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**Theory of the Arc Discharge in
Air Blast Breakers**

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Theory of the Arc Discharge in Air Blast Breakers

Herbert Friedrich Vogel

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THEORY OF THE ARC DISCHARGE
IN AIR BLAST BREAKERS
Herbert Friedrich Vogel, Ph.D.

ABSTRACT

We study the airblast breaker, noted for its large current interrupting capacity. Our theoretical model is for the so-called double jet configuration in which a quasicylindrical arc develops during current interruption between the parting electrodes. The arc is cooled by a jet stream of air expanding from a pressurized reservoir, flowing radially inward in the space opening between the parting electrodes, and finally ejecting axially through the center bore of the electrodes.

In developing arc plasma models, recourse is generally taken to heuristic formulations, entirely on the macroscopic level. There is one notable exception, in which a study of the hydrodynamic aspects is reported in an application of a 13 moment method for approximating Boltzmann's kinetic equation.

Our own formalism is based on the normal Enskog-Chapman approximation in deriving the hydrodynamic equations from the Boltzmann equation. This method is much simpler to present than a 13 moment method. Because single charged positive nitrogen ions and atoms, in addition to electrons, are the predominant species, we consider such a three-component plasma. We use a reported simple parameterization of the heat transport by turbulent air and study the arc plasma's evolution, from the end of one relaxation time interval.

We show that the Boltzmann approximation applies and the arc plasma is in thermodynamic equilibrium, with characteristic times of less than a few tens of nanoseconds. We minimize the Helmholtz free energy on the microscopic and macroscopic scales to determine the arc's stable equilibrium and obtain relationships for closing our system of hydrodynamic equations. Using reported experimental values of pressure and air velocity to test our equations, we calculate the voltage

gradient, arc radius, and temperature. Our calculations agree with observation, when we use two fitting parameters. Those are the flow density of the surface heat dissipated by the turbulent air and the opacity. Our results indicate those two as being the dominant heat transport mechanisms.

The radial component in the temperature gradient is negligible in the arc's interior, because of the high degree of transparency and the magnetohydrodynamic $m = 0$ turbulence. In thermodynamic equilibrium, the $m = 0$ turbulence is shown to evolve on the time scale of one half to a few microseconds. Because the external gas pressure exceeds the electromagnetic pressure by at least one order of magnitude, the arc is wall confined. Pressure gradients deriving from intense surface cooling can cause the wall confinement to become upset when the Kadomtsev stability criterion is locally no longer satisfied, the $m = 0$ instability thus resulting. The set of equations obtaining in the arc's length element is thus developed, taking into account the significant boundary effects. The arc length is determined by the evolution of the state variables in the external circuit equations, closed by our relationships for the radius and voltage gradient, and an expression for the inductance variation as a function of radius and length.

We minimize the Helmholtz free energy on the microscopic and macroscopic scales to obtain relationships for closing our system of differential equations. Using the reported experimental pressure and air velocity vs the axial coordinate to test our equations, we calculate the voltage gradient, arc radius and temperature. Our calculations evidently agree with observation when we use two fitting parameters. These are the heat flow parameter for the turbulent transport, and the opacity of gray body radiation. Our results seem to indicate that these two are the dominant heat transport mechanisms.

Regarding the microscopic basis for our formalism, we show that the Boltzmann approximation is applicable to provide us with a microscopic picture of the arc plasma. We show that the stationary condition is asymptotically stable, with the characteristic times of typically less than a few tens of nanoseconds.

In summary, the complete set of equations obtaining in the arc's length element are given. The arc length is determined when the external circuit equations are closed by an expression for the arc inductance as a function of the radius and length, in addition to our relationships for the radius and voltage gradients.

