2 + 1 DIMENSIONAL QUANTUM GRAVITY AND THE BRAID GROUP*

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Steven Carlip

The Institute for Advanced Study
Princeton, NJ 08540 U.S.A.

Elsewhere in these proceedings [1], Jackiw has described an interesting calculation of scattering amplitudes in 2+1 dimensional gravity. His approach [2], like that of 't Hooft [3], starts from the observation that there is no classical gravitational radiation in 2+1 dimensions: solutions of the field equations are locally flat outside sources, and the gravitational interaction depends only on the global geometry. In the absence of propagating gravitational degrees of freedom, one way to quantize the theory is to use the field equations to eliminate the metric, and to then quantize the remaining matter degrees of freedom. One then has the physically intuitive picture of quantized particles each moving in the background conical geometry induced by the others. For two-particle scattering, this picture is especially simple, since one may use center of mass coordinates [3] to reduce the problem to one of solving the Schrödinger equation on a cone.

While this approach is attractive, it is not obvious that it is equivalent to ordinary canonical quantization. Further, since the total deficit angle is determined classically by the energy, we are forced to work in a fixed eigenstate of the Hamiltonian, and the superposition of energy eigenstates becomes difficult to understand [4]. In addition, by assuming the form of the background geometry, we evade some of the important conceptual issues of quantum gravity, such as the role of time in a theory in which time translations are symmetries [5] and problem of constructing diffeomorphism-invariant observables [6]; while this simplifies calculations, it makes the generalization to 3+1 dimensions less clear.

A conventional canonical treatment of 2+1 dimensional quantum gravity is therefore of interest. Such a quantization is now possible, thanks to the recent observation of Witten [7] that 2+1 dimensional gravity can be treated as a Chern-Simons gauge theory. By comparing such an exact quantization to the physically clear picture of Jackiw *et al.*, we may hope to gain some insight into the quantization of gravity in 3+1 dimensions, while at the same time testing the 2+1 dimensional canonical framework.

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MASTER

1. 2 + 1 GRAVITY AS A CHERN-SIMONS THEORY

Let us begin by reviewing Witten's treatment [7] of 2 + 1 dimensional gravity as a Chern-Simons theory for the 2 + 1 dimensional Poincaré group $ISO(2, 1)$. This group is characterized by two sets of generators, the local Lorentz generators $\mathcal{J}^a$ and the translation generators $\mathcal{P}_a$, obeying the usual algebra

$$\begin{aligned} [\mathcal{P}_a, \mathcal{P}_b] &= 0 \\ [\mathcal{P}_a, \mathcal{J}_b] &= \epsilon_{abc} \mathcal{P}^c \\ [\mathcal{J}_a, \mathcal{J}_b] &= \epsilon_{abc} \mathcal{J}^c \end{aligned} \quad (1.1)$$

To construct a Chern-Simons theory, we associate gauge fields $A_\mu = \{e^a_\mu, \omega_{a\mu}\}$ to the symmetry generators $\{\mathcal{P}_a, \mathcal{J}^a\}$, and form the standard Chern-Simons action $S = \int \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$. To make sense of this expression, we must define "trace" for $ISO(2, 1)$, i.e., we must specify a nondegenerate group-invariant pairing of the generators. Such a pairing is almost unique [7]: it is

$$\begin{aligned} \text{Tr}(\mathcal{J}^a \mathcal{P}^b) &= \eta^{ab} \\ \text{Tr}(\mathcal{J}^a \mathcal{J}^b) &= \text{Tr}(\mathcal{P}^a \mathcal{P}^b) = 0 \end{aligned} \quad (1.2)$$

It is then easy to check that

$$S = \frac{1}{2} \int_M \epsilon^{\rho\mu\nu} e^a_\rho (\partial_\mu \omega_{a\nu} - \partial_\nu \omega_{a\mu} + \epsilon_{abc} \omega^b_\mu \omega^c_\nu) \quad (1.3)$$

which can be recognized as the standard first order form of the Einstein action if we identify $\omega_{a\mu}$ with the spin connection $\frac{1}{2} \epsilon_{abc} \omega_\mu^{bc}$.

