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NON-LINEAR SATURATION OF THE TEARING
MODE IN A REVERSED FIELD PINCH*

by

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ABSTRACT

We use a non-linear, two-dimensional, resistive MHD computer code to study the non-linear behavior of the $m = 0, 1$, and 2 tearing modes in a Reversed Field Pinch. For the cases $m \neq 0$ we employ a coordinate transformation which assumes helical symmetry to reduce the dimensionality of the problem from three to two. We use the force-free Bessel function model equilibrium. We find that the most dangerous resistive instability in this case is the $m = 1$ tearing mode, due to its larger growth rate and extended period of exponential growth.

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1. INTRODUCTION

A recent study by Taylor [1] has dealt with the relaxation of toroidal plasma discharges to final quiescent states. These final configurations of a slightly resistive plasma are found to minimize the energy subject to the constraint that

$$K_0 = \int_{V_0} \underline{A} \cdot \underline{B} \, d\tau \quad (1)$$

be invariant during the discharge. Here, $\underline{A}$ is the vector potential, $\underline{B}$ is the magnetic flux density, and the integral is taken over the entire volume of the plasma. Then it is shown that these final states are characterized by the force-free condition

$$\nabla \times \underline{B} = \mu \underline{B} \quad (2)$$

where μ is a constant.

In the cylindrical approximation, one solution of equation (2) is

$$B_r = 0 \quad (3a)$$

$$B_\theta = a_0 J_1(\mu r) \quad (3b)$$

$$B_z = a_0 J_0(\mu r) \quad (3c)$$

where a_0 is constant, J_0 and J_1 are the zero and first order Bessel functions, and the parameter μ is related to the pinch ratio $\chi \equiv 2I/aB_0$ by $\chi = \mu a/2$ [1]. This magnetic field configuration is referred to as the Lundquist [2], or force-free Bessel function model. We note that for $\mu a > 2.4$ the axial field reverses. This corresponds to a pinch ratio $\chi = 1.2$ which is consistent with the Reversed Field Pinch regime $\chi \geq 1$.

The linear stability of this equilibrium against tearing modes [3] is reviewed by Killeen in these proceedings. In this paper we examine the non-linear properties of these modes by numerically solving the resistive MHD equations. In section 2 we describe the computational model, and in section 3 we present results relevant to the non-linear behavior of the $m = 0, 1$, and 2 tearing modes in the force-free Bessel function equilibrium. We conclude that the $m = 1$ mode is the most dangerous resistive instability due to its large growth rate and extended period of exponential growth.

2. COMPUTATIONAL MODEL

The computational model employed in this work has been presented in detail elsewhere [4,5]. In this section we briefly review its relevant features.

The equations appropriate for the description of low frequency, long wavelength phenomenon in a conducting fluid are the resistive MHD equations. As these equations contain both convective and diffusive time scales, it proves convenient to introduce dimensionless variables in such a way as to separate their effect. When this is done, the relevant equations can be cast as the set of conservation laws

$$\frac{\partial \underline{B}}{\partial t} = \kappa (\underline{v} \times \underline{B} - \frac{r}{2} \nabla \times \underline{B}) \quad (4a)$$

$$\frac{\partial (\rho \underline{v})}{\partial t} = -\nabla \cdot [\rho \underline{v} \underline{v} - \underline{B} \underline{B} + \frac{1}{2} (p + B^2) \underline{I}] \quad (4b)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v}) \quad (4c)$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot [(u+p)\underline{v}] + (\underline{B}^2 \underline{1} - 2\underline{B}\underline{B}) \cdot \underline{v} \quad (4d)$$

$$+ \frac{2\eta}{S} (\underline{B} \cdot \nabla \underline{B} - \nabla \underline{B} \cdot \underline{B})] \quad (4e)$$

where $\underline{B}$ is the magnetic flux density, $\underline{v}$ is the velocity, ρ is the mass density, $u = p_e + B^2 + \rho v^2$ is the total energy density, and e is the specific internal energy. The pressure p is eliminated by employing the perfect gas law, so that $p_e = p/(\gamma-1)$, where $\gamma = C_p/C_v$ is the ratio of specific heats. The non-dimensional parameter $S = t_R/t_H = 4\pi a V_A / c^2 \eta_0$ is the magnetic Reynolds number, and is the ratio of the resistive time scale to the convective time scale.

