TUNE SHIFTS FOR COASTING BEAMS CROSSING AT SMALL ANGLES

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Abstract

Results are presented for the beam-beam v -shifts for beams crossing at a small angle α . The beams are not round and have elliptic gaussian charge distributions. The conditions under which the results are valid are discussed. The results include long range effects, and effects of the cut-off of the beam-beam interaction. The theoretical results are compared with those obtained by numerical computation.

1. Introduction

This paper presents results for the beam-beam v-shifts which include long range effects, effects of the cut-off of the beam-beam interaction, and the effects of beams not being round in the interaction region. These results are found for an elliptic gaussian beam for which $\sigma_y \neq \sigma_x$.

When the beam crossing angle in a proton storage accelerator is made smaller, or β_x, β_y are made smaller, as is desired for high luminosity, the beam-beam v-shift tends to get larger primarily because of certain long range effects. Because the important effects are long range effects which are generated when the beams are well separated, these effects do not depend strongly on the actual transverse beam shape, and it is possible to find fairly simple expressions for the beam-beam v-shifts for non-round elliptical gaussian beams. The case of round beams was previously treated by Keil, Pellegrini, and Sessler (1).

The beam-beam v-shifts are given by

$$\Delta v_x = -\Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y \beta_x} (1 - \beta_x^2 / \ell^2), \quad (1-1a)$$

$$\Delta v_y = \Delta v \left\{ 1 + \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y^2} (1 - \beta_y^2 / \ell^2) \right\}. \quad (1-1b)$$

$\pm \ell$ is the beam-beam interaction region. Δv is the simple result for Δv_y found for constant β_x, β_y and infinite ℓ . Δv is given by

$$\Delta v = - \frac{1+\beta^2}{\beta^2} \frac{I \times 10^{-6}}{B\rho} \frac{\beta_y}{(2\pi)^{1/2} \alpha \sigma_y}, \quad (1-1c)$$

where I is in amperes, $B\rho$ is in KG-m, and all lengths are in m. β_y and β_x are the beta-functions at the beam crossing points.

The above expressions are valid when

$$\alpha \ell \gg \sigma,$$

$$\alpha \beta \gg \sigma,$$

where σ can be either σ_x or σ_y , and β can be either β_x or β_y . $\alpha \ell \gg \sigma$ says the beam separation is large compared to the beam size.

Eq (1.1) may be written as

$$\Delta v_y = \Delta v + \Delta v_{y,L} + \Delta v_{y,C} \quad (1-2a)$$

$$\Delta v_{y,L} = \Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y^2}$$

$$\Delta v_{y,C} = \Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y^2} (-\beta_y^2 / \ell^2)$$

where $\Delta v_{y,L}$ is the long range contribution due to $\beta_y \sim S^2 / \beta_y^*$ and $\Delta v_{y,C}$ is the cut-off contribution due to the finite interaction length, $\pm \ell$.

In a similar way Δv_x can be written

$$\Delta v_x = \Delta v_{x,L} + \Delta v_{x,C}, \quad (1-2b)$$

$$\Delta v_{x,L} = -\Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y \beta_x},$$

$$\Delta v_{x,C} = -\Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\sigma \beta_y \beta_x} (-\beta_x^2 / \ell^2).$$

The result for the long range term, $\Delta v_{x,L}$, has also been derived by E.D. Courant (private communication).

The simple results $\Delta v_y = \Delta v$, $\Delta v_x = 0$, are obtained when the interaction length $\ell \rightarrow \infty$, and β_x , β_y are constant, independent of s . This simple result is modified by the long range effect due to β_x and β_y increasing like s^2 ; and by the cut-off effect due to the finite interaction length ℓ . These two effects modify the simple result $\Delta v_y = v$, $\Delta v_x = 0$ and give rise to the additional terms shown in Eq (1.1). One may note that since $\Delta v \propto 1/\sigma_y$, the additional terms in Eq (1.1) are independent of σ_x , σ_y , the beam dimensions. The contribution for the above two effects, the long range effect and the cut off effect, come largely from the region where the beams are well separated, assuming that $\alpha \ell \gg \sigma$ and $\alpha \beta \gg \sigma$, and these two effects are not sensitive to the beam dimensions σ_x , σ_y .

A third effect, which has not been included in the above results, is the effect due to variation of the fields along the s direction. The field components E_x and E_y vary with s because the beam dimensions σ_x , σ_y depend on s . Numerical results presented below indicate that the contribution from this effect, called the longitudinal effect, is not large but appreciable. In particular, the numerical results indicate that in the interesting cases where Δv_x or Δv_y

become large, the important effects are the two long range effects mentioned above, and the contribution of the longitudinal effect is small. For this reason, it is believed that above results for Δv_x , Δv_y may be useful.