CHAPTER 1

INTRODUCTION

1.1 State of the Art

The electric arc is one of the phenomena of great importance in the operation of practical electrical systems. Yet, its underlying physical mechanisms are still mystery to most engineers. That observation is taken verbatim from the two pertinent texts, that is W. Rieder, Plasma and Arc (1967, in German), and T. H. Lee, Physics and Engineering of High Power Switching Devices (1975). The two authors make a laudable attempt to introduce some of the important physics background to an engineering audience. A well written overview is offered in Chap. 3 of the book of A. Erk and M. Schmelzle, Elementary Switching Technology (1974, in German) which is worth reading, partly because it is free of the many inaccuracies encountered in the other texts. No closed mathematical form seems to be presented in any of the material mentioned thus far. Not surprisingly, therefore, A. Grütz is led to conclude in his doctoral dissertation on A Computer Study of Circuit Breakers, Using a Generalized Arc Theory (Technical University Aachen, 1970) that a) it is impossible to make an adequate estimate of the current interrupting capacity of a given circuit breaker interacting with a given circuit; b) the microscopic processes elude mathematical analysis for reasons of their complexity and multitude; and c) a useful heuristic method is developed, not considering the microscopic processes. We mention all this because of the great significance which the empirical methods have gained over the years, while the microscopic processes have consistently been regarded as elusive of mathematical treatment. From a practical point of view, that emphasis of the heuristic methods is not necessarily a disadvantage, because one is interested primarily in the resistance changes as a function of time, as in the electric spark, when the voltage between the electrodes breaks down. When air and airblast circuit breakers are concerned, one can take the presence of hot arc spots for granted as the source of an ample supply of charge carriers. The arc resistance follows, thus, from

certain empirical assumptions on the arc column radius, length, and temperature, which are the bases of the frequently cited arc models such as those of O. Mayr, A. M. Cassie, and other more refined ones.

In a way, one might compare this state of affairs with that in thermodynamics of times past, before the Austrian physicist Ludwig Boltzmann had published the first microscopic, i.e., kinetic, equation in 1872. James Clerk Maxwell had published his macroscopic, i.e., hydrodynamic, equation a few years earlier, immediately leading to the fluid dynamic equations of change as they had been known at the time (Euler, Navier, Stokes), as well as to those developed later (Burnett). Thus, the two greatest physicists of the century developed the mathematical relationships between the microscopic world of Hamilton's equations of motion, governing the trajectories of gas molecules, and the macroscopic fluid dynamic equations. These are known as Euler's equations and, to a higher order, the Navier-Stokes and Burnett equations. Realizing the complete hopelessness ever to integrate the set of equations of motion of the cosmological number of molecules contained in a practical volume, Boltzmann developed the statistical formalism by which the unique equivalence between Hamilton's and Boltzmann's equations is established, and which exhibits the important further property of time irreversibility. Thus, the Boltzmann equation defines the evolution of the statistically defined quantity called the distribution function which tends toward the equilibrium of maximum randomness and entropy, according to the equation, which was expected from thermodynamics. The equilibrium distribution is known as the Maxwell-Boltzmann velocity distribution which has the form of a Gaussian distribution of molecular speeds. The microscopically conserved quantities, called the dynamical functions, are averaged against that distribution function (whether equilibrium or not). Maxwell's hydrodynamic equation is a differential equation relating the velocity-averaged dynamical functions to the configuration space and time. The microscopically conserved quantities are the mass, momentum, and energy. The distribution function is defined as a number density in a six-dimensional space similar to the phase space in mechanics. Such a

six-dimensional volume is shown, generally, to be conserved in an ensemble or "cloud" of molecules evolving in that space, with gains and losses in the ensemble's 6-volume being caused by the collective effects of collisions. Because of the collisional gain versus loss integral, the Boltzmann equation is nonlinear and its integral is obtainable, in its exact form, only in two special cases. One is the noncollisional limit, which exists in the incompressible fluid. The other is that of the Maxwell Molecules for which Maxwell could derive the Boltzmann equation from his hydrodynamic equation. Maxwell Molecules are hypothetical, interacting according to the special force law, $F \propto r^{-5}$, where r is the separation between two isolated molecules. The difficulty in the case of the electric arc, and plasma physics in general, lies in the Coulomb interactions, with their force law $F \propto r^{-2}$, for which the kinetic equations are difficult to integrate.