This set of equations is valid in any coordinate system. Computationally, one must specify a particular coordinate system and expand the indicated vector operations. In this work we keep the resulting equations as general as possible by specifying the metric as

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 \quad (5)$$

where x_1 , x_2 , and x_3 are orthogonal coordinates and the scale factors $h_j = h_j(x_1, x_2)$ are defined such that $h_j dx_j$ represents the differential arc length along the j^{th} coordinate. When this is done we obtain a set of eight coupled differential equations to which finite difference techniques can be applied.

In cylindrical geometry the unstable eigenmodes are generally functions of three spatial dimensions. We reduce the dimensionality of the problem from three to two by employing the coordinate transformation $\phi = m\theta + k_z z$ where m and k_z are the azimuthal and axial mode

numbers. Such a transformation is clearly an approximation since it does not allow for the non-linear generation of modes with different helicity. The resulting set of equations is

$$\frac{\partial B_r}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial \phi} (mE_z - rk_z E_\theta) \quad (6a)$$

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial E_z}{\partial r} - k_z \frac{\partial E_r}{\partial \phi} \quad (6b)$$

$$\frac{\partial B_z}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (rE_\theta) + \frac{m}{r} \frac{\partial E_r}{\partial \phi} \quad (6c)$$

$$\begin{aligned} \frac{\partial(\rho v_r)}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} (rT_{rr}) - \frac{1}{r} \frac{\partial}{\partial \phi} (mT_{r\theta} + rk_z T_{rz}) \\ & - \frac{1}{2} \frac{\partial}{\partial r} (\rho + B_r^2 + B_\theta^2 + B_z^2) + \frac{T_{\theta\theta}}{r} \end{aligned} \quad (6d)$$

$$\begin{aligned} \frac{\partial(\rho v_\theta)}{\partial t} = & \frac{1}{r} \frac{\partial}{\partial r} (rT_{r\theta}) - \frac{1}{r} \frac{\partial}{\partial \phi} (mT_{\theta\theta} + rk_z T_{z\theta}) \\ & - \frac{m}{2r} \frac{\partial}{\partial \phi} (\rho + B_r^2 + B_\theta^2 + B_z^2) - \frac{T_{r\theta}}{r} \end{aligned} \quad (6e)$$

$$\begin{aligned} \frac{\partial(\rho v_z)}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} (rT_{rz}) - \frac{1}{r} \frac{\partial}{\partial \phi} (mT_{z\theta} + rk_z T_{zz}) \\ & \frac{1}{2} k_z \frac{\partial}{\partial \phi} (\rho + B_r^2 + B_\theta^2 + B_z^2) \end{aligned} \quad (6f)$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (r\rho v_r) - \frac{1}{r} \frac{\partial}{\partial \phi} (m\rho v_\theta + rk_z \rho v_z) \quad (6g)$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (rF_r) - \frac{1}{r} \frac{\partial}{\partial \phi} (mF_\theta + rk_z F_z), \quad (6h)$$

where

$$E_r = v_z B_\theta - v_\theta B_z + \frac{\eta}{S} \frac{1}{r} \frac{\partial}{\partial \phi} (m B_z - r k_z B_\theta) \quad (7a)$$

$$E_\theta = v_r B_z - v_z B_r + \frac{\eta}{S} \left(k_z \frac{\partial B_r}{\partial \phi} - \frac{\partial B_z}{\partial r} \right) \quad (7b)$$

$$E_z = v_\theta B_r - v_r B_\theta + \frac{\eta}{S} \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{m}{r} \frac{\partial B_r}{\partial \phi} \right] \quad (7c)$$