The neglect of the longitudinal effect may be understood in the following way. When the beams are close together, beam separations of the order of σ_x , σ_y , then σ_x and σ_y are nearly constant because $\sigma \ll \alpha \beta$. When the beams are well separated, then the field of the beam may be approximated by the field of a line charge which does not depend on σ_x , σ_y .

2. Comparison with Computed Results

Table I compares computed values of Δv_x , Δv_y with the theory results of Eq (1.1). Three sets of runs were done corresponding to values of β_x of $\beta_x = 2$, 10, 40 m. α and β_y were held constant at $\alpha = .002$ and $\beta_y = 2$ m. For each set of runs, the interaction length l was varied.

The agreement between computed and theory results for Δv_y are quite good. In the case of Δv_x , the disagreement is believed to be due to the longitudinal effect. This is supported by numerical results. Δv_x was computed for the case where β_x and β_y are held constant, but σ_x , σ_y are allowed to vary as $\sigma \propto (1+s^2/\beta)^{1/2}$. As $l \rightarrow \infty$, $\Delta v_x \rightarrow 0$ in this case, except for the dependence of σ on s . Computations for the case where $\beta_x = 40$ gave $\Delta v_x = .16 \Delta v$ when $l \rightarrow \infty$ instead of $\Delta v_x = 0$. This $\Delta v_x = .16 \Delta v$ just accounts for the error in the theory result for Δv_x in this case.

Table I. Comparison of theory and computer results for the beam-beam v-shifts. $\Delta v_x, \Delta v_y$ are in units of Δv (see Eq (1-1c)). $\beta_y = 2$ m, $\alpha = .002$, $I = 8$ A and $E = 30$ GeV.

$\beta_x = 40$	$\ell(m)$	4	18	40	60
	Δv_x (computed)	.953	.309	.168	.101
	(theory)	.79	.142	0	-.067
	Δv_y (computed)	1.13	1.72	2.61	3.40
	(theory)	1.12	1.71	2.59	3.39
	σ (mm)	1.79	1.79	1.79	1.79
	σ_x (mm)	1.40	1.40	1.40	1.40
	σ_y (mm)				
	$\alpha \ell$ (mm)	8	36	80	120
$\beta_x = 10$	$\ell(m)$	5	10	20	40
	Δv_x (computed)	.377	.199	.079	-.039
	(theory)	.120	0	-.120	-.300
	Δv_y (computed)	1.16	1.37	1.78	2.58
	(theory)	1.16	1.38	1.79	2.59
	σ (mm)	1.26	1.26	1.26	1.26
	σ_x (mm)	.40	.40	.40	.40
	σ_y (mm)				
	$\alpha \ell$ (mm)	10	20	40	80
$\beta_x = 2$	ℓ	1	2	4	16
	Δv_x (computed)	.145	.026	-.09	-.60
	(theory)	.120	0	-.120	-.630
	Δv_y (computed)	.863	.979	1.096	1.604
	(theory)	.880	1.000	1.120	1.630
	σ (mm)	.4	.4	.4	.4
	σ_x (mm)	.4	.4	.4	.4
	σ_y (mm)				
	$\alpha \ell$ (mm)	2	4	8	32

3. Derivation of Results for $\Delta v_x, \Delta v_y$

For two beams crossing at a small angle, the horizontal v-shift is given by

$$\Delta v_x = - \frac{1}{4\pi} \int ds \frac{\beta_x}{B\rho} \frac{\partial}{\partial x} (F_x / \beta e), \quad (3-1a)$$

where the beam-beam force F_x is given by

$$F_x = (1 + \beta^2) e E_x \quad (3-1b)$$

Thus,

$$\Delta v_x = - \frac{1 + \beta^2}{\beta} \left(\frac{I/v}{4\pi} \right) \int ds \frac{\beta_x}{B\rho} \frac{\partial}{\partial x} (E_x / \lambda), \quad (3-2)$$

when $\lambda = I/v$ is the line charge density

The results will be found for an elliptical gaussian beam with charge density

$$\rho(x, y) = \frac{\lambda}{2\pi\sigma_x\sigma_y} \exp(-x^2/2\sigma_x^2 - y^2/2\sigma_y^2) \quad (3-3)$$

