-x/\beta}}{k^{1/\beta}} \frac{1}{n^{(1-\beta)/\beta}} \rightarrow 0$$

as $n \rightarrow \infty$.

In the case when $\beta > \rho^{j-2}$ we need to consider the three terms given by (A.1.3). For the third term we have

$$\int_{\frac{(\beta-\rho^{j-2})z}{\beta^2}}^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\infty} \lambda e^{-\lambda u} du dv = e^{-\lambda z/\beta} \left[e^{\frac{-\lambda(\beta-\rho^{j-2})z}{\beta^2}} - e^{-\lambda z/\beta} \right]$$

$$\leq e^{-\lambda z/\beta} e^{-\lambda z/\beta} e^{\lambda \rho^{j-2} z/\beta^2}$$

So

$$\overline{\lim}_{n \rightarrow \infty} n \sum_{j=3}^n \frac{e^{-2x/\beta}}{(nk)^{1/\beta}} \frac{e^{\rho^{j-2} x/\beta^2}}{(\beta-\rho^{j-2})/\beta^2}$$

is the maximum contribution of this term to (5.1.2) and this is bounded above by

$$\overline{\lim}_{n \rightarrow \infty} \frac{C_1}{k^{1/\beta} n^{(1-\beta)/\beta}} \sum_{j=2}^n \frac{1}{(nk)(1/\beta - \rho^{j-2}/\beta^2)}$$

where C_1 is some positive constant. This limit is zero.

For the second term in (A.1.3) we get

$$\int_0^{\frac{(\beta-\rho^{j-2})z}{\beta^2}} \lambda e^{-\lambda v} \int_{\frac{z-\beta v}{\rho^{j-2}}}^{\infty} \lambda e^{-\lambda u} du dv = \frac{\rho^{j-2}}{(\beta-\rho^{j-2})}$$

$$\{e^{-2\lambda z/\beta} e^{\lambda \rho^{j-2} z/\beta^2} - e^{-\lambda z/\rho^{j-2}}\} < \frac{\rho^{j-2}}{(\beta-\rho^{j-2})} e^{-2\lambda z/\beta} e^{\lambda \rho^{j-2} z/\beta^2}$$

when $j-2 > \ell$. So this term contributes less than

$$n \sum_{j=3}^n \frac{C_2 \rho^{j-2}}{(nk) (2/\beta - \rho^{j-2}/\beta^2)} = \frac{C_2}{n^{(1-\beta)/\beta}} \sum_{j=2}^n \frac{\rho^{j-2}}{(nk) (1/\beta - \rho^{j-2}/\beta^2)}$$

where C_2 is a positive constant and clearly this bound also tends to zero as $n \rightarrow \infty$.

We now consider the first term in (A.1.3). Since the following equality holds

$$\begin{aligned} e^{-\lambda \left(\frac{z-\beta v-\rho^{j-2}u}{\rho^{j-1}} \right)} + \frac{1}{(1-\rho^{j-1})} e^{-\lambda(z-\beta v-\rho^{j-2}u)} \left[1 - e^{-\lambda \left(\frac{1-\rho^{j-1}}{\rho^{j-1}} \right) (z-\beta v-\rho^{j-2}u)} \right] \\ = \frac{1}{(1-\rho^{j-1})} e^{-\lambda(z-\beta v-\rho^{j-2}u)} - \frac{\rho^{j-1}}{(1-\rho^{j-1})} e^{-\lambda \left(\frac{z-\beta v-\rho^{j-2}u}{\rho^{j-1}} \right)} \end{aligned}$$

we can bound this part of the integrand by $\frac{1}{(1-\rho^{j-1})} e^{-\lambda(z-\beta v-\rho^{j-2}u)}$.

