MEASURING DISPERSION BY DRIVING RF VOLTAGE*

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Traditionally we have been measuring the energy dispersion η -function using the following procedure: The rf cavity frequency ω_{rf} is manually changed slightly by an amount $\Delta\omega_{rf}$. As a result of this change, the equilibrium beam energy is shifted by $\Delta E/E_0 = -\Delta\omega_{rf}/\alpha\omega_{rf}$, where α is the momentum compaction factor. The difference between closed orbits before and after the rf frequency change at a position monitor, Δx_m , is then related to the energy dispersion function at the monitor, η_m , by $\Delta x_m = \eta_m \Delta E/E_0$. By measuring Δx_m for a known change $\Delta\omega_{rf}$, one obtains the value of η_m .

The accuracy of the above procedure is curtailed by the limited range of $\Delta E/E_0$ that can be tolerated by the beam and by the accuracy of the position monitor readings. It seems there is another, possibly more accurate way of measuring the η -function by amplitude modulating the rf cavity voltage at some frequency ω_d and looking at the response of the beam at the monitor at the same frequency ω_d . For a given allowed range of $\Delta E/E_0$, this method can potentially provide a more accurate η -function measurement because, first of all, the beam response at a particular relatively high frequency ω_d can possibly be picked up more sensitively using narrow-band techniques than the static changes at a position monitor; and secondly, ω_d can be chosen to be near the betatron or the synchrotron frequencies to give a pronounced response of the position monitor signal. The feasibility of this method is looked at in this note.

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In the following, for simplicity, we assume there is only one rf cavity in the storage ring and it is this cavity that we will be driving. We also ignore the vertical motion. The cavity voltage is driven so that the additional energy change due to the driving voltage is, with $\delta = \Delta E/E_0$, given by

$$\Delta\delta = \hat{\Delta\delta} e^{i\omega_d t} \quad (1)$$

We define a 4-dimensional vector

$$X = \begin{bmatrix} x \\ x' \\ z \\ \delta \end{bmatrix} \quad (2)$$

where x and z are the horizontal and longitudinal displacements of the bunch center relative to the ideal trajectory; x' is the angle between the bunch's direction of motion relative to the ideal trajectory. Let T_0 be the 4×4 matrix that transforms X for one revolution around the ring starting and ending at the cavity:

$$\begin{bmatrix} r_{11} & r_{12} & 0 & \eta_c - r_{11}\eta_c - r_{12}\eta_c' \\ r_{21} & r_{22} & 0 & \eta_c' - r_{21}\eta_c - r_{22}\eta_c' \\ a & b & 1 & -2\pi R - a\eta_c - b\eta_c' \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

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where

$$r_{11} = \cos \mu_x + \alpha_c \sin \mu_x$$

$$r_{12} = \beta_c \sin \mu_x$$

$$r_{21} = -\frac{1}{\beta_c}(1 + \alpha_c^2) \sin \mu_x$$

$$r_{22} = \cos \mu_x - \alpha_c \sin \mu_x$$

$$a = r_{21} \eta_c - r_{11} \eta_c' + \eta_c'$$

$$b = -r_{12} \eta_c' + r_{22} \eta_c - \eta_c$$

$$\alpha_c, \beta_c, \eta_c, \eta_c' = \alpha\text{-function, } \beta\text{-function, } \eta\text{-function} \\ \text{and } \eta'\text{-function at the cavity}$$

$$R = \text{machine radius}$$

$$\mu_x = 2\pi\nu_x$$

$$\nu_x = \text{betatron tune.}$$

The transformation matrix for the cavity without the drive is

$$T_{\text{cav}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\mu_s^2}{2\pi\alpha R} & 1 \end{bmatrix} \quad (4)$$

$$\mu_s = 2\pi\nu_s$$

$$\nu_s = \text{synchrotron tune}$$

With the driving voltage (1) applied to the beam at the cavity, the beam response in a steady state can be written as

$$X(n) = \hat{\Delta\delta} X_0 e^{in\mu_d} \quad (5)$$

where $X(n)$ is the state of the bunch at the exit of the cavity in the n -th revolution, $\mu_d = 2\pi \frac{\omega_d}{\omega_o}$ (ω_o = revolution frequency) is the phase advance per turn of the drive.