Much of the matters discussed, thus far, is generally familiar from undergraduate studies at the level of Reif, Fundamentals of Statistical Physics and Thermal Physics, which are a part of most physics, mechanical, and chemical engineering curricula. The more challenging problems in the electric arc lie in the selection of an appropriate kinetic equation, determining whether there is statistical equilibrium, finding an appropriate integrating formalism, and closing the system of equations, considering the boundary conditions. In summary, the state of the art may be described by a variety of heuristic equations, as contrasted by a lack of closed systems of integrated kinetic equations. One important exception exists in the latter category, namely the work of Suchy and Thiel, in which the Boltzmann Equation is integrated, as discussed in the paragraph immediately following.

Instead of integrating both sides of the Boltzmann Equation, one can derive the integral of the left hand side (l.h.s.), which is the set of terms called the streaming function and take for the collision integral [that is the right hand side (r.h.s.)] the collision frequency formulas. The latter are known for certain idealizations, such as the fully ionized plasma. Formulas of that kind are related to the Spitzer

resistivity and thermal conductivity, published in many plasma physics texts. One part of our work is to develop the l.h.s., done in Chaps. 3 and 7. Another part is to develop the integral of the r.h.s., done in App. B, and compare our results for the fully ionized plasma with the Spitzer formulas. In extensive papers, Grad (1949), Weitzsch, and Suchy (1964) have reported a formalism known as the Thirteen Moment Method: The distribution function is expanded in a complete set of orthogonal functions, in the velocity, with coefficients $a_m(\underline{r}, t)$, which are functions of the spatial coordinate $\underline{r}$ and time. The Maxwell-Boltzmann distribution is contained as a component in the expansion. By substituting the expansion in the Boltzmann equation, one can obtain an infinite set of nonlinear differential equations for a_m . The various moments, such as the density n , velocity $\underline{v}$, temperature T , momentum $\underline{P}$, and current density $\underline{J}$, are simple functions of the coefficients a_m . In his dissertation, Thiel applies Suchy's coefficients to the special case of the arc plasma, developing the transport coefficients for it. The Thirteen Moment Method is applicable for the equilibrium and nonequilibrium case.

Let us return to a short introductory discussion of our own formalism, relating it to the state of the art. We show that the arc plasma is in statistical equilibrium. We can use, therefore, the normal solution of Enskog and Chapman in the hydrodynamic approximation which permits the much simpler presentation in our App. B, when compared with the more complicated Thirteen Moment Method. In summary, the distribution function is expanded in the uniformity parameter ϵ , measuring the departure of the state from that of local equilibrium, $f=f^{(0)} + \epsilon f^{(1)} + \dots$. Near the local equilibrium, one can define the number density $n(\underline{r}, t)$, flux $n\underline{v}(\underline{r}, t)$, and energy density $3/2 nkT(\underline{r}, t)$, as slowly varying functions of their arguments $(\underline{r}, t)$ and show that ϵ is proportional to the spatial gradient of those quantities. We show the system of differential equations developed in Chaps. 3 and 7 can be closed, which we do by minimizing the free energy defined in thermodynamics.

1.2 Scientific Motivation

The microscopic processes are generally considered to be beyond the mathematical scope of a reasonable arc model. One takes recourse to heuristic formulations, entirely on the macroscopic level. Celebrated are the early empirical models of A. M. Cassie (1939), and O. Mayr (1943). A noted exception is the important work by H. G. Thiel (1970),¹ which is a thorough study of the hydrodynamic aspects and an application of Suchy's 13-moment method for solving the Boltzmann kinetic equation. Aside from this particular application, use of the Weitzsch-Suchy method* seems uncommon, at least in the English language literature.² Yet, familiarity with Suchy's extensive treatise³ is necessary if one is intent on penetrating Thiel's work. However, deviations from statistical equilibrium are small in the electric arc. It follows that only the zeroth and first order of ϵ are needed in the Enskog-Chapman approximation, which is far more amenable to being presented in a transparent and compact form. By closing the equations, considering boundary and energy conditions, we can fill in another gap in the present literature.