It is convenient to rewrite Eq (3-2) in terms of the Δv -parameter,

$$\Delta v = - \frac{1 + \beta^2}{\beta^2} \frac{I \times 10^{-6}}{B\rho} \frac{\beta_y^*}{(2\pi)^{1/2} \alpha \sigma_y^*} \quad (3-4)$$

where I is in amperes and $B\rho$ is in KG-m, and β_y^* and σ_y^* are evaluated at the crossing point. Δv is the vertical v-shift when long range interactions are neglected. Then Δv_x is given by

$$\Delta v_x = \Delta v (2\pi)^{1/2} \frac{\sigma_y^* \alpha}{4\pi\beta_y^*} \int ds \beta_x(s) \frac{\partial}{\partial x} (E_x/\lambda) \quad (3-5)$$

with a similar expression for Δv_y .

It will be assumed that in the interaction region

$$\beta_x(s) = \beta_x^* (1 + s^2/\beta_x^{*2}), \quad (3-6)$$

$$\beta_y(s) = \beta_y^* (1 + s^2/\beta_y^{*2})$$

Note that σ_x, σ_y also vary with s since $\sigma_x \propto \beta_x^{1/2}$ and $\sigma_y \propto \beta_y^{1/2}$

Δv_x can then be written as

$$\Delta v_x = \Delta v_{xS} + \Delta v_{xL}, \quad (3-7a)$$

where Δv_{xS} and Δv_{xL} are the short and long range contributions to Δv_x and are given by

$$\Delta v_{xS} = \Delta v (2\pi)^{1/2} \frac{\sigma_y^* \alpha}{4\pi\beta_y^*} \int ds \beta_x^* \frac{\partial}{\partial x} (E_x/\lambda). \quad (3-7b)$$

$$\Delta v_{xL} = \Delta v (2\pi)^{1/2} \frac{\sigma_y^* \alpha}{4\pi\beta_y^*} \int ds \frac{s^2}{\beta^{*2}} \frac{\partial}{\partial x} (E_x/\lambda). \quad (3-7c)$$

The integral over s is done over $\pm l$, where $-l$ to l is the interaction region for the crossing beams. From now on, the asterisk will be dropped from

β_x^* and β_y^* , and β_x , β_y refer to the beta functions at the crossing points.

Similar results apply to $\Delta v_y = \Delta v_{yS} + \Delta v_{yL}$.

3.1 Δv_x Derivation

For $\ell \rightarrow \infty$, and if σ_x does not depend on s , then $\Delta v_{xS} = 0$. One can write

$$\frac{d}{ds} E_x = \frac{\partial E_x}{\partial x} \frac{dx}{ds} + \frac{\partial E_x}{\partial y} \frac{dy}{ds} + \frac{\partial E_x}{\partial s} \quad (3.1-1)$$

Note E_x depends on s through $\sigma_x(s) \sim \beta_x(s)^{1/2}$, $\sigma_y(s) \sim \beta_y(s)^{1/2}$. For $x = \alpha s$, $y = 0$

$$\frac{d}{ds} E_x = \alpha \frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial s}, \quad (3.1-2)$$

and

$$\begin{aligned} \Delta v_{xS} = \Delta v (2\pi)^{1/2} \frac{\sigma_y \alpha}{4\pi} \left\{ \frac{2\beta_x}{\alpha} (E_x/\lambda)_{x=\ell} \right. \\ \left. - \beta_x \int \frac{dx}{\alpha} \frac{\partial (E_x/\lambda)}{\partial s} \right\} \end{aligned} \quad (3.1-3)$$

To evaluate Δv_{xS} , it will be assumed that in the first term of Eq (3.1-3), the asymptotic result for E_x/λ can be used. The second term represents the contribution of the longitudinal effect due to σ_x and σ_y not being constant but varying with s . This second term will be neglected as discussed in section 2.

The asymptotic result for E_x/λ is (see table below for when this result is valid)

$$E_x/\lambda = \frac{2}{x}, \quad (3.1-4)$$

$$\frac{\partial}{\partial x} (E_x/\lambda) = -\frac{2}{x^2}$$

The asymptotic results for E_x are valid when $x \gg \sigma$, $\sigma = \max \{\sigma_x, \sigma_y\}$. Thus when $\ell \gg \sigma$,

$$\Delta v_{xS} = \Delta v \frac{(2\pi)^{1/2} \sigma_y \alpha}{4\pi \beta_y} \left(\frac{2\beta_x}{\alpha} \frac{2}{\alpha \ell} \right), \quad (3.1-5)$$

$$\Delta v_{xS} = \Delta v \left(\frac{2}{\pi} \right)^{1/2} \frac{\sigma_y \beta_x}{\alpha \beta_y \ell},$$