S is invariant under the gauge transformations

$$\begin{aligned} \delta e^a_\mu &= -\partial_\mu \rho^a - \epsilon^{abc} \omega_{b\mu} \rho_c - \epsilon^{abc} e_{b\mu} \tau_c \\ \delta \omega^a_\mu &= -\partial_\mu \tau^a - \epsilon^{abc} \omega_{b\mu} \tau_c \end{aligned} \quad (1.4)$$

The τ^a parameterize the local Lorentz transformations, while the ρ^a transformations are equivalent on shell to the diffeomorphisms. Note that this means we can either gauge-fix the diffeomorphisms or the ρ transformations: their difference is a trivial invariance of the kind discussed by Henneaux at this school [8].

We now have a Chern-Simons description of 2 + 1 dimensional gravity without sources. There are various methods of incorporating point particles: by adding source terms to the action, by including timelike Wilson lines [9], or, most simply, by deleting particle world lines from our spacetime manifold and investigating the surrounding geometry. Let us choose this latter approach: we shall study quantum gravity on the manifold $\mathbb{R} \times (\Sigma - \{x_a\})$, where Σ is a two-surface (which we will often take to have the topology $\mathbb{R}^2$) and $\{x_a\}$ are the locations of sources.

One slight complication must be addressed if we wish to consider scattering problems. The classical solution of the 2 + 1 dimensional Einstein equations in the presence of a point source contains a conical singularity at the location of the source. At such a point, the group of rotations of the tangent space changes: if we

denote the deficit angle of a cone by β , a vector at the apex of the cone returns to its original position after a rotation by $2\pi - \beta$ rather than 2π . To allow for this kind of singular behavior, we should replace $ISO(2, 1)$ by its universal covering space $\widetilde{ISO}(2, 1)$. We can always recover the original $ISO(2, 1)$ theory by an appropriate projection, but, as we shall see below, by allowing the projection to vary at chosen points we can also obtain the appropriate singular behavior.

Unfortunately, the passage from $ISO(2, 1)$ to $\widetilde{ISO}(2, 1)$ is not unique, and we will also find a number of ambiguities of 2π in the final theory. For example, the deficit angle of the conical singularity for a classical point source is equal to the mass of the source; but deficit angles are periodic, while masses are not, so the geometry alone is insufficient to determine the physics. A similar ambiguity occurs in the Regge calculus approach to gravity, in 2+1 or 3+1 dimensions [10]. While this ambiguity can be resolved classically by an appeal to the low-mass limit, its role in the quantum theory is not well-understood.

2. QUANTIZATION

Our goal is to quantize the system described by the action (1.3) on a manifold $\mathbb{R} \times (\Sigma - \{x_a\})$. Decomposing our fields into space and time components, we find that

$$S = \int dt \int_{\Sigma} \left(-\epsilon^{ij} e^a_i \frac{d}{dt} \omega_{aj} + e^a_0 \tilde{N}_a + \omega_{a0} N^a \right) \quad (2.1)$$

where

$$N^a = \frac{1}{2} \epsilon^{ij} (\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc} (\omega_{bi} e_{cj} - \omega_{ci} e_{bj})) \\ + \text{boundary terms} \quad (2.2)$$

$$\tilde{N}^a = \frac{1}{2} \epsilon^{ij} (\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc} \omega_{bi} \omega_{cj})$$

Since no time derivatives of e^a_0 or ω_{a0} appear in the action, the N^a and $\tilde{N}^a$ are constraints. From the $e\omega$ term, on the other hand, we can read off the commutators

$$[\omega_{ai}(x), e^b_j(x')] = i\epsilon_{ij} \delta^b_a \delta^2(x - x') \\ [\omega_{ai}(x), \omega_{bj}(x')] = [e^a_i(x), e^b_j(x')] = 0 \quad (2.3)$$