1.3 Objectives

As in Thiel's dissertation, our focus is on the air blast breaker, noted for its enormous interrupting capacity and the resulting popularity among users in the electric utility industry, second only to the SF₆ breaker. Because of the large number of ionized species present in the SF₆ arc, we choose the air blast breaker, being more tractable as a three component plasma device. The configuration of the parting electrodes causes a quasi-cylindrical arc to evolve, as shown in the three configurations of Fig. 1a-c. An actuating mechanism drives the electrodes. The arc is cooled by a jet stream of air expanding with

*Weitzsch was probably first in extending Grad's well known 13-moment method mentioned in Sec. 3.2 to pronounced nonequilibrium. Suchy's paper offers a thorough analysis of the 13-moment methods.

Fig. 1 Contacts in air blast breakers.

- a) Single jet contacts
- b) Single jet contacts with one tubing type contact
- c) Double jet contacts
- d) Idealization of double jet contacts, for model

nearly sonic velocity from a pressurized reservoir into the atmosphere. The arrangement in Fig. 1c is symmetrical about the midplane, with the compressed air streaming radially inward, forming a stagnation point in the center, and ejecting axially through the center bore of the electrodes. The reservoir pressure is approximately 30 to 50 bar in a modern airblast breaker.

It is our objective to derive, for the configuration of Fig. 1c, a set of equations from first principles and in a well understood formalism and then, to simplify the equations as drastically as possible, still yielding agreement with known observation. For the stationary arc, our results are compared with observation.

Because the voltage drop at the electrode spots is negligible when compared with that in the arc column, we neglect all electrode voltage drops. The great magnitude of the arc drop in compressed air is indeed one of the air blast breaker's attractive features. It serves in the field to limit the current in many applications. We shall not consider afterglow phenomena.

1.4 Method and Conventions

From Fig. 1c we get, by idealization, the axisymmetric geometry depicted in Fig. 1d, where we assume the arc radius r to be fairly constant along the arc's center portion. Our idealization of neglecting the electrode drop is indicated schematically in Fig. 1d, by the two fictitious disks of arc plasma.

We discuss the Boltzmann equation. Its application is justified in our plasma parameters. We develop the hydrodynamic equations, using the normal solution method for the collision integral. Because singly charged positive ions and neutral atoms, in addition to the electrons, are the predominant species, we are considering such a three component plasma. We use Thiel's parameterization of the heat transport by turbulent air.

Because of the prevailing air pressure being at least atmospheric in arc chambers, the attendant electron densities are such as to imply Debye lengths of 0.5 to 2.5 μ s, and relaxation times of

subnanosecond duration. One of the important criteria for the validity of the Boltzmann equation is thus satisfied in our application, i.e., the existence of a time interval Δt , long compared with the collision time τ_c , and short compared with the relaxation time τ_r , where τ_r is on the order of the mean time interval separating subsequent collisions. We will show in Chap. 2 that $\tau_c \ll \Delta t \ll \tau_r$ is indeed satisfied. Physically, this condition provides the assurance that the collisions are well defined in space and time, with no overlapping between successive collisions of the same particle. This is an essential prerequisite for the idealization made in Boltzmann's molecular chaos assumption. It permits the treatment by simple binary collisions and evaluation of the loss-gain balance by simply counting collisions. Another important consequence of this two-body picture is its Markovian character. In other words, the evolution of the quantities of interest, depends only on their instantaneous values and not on previous history--as it would if many body collisions had to be considered. Hence, the system evolves asymptotically as long as it is not far removed from equilibrium. Knowledge of the arc ignition phase is, therefore, redundant to the study of the evolution.

As a final prerequisite⁴ for validity in applying the Boltzmann equation, we find the Debye length to be small, being certainly several orders of magnitude less than the smallest linear dimension of practical interest in the arc column. Hence, the distribution function does not change appreciably over a range of distances travelled by an average particle in a time interval Δt . As a further consequence of this condition, the generic properties of the arc are those of a plasma and not an ionized gas. The electron and ion temperatures must therefore be expected to be nearly the same.