The treatment of Δv_{xL} depends on breaking the region of integration up into two parts, from $x = 0$ to x_1 and from x_1 to $\alpha \ell$, where $x_1 = 2\sigma_x$. We now write

$$\begin{aligned} \int_0^\ell ds \, s^2 \frac{\partial}{\partial x} (E_x / \lambda) &= \frac{1}{\alpha^3} \left\{ \int_0^{x_1} dx \, x^2 \frac{\partial}{\partial x} (E_x / \lambda) \right. \\ &\quad \left. + \int_{x_1}^{\alpha \ell} dx \, x^2 \frac{\partial}{\partial x} (E_x / \lambda) \right\} \end{aligned} \quad (3.1-6)$$

We will show that under certain conditions, the $\int_0^{x_1} dx$ can be neglected compared to $\int_{x_1}^{\alpha \ell} dx$, and that the $\int_{x_1}^{\alpha \ell} dx$ can be evaluated by replacing $\partial E_x / \partial x$ by its asymptotic value $-2\lambda/x^2$ provided $\alpha \ell \gg \sigma_x$. The error made in replacing $\partial E_x / \partial x$ by its asymptotic form in the region $x = 2\sigma_x$ to $x = \alpha \ell$ is discussed later on in this section.

In the region $x_0 = 0$ to $x_1 = 2\sigma_x$, $\partial(E_x/\lambda)/\partial x$ and $-\partial(E_y/\lambda)/\partial y$ may be estimated by $4/\sigma_x^2$. This estimate is obtained from the following table which lists the field gradients at the beam center, $x = 0$, for various values of σ_y/σ_x .

σ_y/σ_x	$\sigma_x^2 \frac{\partial(E_x/\lambda)}{\partial x}$	$\sigma_x^2 \frac{\partial(E_y/\lambda)}{\partial y}$
1	1	1
.5	1.32	2.64
.333	1.50	4.4
.2	1.66	7.94

The estimate of $4/\sigma_x^2$ for the field gradients will hold if the beam is no flatter than $\sigma_y/\sigma_x = .333$.

Thus we find,

$$\begin{aligned}
 \int_0^{\alpha l} dx \, x^2 \frac{\partial}{\partial x} (E_x/\lambda) &= \int_0^{x_1} dx (\quad) + \int_{x_1}^{\alpha l} dx (\quad) \\
 &\approx \frac{4}{\sigma^2} \frac{8\sigma^3}{3} - 2 (\alpha l - x_1) \\
 &\approx \frac{32}{3} \sigma - 2 \alpha l + 4\sigma \\
 &\approx -2 \alpha l
 \end{aligned}$$

provided $\alpha l \gg 7\sigma$ where $\sigma = \sigma_x$.

Thus Δv_{xL} is given by

$$\Delta v_{xL} = - \Delta v (2\pi)^{1/2} \frac{\sigma_y \alpha}{4\pi \beta_y \beta_x} \left(\frac{4}{\alpha^3} \alpha l \right),$$

$$\Delta v_{xL} = - \Delta v \left(\frac{2}{\pi} \right)^{1/2} \frac{\sigma_y l}{\alpha \beta_y \beta_x}.$$

Adding the results for Δv_{xS} and Δv_{xL} , one gets for Δv_x

$$\Delta v_x = - \Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y l}{\alpha \beta_y \beta_x} (1 - \beta_x^2 / l^2) \quad (3.1-7)$$

We will now return to a discussion of the error made by replacing

$\partial(E_x/\lambda)/\partial x$ by $-2/x^2$ for $x \geq 2\sigma_x$ in the second term of Eq (3.1-6). This error will be investigated empirically by numerically computing E_x and $\partial E_x/\partial x$ for $x \geq 2\sigma_x$ and $y = 0$ for a gaussian beam for various values of σ_y/σ_x . These numerical results are given in the following table which shows when the gradients reach their asymptotic values for various σ_y/σ_x .

	x/σ_x	$\frac{\partial(E_x/\lambda)}{\partial x} / (-\frac{2}{x^2})$	$\frac{\partial(E_y/\lambda)}{\partial y} / (-\frac{2}{x^2})$
$\sigma_y = \sigma_x$	2	.339	.910
	3	.915	1.02
	4	1.01	1.02
	5	1.01	1.01
	6	1.01	1.01
$\sigma_y = .5\sigma_x$	2	.571	1.65
	3	1.21	1.40
	4	1.18	1.19
	5	1.11	1.11
	6	1.07	1.07
$\sigma_y = .2\sigma_x$	2	.842	3.44
	3	1.45	1.92
	4	1.27	1.30
	5	1.15	1.15
	6	1.09	1.09