Now

$$\int_0^{\frac{(\beta-\rho^{j-2})z}{\beta^2}} \lambda e^{-\lambda v} \int_{\frac{z}{\beta}}^{\frac{(z-\beta v)}{\rho^{j-2}}} \frac{e^{-\lambda(z-\beta v-\rho^{j-2}u)}}{(1-\rho^{j-1})} \lambda e^{-\lambda u} du dv$$

is bounded above by

$$\frac{1}{(1-\beta)} \frac{e^{-\lambda z} e^{-\lambda(1-\rho^{j-2})z/\beta}}{(1-\rho^{j-1}) (1-\rho^{j-2})}.$$

The contribution from this term is no more than

$$\lim_{n \rightarrow \infty} n C_3 \sum_{j=3}^n \frac{1}{nk} \frac{1}{(nk)(1-\rho^{j-2})/\beta} = \lim_{n \rightarrow \infty} \frac{C_3}{k} \sum_{j=3}^n \frac{1}{(nk)(1-\rho^{j-2})/\beta}$$

$$< \frac{C_3}{k^{2-\rho}} \lim_{n \rightarrow \infty} \sum_{j=3}^n \frac{1}{n(1-\rho^{j-2})} < \frac{3C_3}{k^{2-\rho}}$$

where C_3 is a positive constant. So this term contributes at most $o(\frac{1}{k})$ and (5.1.2) is verified.

We must now verify (5.1.3). We consider

$$\int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_0^{\frac{z}{\beta}} P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] \lambda e^{-\lambda u} du dv.$$

When $\beta v > z - \rho^{j-2} u$, $\rho^{j-1} A_0 + Z_{j-1}^* > 0 > z - \beta v - \rho^{j-2} u$. So when $u > (z - \beta v)/\rho^{j-2}$, $P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] = P[A_0 > z - \beta u] = e^{-\lambda(z - \beta u)}$.

We shall consider the following two cases: Case i) $\beta \leq \rho^{j-2}$, and Case ii) $\beta > \rho^{j-2}$. In Case i)

$$\frac{z}{\beta} > v \geq 0 \geq \frac{z(\beta - \rho^{j-2})}{\beta^2}$$

and this implies that $\frac{z}{\beta} > \frac{z - \beta v}{\rho^{j-2}} \geq 0$. So for $\frac{z}{\beta} > u > \frac{z - \beta v}{\rho^{j-2}}$, $P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] = e^{-\lambda(z - \beta u)}$.

We now consider Case i). We have $\rho^{j-2} \geq \beta \Leftrightarrow (j-2) \log \rho \geq \log \beta$
 $\Leftrightarrow j-2 \leq \frac{\log \beta}{\log \rho}$ (note $\log \rho < 0$). We split the integral into two pieces integrating u from $\frac{z - \beta v}{\rho^{j-2}}$ to $\frac{z}{\beta}$ in the first piece.

$$\begin{aligned}
& \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} \int_{\frac{z-\beta v}{\rho^{j-2}}}^{\frac{z}{\beta}} \lambda e^{-\lambda u} e^{-\lambda(z-\beta u)} du dv \\
&= \int_0^{\frac{z}{\beta}} \lambda \frac{e^{-\lambda v} e^{-\lambda z}}{(1-\beta)} \left[e^{-\lambda(1-\beta) \left(\frac{z-\beta v}{\rho^{j-2}} \right)} - e^{-\frac{\lambda(1-\beta)z}{\beta}} \right] dv \\
&< \frac{e^{-\lambda z}}{(1-\beta)} e^{-\lambda(1-\beta)z/\rho^{j-2}} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda v} e^{\lambda\beta(1-\beta)v/\rho^{j-2}} dv \\
&= \frac{e^{-\lambda z}}{(1-\beta)} e^{-\lambda(1-\beta)z/\rho^{j-2}} \frac{\rho^{j-2}}{(\rho^{j-2} + \beta^2 - \beta)} \left(1 - e^{-\lambda \left(\frac{\rho^{j-2} + \beta^2 - \beta}{\rho^{j-2}} \right) \frac{z}{\beta}} \right) \\
&< \frac{\rho^{j-2}}{(1-\beta)(\rho^{j-2} + \beta^2 - \beta)} e^{-\lambda z} e^{-\lambda(1-\beta)z/\rho^{j-2}} .
\end{aligned}$$