Moment equations are not closed. They are the transport equations, i.e., relationships between hydrodynamic variables, leaving us with too many unknown functions. Additional relationships are needed for closure. One equation is created by the assumption that the plasma as a whole be stationary, which means that a stationary state is rapidly attained in which the sums of the electrons and ions possess equal, but

oppositely directed, momenta. Another relationship is contributed by the boundary condition that, the total current equal that of the external circuit. Two additional relationships are obtained from the requirement of a stationary free energy. The free energy has an extremum for a thermodynamic equilibrium system of fixed volume and mass, i.e., fixed density or specific volume. Extrema are determined with respect to an ionization process and the Coulomb interactions. The former yields the Saha equation, i.e., the law of mass action for the chemical equilibrium, and the latter results in two separate conditions, i.e., a microscopic and a macroscopic one. In the microscopic system, the Debye theory is usually used with the result that each electron and ion is effectively surrounded by a cloud of shielding charges with a spatial distribution function such that the free energy of the Coulomb interaction is an extremum. In the macroscopic system, we gain a relationship between the thermodynamic variables of random and drift velocities, i.e., between the electron temperature and the current. We determine the locus for this condition in Sec. 9.2, and Chap. 13.

We are thus left with only three external quantities, i.e., the gas kinetic pressure, the current, and the external heat dissipation parameter, complemented by our presumed existence of a constant current density profile within a cylindrical arc envelope. We show in Chap. 6 that a cylindrical arc envelope is indeed associated with a constant current density, provided the heat transfer function within the arc column is independent of the radius. We can show, in Sec. 11.3, that the radial temperature gradient is indeed negligible, because of magnetohydrodynamic turbulence evolving on the time scale of 0.5 to a few microseconds, under the existing boundary conditions on the arc surface. Consequently, we have a quasi-cylindrical arc envelope and a nearly flat current profile.

In summary, we study the arc's evolution, starting at the end of one relaxation time interval. Because we can close the set of hydrodynamic relationships, we obtain, in the stationary condition, a determined system of algebraic relationships, with characteristic times $\tau \approx 0.1 \text{ ns} \gg \tau_r$.

We use Gauss' symmetrical units or mks units throughout. Gaussian units in the case of our equations are identical with the esu. In App. A we show the conversion from cgs to mks. In addition, esu or mks is indicated in front of each chapter.

Only when referring to some Eq. (j) in a different chapter or appendix, i, we denote the reference as Eq. (i-j). We omit i in references within the same chapter or appendix, simply saying Eq. (j).

CHAPTER 2
CHARACTERIZATION OF ARC PLASMA
(esu)

Boltzmann's molecular chaos assumption means that the collision frequency can be calculated as long as time intervals Δt are considered for which the mean flight time $\gg \Delta t$ > average collision time. In other words, the dynamic relations between successive collisions are easily lost in chaos because the scatterers are so numerous and randomly distributed in space. This fundamental assumption imposes a limitation on the validity of many calculations in plasma physics, especially when low density plasmas are concerned. It imposes no limitation of consequence on arc plasma calculations. Three types of Coulomb interaction are considered in plasma physics, i.e., the short range, long range, and very long range interactions. In the case of the short range interactions, particles approach one another with an impact parameter of

$$b_0 = e^2 / 3 kT \quad (2-1)$$

or less, so they will be scattered through large angles, of at least 90° , in any single encounter. The cross section for this type of event is given by $\sigma_s = \pi b_0^2$ and it is easy to verify that the corresponding collision frequency is given by

$$\nu_s = \pi n e^4 m_e^{-1/2} (3 kT_e)^{-3/2} .$$

The long range interactions are limited by the shielding distance

$$\lambda_d = (kT_e / 4\pi n)^{1/2} / e .$$

The corresponding interaction time in a single encounter, $\tau_c = \lambda_d / \bar{v}$ is

determined by the distance λ_d over which particles may interact by their Coulomb fields. Hence, it is easily verified that

$$\tau_c = \frac{1}{e} \left(\frac{m_e}{12\pi n_e} \right)^{1/2} .$$

Rutherford scattering determines the distant collisions between electrons and ions. It is discussed in Chap. 5 and App. B-5. The corresponding collision frequency $\nu_r \equiv \nu_{ei}$ is shown in Table I to be much greater than the short range collision frequency.