The condition that $X(n)$ is the steady state requires

$$X(n) = T_{cav} T_o X(n-1) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{in\mu_d} \end{bmatrix} \quad (6)$$

Here we have assumed that the phase advance of the drive from turn to turn is always given by μ_d . In reality, it is modified by the time of arrival of the bunch. However, this is an effect which is second order in $\Delta\hat{\delta}$ and a computer tracking program written to study this effect indeed shows that this effect can be ignored.

Substituting (5) into (6), one obtains

$$X_o = e^{-i\mu_d} T_{cav} T_o X_o + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

or, solving for X_o ,

$$X_o = \left(1 - e^{-i\mu_d} T_{cav} T_o \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (8)$$

Substituting (3), (4) into (8), one obtains after some algebra,

$$\mathbf{x}_0 = \begin{bmatrix} x_0 \\ x_0' \\ z_0 \\ \delta_0 \end{bmatrix} \quad (9)$$

with

$$\begin{aligned} x_0 &= \left\{ (1 - \cos \mu_d) \sin \mu_x (\beta_c \eta_c' + \alpha_c \eta_c) + i \sin \mu_d (1 - \cos \mu_x) \eta_c \right\} / D \\ x_0' &= \left\{ i \sin \mu_d (1 - \cos \mu_x) \eta_c' - \frac{1}{\beta_c} [\eta_c (1 + \alpha_c^2) + \alpha_c \beta_c \eta_c'] (1 - \cos \mu_d) \sin \mu_x \right\} / D \\ z_0 &= \left\{ (\cos \mu_x - \cos \mu_d) 2\pi \alpha R - i \sin \mu_d \sin \mu_x \left[\frac{\eta_c^2 + (\alpha_c \eta_c + \beta_c \eta_c')^2}{\beta_c} \right] \right\} / D \\ \delta_0 &= (1 - e^{i\mu_d}) (\cos \mu_x - \cos \mu_d) / D \\ D &= 2(\cos \mu_d - \cos \mu_x) \left[\cos \mu_d - (1 - \frac{1}{2} \mu_s^2) \right] \\ &\quad + \frac{\mu_s^2 \sin \mu_x}{2\pi \alpha R} \left[\frac{\eta_c^2 + (\alpha_c \eta_c + \beta_c \eta_c')^2}{\beta_c} \right] (1 - \cos \mu_d) \end{aligned} \quad (10)$$

The denominator D shows the resonance behavior as $\cos \mu_d$ approaches $\cos \mu_x$ and $1 - \frac{1}{2} \mu_s^2 \approx \cos \mu_s^{**}$. The second term in D is the synchro-betatron coupling term. It slightly shifts the resonant tunes away from ν_x and ν_s .

Note that if $\eta_c = 0$ and $\eta_c' = 0$, we have

$$x_0 = 0$$

$$x_0' = 0$$

(**) The fact that this approximation has to be made can be traced back to Eq. (4), for which a better representation of the matrix element $\mu_s^2/2\pi \alpha R$ is $(1 - \cos \mu_s)/\pi \alpha R$.

$$z_0 = -\pi\alpha R / (\cos\mu_d - \cos\mu_s)$$

$$\delta_0 = - (1 - e^{i\mu_d}) / (\cos\mu_d - \cos\mu_s) \quad (11)$$

which shows that the drive does not affect the horizontal motion and that only the synchrotron resonance is driven if the η -function and its derivative are zero at the cavity.

If η_c and η_c' are non-zero, then both the betatron coordinates x , x' and the synchrotron coordinates z , δ are driven resonantly near $\mu_d = \mu_s$. Near $\mu_d = \mu_x$, on the other hand, the response in the synchrotron coordinates is much weaker than that in the betatron coordinates. Indeed, near $\mu_d = \mu_x$, the resonant response in z is weaker than that in x by roughly a factor of η_c/β_c , and the response in δ is completely negligible. These effects are more explicitly shown in Fig. 1.

In Fig. 1, we have plotted the amplitudes of x_0 , x_0' , z_0 and δ_0 versus the tune of the drive $\nu_d = \mu_d/2\pi$. The other parameters for Fig. 1 are chosen to be $\nu_s = 0.05$, $\nu_x = 0.2$, $\beta_c = 15$ m, $\eta_c = 0.1$ m, $2\pi\alpha R = 6.6$ m, $\eta_c' = 0$ and $\alpha_c = 0$.