This part contributes less than

$$C_4 n \sum_{j=2}^{l+2} \frac{\rho^{j-2}}{nk (nk) (1-\beta)/\rho^{j-2}}$$

which tends to zero as $n \rightarrow \infty$. C_4 is a positive constant. In Case i)

we need to also compute

$$\int_0^{\frac{z}{\beta}} \int_0^{\frac{z-\beta v}{\rho^{j-2}}} P[A_0 > z-\beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z-\beta v - \rho^{j-2} u] \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv .$$

We shall see that the computation is the same as in Case ii) so we shall now consider Case ii).

In Case ii) $\beta > \rho^{j-2}$ so $j-2 \geq k+1$.

$$P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u]$$

$$= \int_0^\infty P[A_0 > z - \beta u, A_0 > \frac{z - \beta v - \rho^{j-2} u - w}{\rho^{j-1}}] \lambda e^{-\lambda w} dw.$$

When $z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u) \leq 0$, we have $P[A_0 > z - \beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z - \beta v - \rho^{j-2} u] = e^{-\lambda(z - \beta u)}$ and when $z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u) > 0$ we get that this probability is equal to

$$\begin{aligned} & \int_{z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u)}^\infty \lambda e^{-\lambda(z - \beta u)} e^{-\lambda w} dw \\ & + \int_0^{z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u)} e^{-\lambda(\frac{z - \beta v - \rho^{j-2} u - w}{\rho^{j-1}})} \lambda e^{-\lambda w} dw \\ & = e^{-\lambda(z - \beta u)} e^{-\lambda(z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u))} \\ & + \frac{\rho^{j-1}}{(1 - \rho^{j-1})} e^{-\lambda(\frac{z - \beta v - \rho^{j-2} u}{\rho^{j-1}})} \left\{ e^{\lambda(\frac{1 - \rho^{j-1}}{\rho^{j-1}}) [(1 - \rho^{j-1})z - \beta v - \rho^{j-2}(1 - \beta\rho)u]} - 1 \right\}. \end{aligned}$$

For $z - \beta v - \rho^{j-2} u - \rho^{j-1}(z - \beta u) \leq 0$, $v > \frac{(1 - \rho^{j-1})z - \rho^{j-2}(1 - \beta\rho)u}{\beta}$ and

we get

$$\begin{aligned}
& \int_0^{\frac{z}{\beta}} \int_0^{\frac{z}{\beta}} \frac{\lambda e^{-\lambda v} \lambda e^{-\lambda u} e^{-\lambda(z-\beta u)}}{(1-\rho^{j-1})z - \rho^{j-2}(1-\beta\rho)u} dv du \\
&= e^{-\lambda z} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda(1-\beta)u} \left\{ e^{-\lambda \left[\frac{(1-\rho^{j-1})z - \rho^{j-2}(1-\beta\rho)u}{\beta} \right]} - e^{-\lambda z/\beta} \right\} du \\
&< e^{-\lambda z} e^{-\lambda(1-\rho^{j-1})z/\beta} \int_0^{\frac{z}{\beta}} \lambda e^{-\lambda \left(1-\beta-\frac{\rho^{j-2}}{\beta} + \rho^{j-1} \right)u} du.
\end{aligned}$$

For large j , $1-\beta-(\rho^{j-2}/\beta)(1-\beta\rho) > 0$ and for small j , $1-\beta-(\rho^{j-2}/\beta)(1-\beta\rho) \leq 0$.

Let ℓ_1 be such that $1-\beta-(\rho^{\ell_1}/\beta)(1-\beta\rho) > 0$ but $1-\beta-(\rho^{\ell_1-1}/\beta)(1-\beta\rho) \leq 0$.