are the components of the electric field,

$$T_{\alpha\beta} = \rho v_\alpha v_\beta - B_\alpha B_\beta \quad (8)$$

($\alpha, \beta = r, \theta, z$) are the components of the symmetric stress tensor, and

$$F_r = (u + p - B_r^2 - B_\theta^2 + B_z^2) v_r - 2B_r (B_\theta v_\theta + B_z v_z) + \frac{2\eta}{S} \left(\frac{m B_\theta}{r} + k_z B_z \right) \frac{\partial B_r}{\partial \phi} - \frac{B_\theta}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{2} \frac{\partial}{\partial r} B_z^2 \quad (9a)$$

$$F_\theta = (u + p + B_r^2 - B_\theta^2 + B_z^2) v_\theta - 2B_\theta (B_r v_r + B_z v_z) + \frac{2\eta}{S} \left[\frac{B_r}{r} \frac{\partial}{\partial r} (r B_\theta) + k_z B_z \frac{\partial B_\theta}{\partial \phi} - \frac{m}{2r} \frac{\partial}{\partial \phi} (B_r^2 + B_z^2) \right] \quad (9b)$$

$$F_z = (u + p + B_r^2 + B_\theta^2 - B_z^2) v_z - 2B_z (B_r v_r + B_\theta v_\theta) + \frac{2\eta}{S} \left[B_r \frac{\partial B_z}{\partial r} + \frac{m B_\theta}{r} \frac{\partial B_z}{\partial \phi} - \frac{1}{2} k_z \frac{\partial}{\partial \phi} (B_r^2 + B_\theta^2) \right] \quad (9c)$$

are the components of the energy flux.

The spatial differencing of these equations is fully conservative on a variably zoned Eulerian grid, with densities defined at grid points

and fluxes at half grid points. The temporal differencing is Alternating Direction Implicit (ADI), with the mixed derivatives are always treated explicitly. The resulting difference equations are non-linear and must be solved iteratively.

To effect a solution, boundary conditions must be specified. At a perfectly conducting wall, conditions on the field and flow variables are well known. For the thermodynamic variables we impose conditions which require that mass and energy be conserved. For example, the mass density at the wall is advanced in such a way as to exactly balance the flux of mass entering from the neighboring interior cell.

Singular boundaries (e.g., $r = 0$) present a particular problem, for unless symmetry conditions exist there is no natural boundary condition to be imposed. For problems characterized by axial symmetry (i.e., $m = 0$), we can employ a modification of the conservative boundary conditions imposed at a conducting wall. For cases where $m^2 > 0$, this boundary becomes degenerate in that it represents the same point in space for all paths of approach. Thus all scalar quantities must be unique there, and we derive special boundary conditions based on this result. Vector components, however, need not be unique. For $m^2 > 1$ boundary conditions on these variables are derived based on an averaging procedure. For $m = 1$, we employ a condition based on symmetry about a plane passing through the origin.

3. TEARING MODE IN THE BESSEL FUNCTION MODEL

We apply the numerical methods described in section 2 to the study of the tearing mode in the force-free Bessel function equilibrium given

in equation (3). The linear growth of the tearing mode for this model has been extensively studied by Dibiase [6] using two computer codes which solve the linearized, Fourier analyzed, resistive MHD equations. One of these codes uses an incompressible model [7], while the other retains such effects as compressibility, viscosity, and thermal conductivity [8]. We use the incompressible code to generate the initial perturbation for the non-linear code described above, and follow the eigenfunctions into the non-linear regime. Due to the large amount of computer time required we have not performed an extensive parameter study of the non-linear behavior of these modes. Instead, we concentrate on the qualitative long-time behavior of solutions characterized by azimuthal mode numbers $m = 0, 1$, and 2 .