It may then be checked that the constraints obey the $ISO(2, 1)$ algebra (1.1) and that the generator of gauge transformations (1.4) is

$$G = - \int_{\Sigma} (\tau_a N^a + \rho_a \tilde{N}^a) \quad (2.4)$$

We must next construct a Hilbert space $\mathcal{H}$ on which operators e and ω act. Because of the structure of the constraints, it turns out to be easiest to work in a “momentum representation,” in which the Hilbert space is built out of functionals of the ω_{ai} . To impose the constraints $N^a = \tilde{N}^a = 0$, we have two options: as Seiberg has explained in his lectures here [11], such constraints may either be solved before quantization or imposed as operator equations after quantization.

Figure 1. A flat connection on a punctured plane is determined by its holonomies around the punctures.

Let us choose the former approach. To solve the constraints, we observe that the condition $\tilde{N}^a = 0$ is simply the statement that ω_{ai} is a flat $\widetilde{SO}(2, 1)$ connection on $\Sigma - \{x_\alpha\}$. N^a then generates the $\widetilde{SO}(2, 1)$ transformations of ω , so the constraint $N^a = 0$ requires us to identify connections which are gauge-equivalent. Hence the physical Hilbert space is the space of square integrable functions on the moduli space $\mathcal{N}$ of flat $\widetilde{SO}(2, 1)$ connections on $\Sigma - \{x_\alpha\}$.

More concretely, a flat connection is determined up to gauge transformations by its holonomies around the nontrivial loops in $\Sigma - \{x_\alpha\}$. To specify these holonomies, we must associate an element of the gauge group with each homotopy class of loops; in other words, we must give a group homomorphism $\pi_1(\Sigma - \{x_\alpha\}, *) \rightarrow \widetilde{SO}(2, 1)$, where $*$ is a base point for the fundamental group of $\Sigma - \{x_\alpha\}$. Under a gauge transformation $\mathcal{G}: \Sigma - \{x_\alpha\} \rightarrow \widetilde{SO}(2, 1)$, the holonomies are conjugated at $*$ by $\mathcal{G}(*)$. Hence the moduli space of flat connections may be written as

$$\mathcal{N} = \text{Hom}(\pi_1(\Sigma, *), \widetilde{SO}(2, 1)) / \sim \quad (2.5)$$

where two homomorphisms are identified under $\sim$ if they differ by conjugation.

If Σ is not compact, we must modify this result slightly, since the action (1.3) is then only invariant under gauge transformations which fall off sufficiently fast at infinity. The simplest approach is to choose the base point $*$ to be at infinity, so gauge transformations are the identity at $*$, and to omit the equivalence relation $\sim$ from the definition of $\mathcal{N}$. In particular, if Σ has the topology $\mathbb{R}^2$, a point in $\mathcal{N}$ is determined by a set of holonomies

$$\Lambda_a = e^{-ip_{aa}\mathcal{J}^a} \in \widetilde{SO}(2, 1) \quad (2.6)$$

around the punctures $\{x_\alpha\}$ (see figure 1).

We are thus led to consider a Hilbert space of square integrable functions $\psi(p_{aa})$. One cautionary note is needed, however. As noted above, we must allow our holonomies to be elements of $\widetilde{SO}(2, 1)$, not just $SO(2, 1)$. This means in effect that the wave functions ψ must be multivalued: the 2π rotation operator $\exp\{2\pi i\mathcal{J}_a^0\}$ need not act as the identity, but may change the phase of ψ . We therefore obtain a quantum theory described by a Hilbert space $\mathcal{H}$ of multivalued square integrable functions of the parameters p_{aa} which label the holonomies of ω_a .