When $j-2 \geq \ell_1$ we get the bound

$$\frac{e^{-\lambda z} e^{-\lambda(1-\rho^{j-1})z/\beta}}{(1-\beta-\frac{\rho^{j-2}}{\beta} + \rho^{j-1})}$$

and so this term contributes less than

$$\begin{aligned}
\overline{\lim}_{n \rightarrow \infty} n C_5 \sum_{j=2}^n \frac{1}{n_k (n_k) (1-\rho^{j-1})/\beta} &< \overline{\lim}_{n \rightarrow \infty} \frac{C_5}{k \left(1 + \frac{(1-\rho)}{\beta}\right)} \sum_{j=2}^n \frac{1}{n \frac{(1-\rho^{j-1})}{\beta}} \\
&< \frac{C_5}{k \left(1 + \frac{(1-\rho)}{\beta}\right)} \overline{\lim}_{n \rightarrow \infty} \sum_{j=2}^n \frac{1}{n (1-\rho^{j-1})} < \frac{2 C_5}{k \left(1 + \frac{(1-\rho)}{\beta}\right)} = o\left(\frac{1}{k}\right).
\end{aligned}$$

for $j-2 < \ell_1$ we get the bound $\frac{\lambda z}{\beta} e^{-\lambda z/\beta} e^{-\lambda(1-\frac{\rho^{j-2}}{\beta})z/\beta}$ and the contribution is less than

$$C_6 \left(\frac{|x|}{\beta} + \frac{\ell_n(nk)}{\beta} \right) \frac{n}{(nk)^{1/\beta}} \sum_{j=\ell+3}^{\ell_1+1} \frac{1}{[nk]^{(1-\rho^{j-2})/\beta}} \frac{C_6}{(k)^{1/\beta}} \left(\frac{|x|}{\beta} + \frac{\ell_n(nk)}{\beta} \right) \frac{\ell_1}{n^{(1-\gamma)/\beta}}$$

where C_6 is a positive constant, and $\gamma = \frac{\rho^{\ell+1}}{\beta}$ and this tends to zero as $n \rightarrow \infty$ since by L'Hospital's rule

$$\lim_{s \rightarrow \infty} \frac{\ell_n s}{s^{(1-\gamma)/\beta}} = \lim_{s \rightarrow \infty} \frac{\frac{1}{s}}{\left(\frac{1-\gamma}{\beta}\right) s^{(1-\gamma)/\beta}} = \lim_{s \rightarrow \infty} \frac{1}{\left(\frac{1-\gamma}{\beta}\right) s^{(1-\gamma)/\beta}} = 0.$$

When $z - \beta v - \rho^{j-2} u - \rho^{j-1} (z - \beta u) > 0$ we have two terms. For the first term we get

$$\int_0^{\frac{z}{\beta}} \int_0^{\frac{(1-\rho^{j-1})z - \rho^{j-2}(1-\beta\rho)u}{\beta}} e^{-\lambda(z-\beta u)} e^{-\lambda((1-\rho^{j-1})z - \beta v - \rho^{j-2}(1-\beta\rho)u)} \lambda e^{-\lambda v} \lambda e^{-\lambda u}$$

$$dv du = e^{-\lambda z} e^{-\lambda(1-\rho^{j-1})z} \int_0^{\frac{z}{\beta}} \int_0^{\frac{(1-\rho^{j-1})z - \rho^{j-2}(1-\beta\rho)u}{\beta}} \lambda e^{-\lambda(1-\beta)v}$$

$$\lambda e^{-\lambda(1-\beta-\rho^{j-2}(1-\beta\rho))u} dv du$$

$$< \frac{e^{-\lambda z} e^{-\lambda(1-\rho^{j-1})z}}{(1-\beta)(1-\beta-\rho^{j-2}(1-\beta\rho))}$$

for $j-2 \geq \ell_2$ where ℓ_2 is such that $1-\beta-\rho^{\ell_2}(1-\beta\rho) > 0$ and $1-\beta-\rho^{\ell_2-1}(1-\beta\rho) \leq 0$ and for $j-2 < \ell_2$ we can use the bound

$$\frac{e^{-\lambda z/\beta}}{(1-\beta)} e^{-\lambda(1-\rho^{j-2})z} \frac{\lambda z}{\beta}.$$