Since the equilibrium under consideration is force-free, and since the plasma is assumed to fill the cylinder with constant $\rho = 1$ initially, we are free to choose a pressure level arbitrarily. Additional parameters to be specified are the magnetic Reynolds number S , the non-dimensional axial wave number k_z , the wall radius R_w , the radius of the singular surface r_s , and the non-dimensional wave number

$$\alpha = (k_z^2 + \frac{m^2}{r_s^2})^{1/2} . \quad (10)$$

The growth rate of the resistive modes studied here is determined from the instantaneous value of

$$\rho \equiv \omega_{rH} = \frac{1}{\Delta\phi_s} \frac{\partial\Delta\phi_s}{\partial\tau} \quad (11)$$

where $\Delta\phi_s$ is the amplitude of the reconnected flux at the singular surface. This effectively isolates the resistive modes from ideal MHD phenomena since in the absence of resistivity $\Delta\phi_s$ must vanish.

When the azimuthal mode number m vanishes, the problem becomes axially symmetric and can be posed in the (r,z) plane. In this case, the location of the singular surface is determined by the vanishing of the axial magnetic field which occurs at $r_s = 2.405$. We place the wall of $R_w = 5.5$. For computational purposes we take a coordinate system for which $x_1 = z$, $x_2 = r$, $x_3 = \theta$. The corresponding scale factors are $h_1 = 2\pi/\alpha$, $h_2 = 1$, $h_3 = r \equiv x_2$. The reconnected flux is then given by

$$\phi_s(z) = \int_0^{r_s} B_z(r,0)dr - \int_0^z B_r(r_s,z')dz' \quad (12)$$

with $\Delta\phi_s = \phi_{s_{\max}} - \phi_{s_{\min}}$.

We follow the mode into the non-linear regime for the case of $S = 96.65$, $B = .3164$. In figure (1) we plot the exponential growth rate as a function of time, and we note the non-linear saturation. Figure (2) shows the magnetic flux surfaces in the (z,r) plane after four Alfvén transit times. Note the well formed magnetic island.

For cases where $m \neq 0$ we must use the transformation to helical coordinates described in section 2. In this case the calculation of the growth rate is based on the reconnected flux along a singular surface which wraps itself helically around a cylinder. We define the helical flux to be

$$\begin{aligned} \phi(r, \theta) = & - \int_{(0,0)}^{(r,0)} (m\theta_\theta + r'k_z B_z)_{\theta=0} dr' \\ & + r \int_{(r,0)}^{(r,\theta)} B_r(r, \theta') d\theta' \end{aligned} \quad (13)$$

and calculate the amplitude of the reconnected flux as

$$\Delta\phi_s = \max_{0 \leq \theta \leq \frac{2\pi}{m}} [\phi(r_s, \theta)] - \min_{0 \leq \theta \leq \frac{2\pi}{m}} [\phi(r_s, \theta)] \quad (14)$$

where we have employed symmetry to reduce the range of θ .

For the cases presented below, we have taken $S = 100$ and have set $R_w = 5.5$. We set $\delta = 0.6$, and choose a non-dimensional axial wave number $k_z = 1/3$, which corresponds to an axial mode number $n = 1$ and an aspect ratio $R/a = 3$.

For the case $m = 2$, the singular surface occurs at $r_s = 3.389$, giving a value of $\alpha = 0.662$. We put 31 mesh uniformly spaced mesh points in the range $0 \leq \theta \leq \pi$, and put 51 mesh points in the range $0 \leq r \leq 5.5$. The radial grid is non-uniform, with $\Delta r = .055$ near the singular surface and increasing away from it. This is the same resolution of this region obtained in the linear code with 101 equally spaced radial mesh points.

In Figure (3a-f) we plot surfaces of constant helical flux (see equation (13)) at various times for this case. The same contour values are retained in each plot so that the evolution of a particular flux surface can be followed in time. Figure (3a) shows the initial

conditions for the run. The perturbation is evident, but the magnetic island is too small to be resolved. Figure (3b) shows the flux surfaces after 0.75 Alfvén transit times, and we see that previously unconnected field lines have now linked to form an island. In Figure (3c) we see that this reconnected flux surface is diffusing away, while two more field lines are on the verge of reconnecting, which is shown in Figure (3d) after 3.75 Alfvén transit times. In Figure (3e) we see that both reconnected flux contours are shrinking, and in Figure (3f) we see that, after 5 Alfvén transit times, the island has completely saturated with no further reconnection taking place.