3. DYNAMICS

With this understanding of the kinematics of 2+1 dimensional quantum gravity, can turn to the question of the dynamics, that is, time translation and the Hamiltonian. If space is compact, this is can be a confusing task: time translations are gauge symmetries, the Hamiltonian is a constraint, and even the definition of time becomes ambiguous. If space is asymptotically locally flat, however, the Hamiltonian includes a nonvanishing boundary contribution, and there is a sensible definition of time translation based on the behavior of the metric at infinity.

To derive this boundary contribution, Regge and Teitelboim [12] give a simple general argument. Let G be the generator of a symmetry such as time translation. For the Poisson bracket of G with the fields to be well-defined, we must be able to take functional derivatives of G ; this will be possible only if variations δG contain no boundary contributions. While boundary contributions which do occur can sometimes be eliminated by restricting the allowed variations of the fields, certain variations — those corresponding to asymptotic symmetries — must be permitted on physical grounds, and the unwanted boundary variations must instead be cancelled by adding terms to G .

For 2 + 1 dimensional gravity, the classical metric for isolated sources is asymptotically conical [13]:

$$ds^2 \sim \left(dt - \frac{\alpha}{2\pi - \beta} d\phi\right)^2 - dr^2 - r^2 d\phi^2 \quad 0 \leq \phi < 2\pi - \beta \quad (3.1)$$

where β is the total deficit angle and α is proportional to the total angular momentum. Such a metric admits two asymptotic $ISO(2, 1)$ symmetries, time translation

$$\rho^0 \sim -1 \quad (3.2)$$

and spatial rotation

$$\begin{cases} \tau^0 \sim 1 \\ \rho^2 \sim -\tau \end{cases} \quad (3.3)$$

For the generator G of gauge transformations defined by equation (2.4),

$$\delta G = - \int_{\partial\Sigma} \rho_a \delta \omega^a_{\parallel} + \text{volume terms} \quad (3.4)$$

where the subscript $\parallel$ denotes the component tangent to $\partial\Sigma$. (Because of the boundary terms in (2.2), there is no τ_a contribution.) Hence for an asymptotic time translation (3.2), we must cancel the boundary variation by adding a term

$$H = - \int_{\partial\Sigma} \omega^0_{\parallel} \quad (3.5)$$

to G .

Figure 2. The contour defining H can be deformed to give a product of holonomies.

Classically, $H = \beta$, the total deficit angle. When $\Sigma = \mathbb{R}^2 - \{x_\alpha\}$, there is a useful trick for evaluating this quantity. In the frame for which (3.1) holds, the only component of ω at infinity lies in the $a = 0$ direction. Hence

$$\text{Tr } e^{-iH\mathcal{J}_0} = \text{Tr } P \exp \left\{ i \int_{\partial\Sigma} \omega^a_i dx^i \mathcal{J}_a \right\} = \text{Tr} \left(\prod_{\alpha} \Lambda_{\alpha} \right) \quad (3.6)$$

where the holonomies Λ_{α} are given by (2.6), and the last equality comes from deforming the contour of integration through regions of zero curvature (see figure 2). On the other hand, in the fundamental representation of $SO(2, 1)$,

$$\text{Tr } e^{-iH\mathcal{J}_0} = 1 + 2 \cos H \quad (3.7)$$

For the case of two Wilson lines, for instance, this means that

$$\cos \frac{H}{2} = \cos \frac{m_1}{2} \cos \frac{m_2}{2} - \sin \frac{m_1}{2} \sin \frac{m_2}{2} \left(\frac{p_1 \cdot p_2}{m_1 m_2} \right) \quad (3.8)$$

where

$$m_{\alpha}^2 = p_{\alpha}^2 \quad (3.9)$$

This expression agrees with the result of Deser *et al.* [14] for the addition of deficit angles. From the classical solutions described in [14], it may be checked that the symbols p and m , which we introduced to parameterize the holonomies, really do have interpretations as momenta and masses. Note also that in the fundamental representation of $SO(2, 1)$, the Hamiltonian H — and for that matter the masses m_{α} — are only determined mod 2π . This is an early sign of the ambiguity in the lifting from $ISO(2, 1)$ to $\tilde{ISO}(2, 1)$; it will reappear when we try to evaluate scattering amplitudes.