To measure the η -function at a monitor, the x response of the beam is measured at the monitor. In the steady state, the x response at the monitor is obtained by transporting the vector $X(n)$ of Eq. (5) from the cavity to the monitor:

$$x_m(n) = \Delta\hat{\delta} e^{in\mu_d} [R_{11} x_0 + R_{12} x_0' + (\eta_m - R_{11}\eta_c - R_{12}\eta_c')\delta_0] \quad (12)$$

where

$$R_{11} = \sqrt{\frac{\beta_m}{\beta_c}} [\cos \Delta\psi + \alpha_c \sin \Delta\psi]$$

$$R_{12} = \sqrt{\beta_c \beta_m} \sin \Delta\psi$$

$$R_{21} = - \frac{1}{\sqrt{\beta_c \beta_m}} \left[(1 + \alpha_c \alpha_m) \sin \Delta\psi - (\alpha_c - \alpha_m) \cos \Delta\psi \right]$$

$$R_{22} = \sqrt{\frac{\beta_c}{\beta_m}} \left[\cos \Delta\psi - \alpha_m \sin \Delta\psi \right] \quad -$$

$\alpha_m, \beta_m, \eta_m = \alpha$ -function, β -function and η -function at the monitor

$\Delta\psi =$ betatron phase advance from cavity to monitor

or more explicitly by substituting (10) into (12),

$$x_m(n) = \Delta \hat{\phi} e^{i n \mu_d} \left[(\eta_c, \eta_c' \text{ term}) + (\eta_m \text{ term}) \right] \quad (13)$$

where

$$(\eta_c, \eta_c' \text{ term}) = (1 - \cos \mu_d) \left\{ \eta_c \sqrt{\frac{\beta_m}{\beta_c}} \left[e^{i \mu_d} (\cos \Delta\psi + \alpha_c \sin \Delta\psi) \right. \right. \\ \left. \left. - \cos(\mu_x - \Delta\psi) + \alpha_c \sin(\mu_x - \Delta\psi) \right] \right. \\ \left. + \eta_c' \sqrt{\beta_c \beta_m} \left[e^{i \mu_d} \sin \Delta\psi + \sin(\mu_x - \Delta\psi) \right] \right\} / D$$

and

$$(\eta_m \text{ term}) = \eta_m (1 - e^{i \mu_d}) (\cos \mu_x - \cos \mu_d) / D$$

The $(\eta_c, \eta_c' \text{ term})$ in (13) exhibits resonances as ν_d approaches both ν_x and ν_s . The $(\eta_m \text{ term})$ exhibits only a resonance near $\nu_d = \nu_s$ and not near $\nu_d = \nu_x$ due to the factor $(\cos \mu_x - \cos \mu_d)$ in the numerator. These behaviors have been already shown in Fig. 1.

The resonance behavior of the two terms in (13) unfortunately means that the $(\eta_c, \eta_c' \text{ term})$ will contaminate the monitor signal for all values of ν_d including near both resonances. To obtain information of η_m , it is

necessary therefore to minimize η_c and η_c' at the cavity. The procedure of measuring η -function by driving the rf cavity will then be the following:

1. Empirically adjust η_c and η_c' at the driven cavity so that there appears no resonant response in the x-signal at the monitor. This will eliminate the (η_c, η_c') term in (13).(**)
2. Then drive the beam near $v_d = v_s$. The amplitude of the x-signal received at the monitor is related to the drive by

$$\hat{x}_m = \left| \Delta\delta \frac{\mu_s \eta_m}{2(\cos\mu_d - \cos\mu_s)} \right| \quad (14)$$

This gives the magnitude of η_m . If $v_s = 0.05$ and one wants to measure η_m to an accuracy of 5 cm with a drive of $\Delta\delta = 10^{-5}$ and $v_d = 0.055$, the necessary accuracy that one has to measure $\hat{x}_m$ is about 8 μ m.

3. To determine the sign of η_m , it is necessary to measure the relative phase between the x-signal at the monitor and the driving voltage. For $\eta_m > 0$, the phase of x is behind the drive by 90° if $v_d > v_s$ and ahead by 90° if $v_d < v_s$. For $\eta_m < 0$, it is ahead by 90° if $v_d > v_s$ and behind by 90° if $v_d < v_s$.

(**) This does not necessarily mean $\eta_c = 0$ and $\eta_c' = 0$. To eliminate η_c and η_c' , one has to make resonant response near $v_d = v_x$ to disappear simultaneously at two appropriately chosen position monitors.

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