This behavior is summarized in Figure (4) where we plot the exponential growth rate (see Equation (11)) as a function of time. The initial value is in agreement with the results of the linear code. We note that this mode saturates within a few Alfvén transit times, with island growth ceasing shortly thereafter.

For the $m = 1$ mode, we find the singular surface to be located at $r_s = 3.139$, so that $\alpha = .438$, and in Figure (5) we show the initial flux contours. Again, the initial perturbation is too small for the magnetic island to be resolved. In Figure (6) we plot the same flux contours at $t = 4.25t_H$. The island growth is again clear. In Figure (7) we plot the reconnected helical flux at the singular surface as a function of time and note the initial exponential growth (again at a rate in agreement with the linear results) followed by non-linear saturation. In Figure (8) we display the exponential growth rate as a function of time for this case, and we note the extended period of rapid growth.

We were unable to run this case to larger amplitude because of numerical problems encountered at the origin for large times. This may be due to the fact that the boundary conditions for this mode described in section 2 are only first order accurate. This generates errors in the numerical solution which are found to become unstable near the origin. This does not seem to affect the solution in the body of the plasma, so that we have some confidence in the saturation results presented above.

It is clear that the $m = 2$ mode is relatively benign for this configuration due to its low growth rate and rapid saturation. While the $m = 0$ mode can grow to larger amplitude, it still has a lower growth rate and more rapid saturation than does $m = 1$. We conclude that, for the force-free Bessel function model, the $m = 1$ tearing mode is the most dangerous resistive instability.

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REFERENCES

1. J. B. Taylor, in Pulsed High Beta Plasmas, D. E. Evans (Ed.), Pergamon Press, Oxford (1976)
2. S. Lundquist, Phys. Rev. 83, 307 (1951)
3. H. F. Furth, J. Killeen, and M. Rosenbluth, Phys. Fluids 6, 459 (1963)
4. D. Schnack, Ph.D. Thesis, Univ. of Calif., Davis, UCRL-52399 (1978)
5. D. Schnack and J. Killeen, to appear in J. Comp. Phys.
6. J. Dibiase, Ph.D. Thesis, Univ. of Calif., Davis, UCRL-51591 (1974)
7. J. Killeen, in Physics of Hot Plasmas, B. J. Rye and J. C. Taylor (Eds.), pg. 202, Plenum Press, New York (1970).
8. J. Dibiase and J. Killeen, J. Comp. Phys. 24, 158 (1977).

Figure Captions

Figure 1. Exponential growth rate vs. time for $m = 0$.

Figure 2. Magnetic flux surfaces for $m = 0$ at $t = 4t_H$.

Figure 3. Magnetic flux surfaces for $m = 2$.

Figure 4. Exponential growth rate vs. time for $m = 2$.

Figure 5. Magnetic flux surfaces for $m = 1$ at $t = 0$.

Figure 6. Magnetic flux surfaces for $m = 1$ at $t = 4.25t_H$.

Figure 7. Reconnected flux vs. time for $m = 1$.

Figure 8. Exponential growth rate vs. time for $m = 1$.

Fig. 1. Exponential growth rate vs. time for $m = 0$.

Fig. 2. Magnetic flux surfaces for $m = 2$ at $t = 4t_H$.

Fig. 3. Magnetic flux surfaces for $m = 2$.

Fig. 4. Exponential growth rate vs. time for $m = 2$.

Fig. 5. Magnetic flux surfaces for $m = 1$ at $t = 0$.

Fig. 6. Magnetic flux surfaces $f = 1$ at $t = 4.25\tau_H$.

Fig. 7. Reconnected flux vs. time for $\eta = 1$.

Fig. 3. Exponential growth rate vs. time for $m = 1$.