Figure 3. The holonomies around a pair of punctures (a) change under the action of the braid group (b).

4. BRAIDS

We now have a Hilbert space and a Hamiltonian for 2+1 dimensional gravity. The Hamiltonian is essentially the same as that of Deser and Jackiw [2] and 't Hooft [3], and it gives the same differential equations for scattering amplitudes as those proposed by Jackiw. But we have not yet seen any sign of the conical boundary conditions so crucial to these other analyses.

To understand the origin of these boundary conditions, we must examine more carefully the symmetries of the theory. We saw above that diffeomorphisms in 2+1 dimensional gravity are equivalent on shell to ρ gauge transformations, and thus are factored out when we gauge-fix. However, the proof of this equivalence starts with the infinitesimal form (1.4) of the transformations, and, strictly speaking, only holds for those transformations which can be reached by exponentiating infinitesimal ones. There remain the diffeomorphisms which are not isotopic to the identity, which must still be factored out.

For a manifold $\mathbb{R} \times (\mathbb{R}^2 - \{x_\alpha\})$, such diffeomorphisms are generated by Dehn twists around the punctures x_α . At first sight, such transformations appear to have no effect on the holonomies Λ_α , which are defined in terms of coordinate-independent integrals. If we look more carefully, however, we see that the definition (2.6) of Λ_α depends implicitly on a choice of generators for the fundamental group $\pi_1(\Sigma - \{x_\alpha\})$. A Dehn twist has the effect of changing this choice of generators, as is apparent in figure 3a-b: when one puncture is wrapped around another, the relevant holonomies change by conjugation. This is the well-known action of the braid group on the fundamental group of a punctured surface, and it gives the relevant action of the mapping class group on the holonomies.

If we wish to consider the full diffeomorphism group to be a symmetry of our theory, we must demand that the wave functions ψ be invariant under this action. Note that the generalization from $\mathbb{R}^2$ to an arbitrary two-manifold Σ is straightforward; there will be new elements of the braid group coming from the possibility of wrapping punctures around handles, but the properties of the resulting group are well-understood [15,16].

For concreteness, let us specialize to the case of two punctures. Our Hilbert space then consists of square integrable functions of two momenta p_1 and p_2 , the Hamiltonian is given by (3.8), and it is apparent from figure 3 that the action of

the braid group is

$$\begin{aligned}\Lambda_1 &\mapsto \Lambda_2 \Lambda_1 \Lambda_2^{-1} = (\Lambda_2 \Lambda_1) \Lambda_1 (\Lambda_2 \Lambda_1)^{-1} \\ \Lambda_2 &\mapsto (\Lambda_2 \Lambda_1) \Lambda_2 (\Lambda_2 \Lambda_1)^{-1}\end{aligned}\quad (4.1)$$

The momenta p after the transformation will thus differ by an $SO(2,1)$ matrix $\Lambda_2 \Lambda_1$ from the initial momenta. This matrix, in turn, determines an operator $\widehat{\Lambda_2 \Lambda_1}$ on the Hilbert space $\mathcal{H}$, and states must be invariant under its action.

Note that once again a 2π ambiguity has appeared. The action of the braid group on $SO(2,1)$ holonomies is determined geometrically, but its lift to an operator acting on $\mathcal{H}$ is not unique. Instead, $\widehat{\Lambda_2 \Lambda_1}$ generates an $\widetilde{SO}(2,1)$ transformation characterized by a rotation by an angle $2\pi n - H$, with the integer n undetermined. Indeed, a simple calculation shows that