So for $j \geq \ell_2 + 2$ this term contributes less than

$$\begin{aligned} \lim_{n \rightarrow \infty} n C_7 \sum_{j=2}^n \frac{1}{nk (nk)^{1-\rho^{j-1}}} &= \lim_{n \rightarrow \infty} \frac{C_7}{k} \sum_{j=2}^n \frac{1}{(nk)^{1-\rho^{j-1}}} \\ &< \frac{C_7}{k^{2-\rho}} \lim_{n \rightarrow \infty} \sum_{j=2}^n \frac{1}{n^{1-\rho^{j-1}}} < \frac{2 C_7}{k^{2-\rho}} = o\left(\frac{1}{k}\right) \end{aligned}$$

where C_7 is a positive constant.

For $j < \ell_2 + 2$ we get the bound

$$nC_8 \sum_{j=\ell+3}^{\ell_2+1} \frac{\{\ln(nk) + x\}}{(nk)^{1/\beta} (nk)^{1-\rho^{j-2}} \frac{1-\rho^{j-2}}{\beta}} \leq \frac{\ell_2 C_8}{k^{1/\beta}} \frac{\{\ln(nk) + |x|\}}{\frac{1-\rho^{j-2}}{\beta}},$$

where C_8 is a positive constant. This bound goes to zero as $n \rightarrow \infty$ also by L'Hospital's rule.

Let

$$g_j(z, u, v) = \frac{\rho^{j-1}}{(1-\rho^{j-1})} e^{-\lambda(z-\beta v-\rho^{j-2}u)/\rho^{j-1}}$$

$$\cdot \{e^{\lambda((1-\rho^{j-1})/\rho^{j-1})[(1-\rho^{j-1})z-\beta v-\rho^{j-2}(1-\beta\rho)u]} - 1\}.$$

We want to bound

$$(A.1.4) \quad \int_0^{\frac{z}{\beta}} \int_0^{\frac{(1-\rho^{j-1})z - \rho^{j-2}(1-\beta\rho)u}{\beta}} g_j(z, u, v) \lambda e^{-\lambda u} \lambda e^{-\lambda v} dv du .$$

We have the following inequality

$$(A.1.5) \quad g_j(z, u, v) < \frac{\rho^{j-1}}{1-\rho} e^{-\lambda z/\rho^{j-1}} e^{\lambda(1-\rho^{j-1})^2 z/\rho^{j-1}} e^{\lambda\beta v} e^{\lambda\beta u} e^{\lambda\rho^{j-2}u} e^{-\lambda\beta\rho^{j-1}u} .$$

We can bound (A.1.4) above by

$$(A.1.6) \quad \int_0^{\frac{z}{\beta}} \int_0^{\frac{z}{\beta}} \frac{\rho^{j-1}}{1-\rho} e^{-\lambda z/\rho^{j-1}} e^{\lambda(1-\rho^{j-1})^2 z/\rho^{j-1}} e^{\lambda\beta v} e^{\lambda\beta u} e^{\lambda\rho^{j-2}u} e^{-\lambda\beta\rho^{j-1}u} \lambda e^{-\lambda u} \lambda e^{-\lambda v} dv du .$$

(A.1.6) is bounded above by

$$\frac{\rho^{j-1}}{1-\rho} \frac{e^{-\lambda z/\rho^{j-1}} e^{\lambda(1-\rho^{j-1})^2 z/\rho^{j-1}}}{(1-\beta)(1-\rho^{j-2}-\beta+\beta\rho^{j-1})}$$

when $1-\beta-\rho^{j-2}(1-\beta\rho) > 0$ (i.e. $j-2 \geq \ell_2$). So this term contributes less than

$$nC_9 \sum_{j=2}^n \frac{\rho^{j-1}}{(nk)^{2-\rho^{j-1}}} < \frac{C_9}{k} \sum_{j=2}^n \frac{\rho^{j-1}}{(nk)^{1-\rho^{j-1}}}$$

and this tends to zero as $n \rightarrow \infty$. C_9 is a positive constant. For