Interactions over the very distant range $\lambda > \lambda_d$ do not occur, because of shielding, in a plasma with Maxwellian particle distribution in velocity space, as shown by the Debye-Hückel theory. In cases where encounters over the very distant range do have to be considered, the Vlasov equation provides a good model. In Table I we indicate that for extremely small currents, especially if the pressure is relatively small, the relative drift velocity may assume values close to 1, which would mean the electric field gradient was too great for the theory to be applicable. The energy gained in the electric field would then be comparable with the initial, i.e., the thermal energy. It would then be possible for electrons to accelerate to very high velocities and not be scattered at all, because the Rutherford scattering cross section decreases like (energy)⁻². This is the phenomenon studied by Giovanelli and Dreicer.^{5,6} Our discussion is then inadequate because it is based on small deviations from equilibrium, i.e., on elastic collisions. Observe that in some recent arc related work⁷ it seems to be assumed that the drift velocity were of a magnitude comparable with the thermal velocity of the particle.

Finally, we make two important assumptions, i.e.,

- a) that the arc plasma is homogeneous, which has been emphasized already; and

TABLE I
CHARACTERISTIC DATA FOR ARC PLASMA

Gas kinetic pressure		bar	1	24
Equilibrium temperature	T	kK	16	20
Electron density	n	m ⁻³	1.3·10 ²³	2.6·10 ²⁴
Neutral particle density	n ₀	m ⁻³	1.8·10 ²³	4.0·10 ²⁴
Total current	I	kA	0.1 1.3 2.4	1.3 2.4
Current density	J	10 ⁸ Am ⁻²	1.9 0.8 0.7	1.6 1.31
Impact parameter, 90° scattering	b ₀	Å	3.5	2.8
Short range scattering cross section	σ _s	Å ²	38	24
Long range scattering cross section	σ _d	Å ²	1.8·10 ⁵	1.2·10 ⁴
Λ = r _d /r _s		l	70	22
ln Λ		l	4.2	3.1
Collision (i.e., interaction) time	τ _c	ns	2.8·10 ⁻⁵	0.64·10 ⁻⁵
Relaxation time	τ _r	ns	71·10 ⁻⁵	6.7·10 ⁻⁵
Ratio τ _c /τ _r		l	4.0·10 ⁻²	9.5·10 ⁻²
No. of particles in a Debye space		l	7.7	2.4
Mean random speed $\bar{v}_e = \left(\frac{8}{\pi} \frac{kT_e}{m_e}\right)^{1/2}$		m/s	8·10 ⁵	9·10 ⁵
$\bar{v}_i = \left(\frac{8}{\pi} \frac{kT_i}{m_i}\right)^{1/2}$		m/s	5·10 ³	6·10 ³
Drift velocity v _{de} = J _e /en		m/s	9·10 ³ 4·10 ² 3·10 ³	350/400
v _{di} = J _i /en		m/s	20/50	2/5
Characteristic lengths: σ ₀ = e ² /kT		Å	10.4	8.4
d = n ^{-1/3}		Å	200	73
L = (nσ ₀ ² lnΛ) ⁻¹		Å	170	18
Debye r _d ≡ λ _d		Å	242	61
Arc radius		mm	2	0.5

b) that a sufficient flux of electrons and ions is supplied by the cathode.

Condition b) means there will be no arc unless the cathode is sufficiently hot. Small (needle type) cathodes may sustain small-current arcs, whereas large, cooled, cathodes require great current to sustain the arc.