$$\widehat{\Lambda_2 \Lambda_1} = e^{-iP_a(J_1 + J_2)^a} \quad (4.2)$$

where

$$\begin{aligned}\sin \frac{H}{2} \left(\frac{P_a}{|P|} \right) &= \left(\cos \frac{m_2}{2} \sin \frac{m_1}{2} \right) \frac{p_{1a}}{m_1} + \left(\cos \frac{m_1}{2} \sin \frac{m_2}{2} \right) \frac{p_{2a}}{m_2} \\ &+ \left(\sin \frac{m_1}{2} \sin \frac{m_2}{2} \right) \frac{(p_1 \times p_2)_a}{m_1 m_2}\end{aligned}\quad (4.3)$$

and

$$|P| = H - 2\pi n \quad (4.4)$$

Roughly speaking, P_a is the momentum of the center of mass, and in a suitable frame at infinity — i.e., after an appropriate $SO(2,1)$ transformation at $*$ — the condition of invariance under the braid group can be shown to be [9]

$$e^{i(2\pi n - H)\hat{L}}\psi = \psi \quad (4.5)$$

where p is a suitably defined relative momentum and $\hat{L}$ is the angular momentum operator for p in the center of mass frame.

We can now determine the integer n from physical arguments. In the limit of small masses, equation (4.5) should reduce to the ordinary projection from $\widetilde{SO}(2,1)$ to $SO(2,1)$ in order to recover the correct flat space limit. We must therefore choose $n = 1$; any other value will give a wave function defined not on the plane, but on an n -fold branched cover of the plane.

If we now take ψ to be an eigenstate of H with eigenvalue E , we have precisely recovered 't Hooft's description [3] of the Hilbert space and the Hamiltonian for two-particle scattering. Equivalently, viewing $\hat{L}$ as a generator of rotations, equation (4.5) is the condition of invariance under rotation by $2\pi - E$; this is precisely Deser and Jackiw's conical boundary condition. At the same time, we have avoided some of the ambiguities of the earlier approaches: ψ need not be an energy eigenstate, and the generalization to more than two particles and to more complicated topologies is clear. The price we have paid is in 2π ambiguities; other choices of n in (4.5) may eventually be of interest, but for now their significance is not understood.

5. THE EMERGENCE OF SPACE AND TIME

One striking feature of this analysis may have implications for quantum gravity in 3+1 dimensions. A long-standing problem in quantum gravity is the "breaking of general covariance," that is, the appearance of an approximately classical spacetime structure in a theory which is initially diffeomorphism-invariant and which involves a sum over all metrics. The problem can be thought of as one of symmetry-breaking: the metric must acquire an expectation value which is not diffeomorphism-invariant. It is evident that such symmetry-breaking has occurred here, since our scattering amplitudes have clear spacetime interpretations; it is of interest to understand exactly how this has happened.

We can recover the spatial structure more explicitly from our Hilbert space by Fourier transforming the wave functions ψ :

$$\psi(p) = \int \frac{d^2 y}{(2\pi)^2} e^{i p \cdot y} \phi(y) \quad (5.1)$$

The condition of invariance under the braid group then becomes

$$\phi(e^{i(2\pi-E)\tilde{L}} y) = \phi(y) \quad (5.2)$$

which means that y acts as a coordinate on a cone with deficit angle E . Recall, however, that p originated not as a local spatial variable, but as a parameter characterizing a holonomy. We have thus built a spatial geometry out of a set of variables which had no obvious local geometric interpretation.

To understand this better, consider the classical version of equation (5.2). The relevant classical holonomies are $\widetilde{ISO}(2,1)$ matrices

$$\Xi = P \exp \left\{ i \int (\omega_a \mathcal{J}^a + e_a \mathcal{P}^a) \right\} = \exp \left\{ -i(p_a \mathcal{J}^a + J_a \mathcal{P}^a) \right\} \quad (5.3)$$

where for spin zero sources we can take $J = q \times p$. Conjugation of one such a holonomy by another represents parallel transport in $\widetilde{ISO}(2,1)$, that is, translation and rotation of p and q . In particular, for a classical solution with two holonomies Ξ_1 and Ξ_2 , conjugation by Ξ_1 corresponds to the transformation

$$\begin{aligned} q_2 &\rightarrow q_1 + \Xi_1(q_2 - q_1) \\ p_2 &\rightarrow \Xi_1 p_2 \end{aligned} \quad (5.4)$$