One sees from the table, that for the range of σ_y/σ_x considered, the error in using the asymptotic expression for the gradients is no more than 30% for $x \geq 4\sigma_x$. There is an appreciable error for $2\sigma_x < x < 4\sigma_x$, especially for $\sigma_y/\sigma_x = .2$. However the integrand in Eq (2.1) is roughly constant over the range of integration $x = 2\sigma_x$ to $x = \alpha \ell$, and if $\alpha \ell \gg \sigma$ then the error made in the region $x = 2\sigma_x$ to $x = 4\sigma_x$ is small compared to the value of the integrand. Using the numbers in the table, one can estimate the error in the integral made by replacing $\partial(E_x/\lambda)/\partial x$ by $-2/x^2$ as being

$$\text{error} = \frac{(1.5) 2\sigma_x}{\alpha \ell}$$

which is small if $\alpha \ell \gg 3\sigma_x$.

3.2 Δv_y Derivation

As in the case of Δv_x we write Δv_y as (see Eq 3.7a)

$$\Delta v_y = \Delta v_{yS} + \Delta v_{yL}, \quad (3.2-1)$$

$$\Delta v_{yS} = \Delta v (2\pi)^{1/2} \frac{\sigma_y \alpha}{4\pi \beta_y} \int ds \beta_y \frac{\partial}{\partial y} (E_y/\lambda),$$

$$\Delta v_{yL} = \Delta v (2\pi)^{1/2} \frac{\sigma_y \alpha}{4\pi \beta_y} \int ds \frac{s^2}{\beta_y} \frac{\partial}{\partial y} (E_y/\lambda),$$

where β_y is the beta function at the crossing point.

The treatment of Δv_{yL} is similar to that of Δv_{xL} where essentially $\partial(E_y/\lambda)/\partial y$ is replaced by its asymptotic form $2/x^2$. One thus finds for Δv_{yL} (see Eq (3.1-4) for Δv_{xL})

$$\Delta v_{yL} = \Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y \ell}{\alpha \beta_y^2}, \quad (3.2-2)$$

where it is assumed that $\alpha \ell \gg \sigma$.

To compute Δv_{yS} , the divergence equation is used to replace $\partial(E_y/\lambda)/\partial y$ by

$$\frac{\partial(E_y/\lambda)}{\partial y} = \frac{4\pi}{\lambda} \rho - \frac{\partial}{\partial x} (E_x/\lambda) \quad (3.2-3)$$

The $\partial(E_x/\lambda)/\partial x$ gives the same kind of contribution found for Δv_{xS} , Eq (2.1-2), or

$$-\Delta v \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_x}{\alpha \ell},$$

where in Eq (3.1-2), β_x is replaced by β_y , the sign is reversed, and the contribution of the longitudinal effect is again neglected.

The $4\pi\rho/\lambda$ term is what gives the result $\Delta v_y = \Delta v$ when $\ell \rightarrow \infty$ and σ_x, σ_y do not depend on s . This term still gives the result Δv under our assumptions that $\alpha \ell \gg \sigma$ and $\alpha \beta \gg \sigma$. We can write

$$\int ds \rho \sim \int ds \frac{1}{2\pi\sigma_x(s)\sigma_y(s)} \exp(-x^2/2\sigma_x^2(s)).$$

Most of this integral comes from the region $s \lesssim \sigma_x/\alpha$; and in this region σ_x and σ_y are nearly constant since $\alpha \beta \gg \sigma$. Thus σ_x and σ_y can be evaluated at $s = 0$.

Adding the contributions due to the two terms in Eq (3.2-3) one gets

$$\Delta v_{yS} = \Delta v \left\{ 1 - \left(\frac{2}{\pi}\right)^{1/2} \sigma_y/\alpha \ell \right\}. \quad (3.2-4)$$

Adding the results for Δv_{yS} and Δv_{yL} , one gets for Δv_y

$$\Delta v_y = \Delta v \left\{ 1 + \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_y l}{\alpha \beta_y^2} (1 - \beta_y^2 / l^2) \right\} \quad (3.2-5)$$

Note that in computing Δv_y , as in computing v_x , we have neglected contributions due the σ_x, σ_y not being constant and varying with s .

References

1. E. Keil, C. Pellegrini and A.M. Sessler, Nucl. Inst. and Methods, 113, 333, (1973)