$j-2 < \ell_2$ we get the bound

$$\frac{\rho^{j-1} \lambda z e^{-\lambda z} e^{-\lambda(1-\rho^{j-2})z/\beta}}{(1-\rho)(1-\beta)\beta}$$

Now this contributes less than

$$\begin{aligned} nC_{10} \sum_{j=3}^{\ell_2+1} \frac{\rho^{j-1} \{\ell_n(nk) + x\}}{nk(nk)(1-\rho^{j-2})/\beta} &= C_{10} \sum_{j=3}^{\ell_2+1} \frac{\rho^{j-1} \{\ell_n(nk) + x\}}{k(nk)(1-\rho^{j-2})/\beta} \\ &< C_{10} \frac{\ell_n(nk)}{k(nk)(1-\rho)/\beta} \sum_{j=3}^{\ell_2+1} \rho^{j-1} + \frac{C_{10} |x|}{k(nk)(1-\rho)/\beta} \sum_{j=3}^{\ell_2+1} \rho^{j-1} \end{aligned}$$

where C_{10} is a positive constant. Both terms tend to zero as $n \rightarrow \infty$ and so Case ii) is completed. For Case i) we need to get an upper bound for

$$\int_0^{\frac{z}{\beta}} \int_0^{\frac{z-\beta v}{\rho^{j-2}}} P[A_0 > z-\beta u, \rho^{j-1} A_0 + Z_{j-1}^* > z-\beta v - \rho^{j-2} u] \lambda e^{-\lambda u} \lambda e^{-\lambda v} du dv.$$

Since $\frac{z-\beta v}{\rho^{j-2}} < \frac{z}{\beta}$ we see that when $j > 2$ the upper bounds in Case ii) apply. Note that the integrand is the same as in Case ii). When $j=2$ a better bound, $\frac{C_{11}}{n(1-\rho)(1-\beta)}$ with $C_{11} > 0$, is obtained. This bound tends to zero as $n \rightarrow \infty$.

We now can conclude that (5.1.3) holds. Since (5.1.2) and (5.1.3) hold, we conclude that

$$\lim_{n \rightarrow \infty} n \sum_{j=2}^n P[X_1 > u_{nk}(x), X_j > u_{nk}(x)] = o\left(\frac{1}{k}\right).$$

Appendix 2

Simulation of the Cauchy Process

A simulation of the Cauchy AR(1) process was done by Professor Faith. A sequence of length one hundred was generated and replicated 2000 times for values of ρ ranging from .1 to .9 in increments of .2. The maximum was determined in each case and for each ρ an estimate of $P[M_n \leq nx]$ was obtained using the empirical distribution of M_n . For i.i.d. Cauchy random variables the limit distribution of M_n^* using this normalization is $e^{-1/\pi x}$. We conjecture that the limit for $P[M_n \leq nx]$ is $e^{-(1-\rho)/(\pi x)}$. When $n = 100$, $P[M_n^* \leq nx]$ is reasonably well approximated by its limiting probability. Table A gives the lower quartile, median, and upper quartile values for x obtained from the empiric distribution and gives a comparison to the values $e^{-(1-\rho)/\pi x}$. Numbers are rounded off to the second decimal place.

TABLE A

		0	.1	.3	.5	.7	.9
sample	lower quartile	-	.20	.16	.11	.07	.02
$e^{-(1-\rho)/\pi x}$	lower quartile	.23	.21	.16	.11	.07	.02
sample	median	-	.40	.33	.22	.14	.05
$e^{-(1-\rho)/\pi x}$	median	.46	.41	.32	.23	.14	.05
sample	upper quartile	-	.97	.79	.54	.35	.12
$e^{-(1-\rho)/\pi x}$	upper quartile	1.11	1.00	.77	.55	.33	.11

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