Invariance under such a transformation means that although q_2 looks like a flat coordinate, it is really only locally flat; in fact, (5.4) is a typical matching condition for a coordinate in a classical solution for 2 + 1 dimensional gravity with point sources [14]. The requirement of invariance under the braid group thus allows us to reconstruct the classical geometry from the parameters q which label $\widetilde{ISO}(2,1)$ holonomies. In the quantum system, the translations $\mathcal{P}^a$ no longer appear in the holonomies, and the conical geometry becomes fuzzy, but its presence can still be felt in the boundary condition (4.5).

Finally, let us turn to the role of time in our quantization of $2 + 1$ dimensional gravity. Our construction of the Hamiltonian, and thus our interpretation of dynamics, was possible because we worked in an asymptotically locally flat space, which allowed us to define time translations as translations at infinity. But $2 + 1$ dimensional gravity is a topological theory, and infinity is not really such a special point. In a compact space Σ , we can instead choose a preferred puncture x_0 to play the same role.

In particular, x_0 can serve as a basepoint for the holonomies of $\omega_{\alpha i}$ around the remaining x_α . Wave functions will now depend on the holonomies Λ_α , $\alpha \neq 0$, and not on Λ_0 , but there is no inconsistency here: for a closed space Σ , the holonomies are not all independent, but satisfy one relation which can be used to eliminate Λ_0 . We can then define time translations as translations at x_0 . Equation (3.6) will again determine the generator of such translations, provided that Λ_0 is omitted from the trace on the right hand side. The world line of x_0 thus serves as a clock with which we can measure time evolution of the rest of the universe. Similar ideas have been proposed in $3 + 1$ dimensions, but it has never been clear how to isolate a "clock" variable.

Canonical quantization of $2 + 1$ dimensional gravity thus successfully reproduces the physically intuitive results of Deser, Jackiw, and 't Hooft for point particle scattering. In principle, it provides a framework in which those results can be unambiguously extended to more complicated topologies and larger numbers of particles. At the same time, it provides a simple model in which some of the deeper conceptual questions of quantum gravity can be addressed.

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REFERENCES

1. R. Jackiw, lectures in these proceedings
2. S. Deser and R. Jackiw, *Commun. Math. Phys.* **118**(1988), 495
3. G. 't Hooft, *Commun. Math. Phys.* **117**(1988), 685
4. R. Jackiw, "Quantum Gravity in Flatland," MIT preprint CTP-1622 (1988)
5. B. DeWitt, *Phys. Rev.* **160**(1967), 1113; W. G. Unruh and R. M. Wald, "Time and the Interpretation of Canonical Quantum Gravity," Santa Barbara preprint ITP-88-190 (1988)
6. L. Smolin, "Nonperturbative Quantum Gravity via the Loop Representation," Syracuse preprint, to appear in *Proceedings of the 1988 Osgood Hill Meeting on Quantum Gravity*, A. Ashtekar and J. Stachel, eds.
7. E. Witten, *Nucl. Phys.* **B311**(1988), 46
8. M. Henneaux, lectures in these proceedings
9. S. Carlip, *Nucl. Phys.* **B324**(1989), 106
10. M. Rocek and R. Williams, to appear
11. N. Seiberg, lectures in these proceedings
12. T. Regge and C. Teitelboim, *Ann. Phys.* **88**(1974), 276
13. M. Henneaux, *Phys. Rev.* **D29**(1984), 2766
14. S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys.* **152**(1984), 220
15. J. S. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Mathematics Studies 82 (Princeton University Press, Princeton, 1975)
16. J. S. Birman, *Comm. Pure Appl. Math.* **22**(1969), 213