The most important difference between the arc studied in the present work and the spark is in the mechanism serving as carrier supply. In sparks, it is a cathodic feedback avalanche, whereas in arcs, one has a simple avalanche supply. In the feedback avalanche, the photon flux emanating from the arc creates new charges by photoemission from the cathode. Once the photon flux is sufficiently intense and energetic to overcompensate the loss of carriers, the emitted flux is driven to saturation by the feedback avalanche. In arcs, on the other hand, one has thermoemission. Atoms and single-charged ions are the predominant species.¹ We confine ourselves to these.

CHAPTER 3
HYDRODYNAMIC EQUATIONS OF MULTICOMPONENT PLASMAS
(mks)

3.1. Introduction

The streaming terms, of which the left side of all kinetic equations are composed, originate with the Liouville equation. The differences among the kinetic equations rest in the collision integral. The Vlasov equation is the simplest kinetic equation of plasmas. Because it is reversible in time, it is not suited to describe an irreversible approach to equilibrium. Consequently, the Vlasov equation is of particular use in treating phenomena with very short characteristic times, such as Landau damping, and free plasma oscillations in which collisional damping (responsible for the relaxation to equilibrium) plays an insignificant role. Another important kinetic equation is the Fokker-Planck equation in which the effect of molecular collisions is treated stochastically. Being itself nonlinear, this equation is most often used in a linear approximation. The divergence at large interparticle distances in the field of the Coulomb interaction potential is solved by the same cut-off procedure as in the Boltzmann equation. the collision term in the Fokker-Planck equation is calculated on the assumption that there are simultaneous random changes of small vector length in the momentum of the particle, due to its interaction with the other particles of the gas. Consequently, the equation is used for some plasmas in which treatment by a binary collisions formalism may not be justified.

For seven decades after 1872, the Boltzmann equation remained the equation with which to describe irreversible processes in a gas. When one applies the Boltzmann or the Fokker-Planck equation to plasmas, a divergence difficulty arises from the long range of the Coulomb interactions. To avoid it, one inserts a correction for the electric charges' collective effect. We do this by introducing a cut-off equal to the Debye-Hückel length, thus imposing an upper limit on the impact parameter.

Let us determine whether Boltzmann's binary collision assumption is justified in the present case. We consider the following characteristic lengths in a plasma with the mean temperature T and mean particle number density $n = d^{-3}$, as in Chap. 6.1 in Wu's book⁸ (q.v.):

$$\sigma_0 = \frac{e^2}{kT}; \quad d = n^{-1/3}; \quad r_D = (kT/4\pi n e^2)^{1/2} \quad (\text{esu})$$

$$L = (n\sigma_0^2 \ln \Lambda)^{-1}.$$

These quantities are listed in Table i; they confirm the existence of interparticle distances r such that $\sigma_0 < r < \sigma$, where $\sigma_0 \ll \sigma \ll d$. This indicates that any two particles "in a Debye sphere" do not "see" other particles between one another, hence their interaction may be considered a binary one for which the molecular chaos assumption is indeed justified. For interparticle distances r such that $\sigma < r < r_D$, there are many particles in a sphere of radius r surrounding the particle; but at $r \gg r_D$, Debye shielding prevails. We have thus shown that it is useful and physically justified for us to proceed with the Boltzmann equation. Methods for solving it were developed by Hilbert (1912), Enskog (1911, 1917), Chapman (1911, 1916), Grad (1949), and others.

3.2. Selection of a Solution Method

Already Boltzmann recognized in his papers on viscosity that his equation admits convenient solutions only in the case of Maxwell molecules (Maxwell, assuming molecules to interact according to his r^{-5} law, showed that all collision integrals can be solved for $s = 5$, with no distribution function being given). Hilbert showed it is possible to obtain the solution by solving an infinite series of linear integral equations of the second kind; although this is not a convenient form for the distribution function f . Enskog determined f from the Boltzmann equation by a modification of Hilbert's method. He succeeded in

deriving general formulas for the three coefficients of viscosity, conduction and diffusion in gases of one or several constituents; and calculated the stress tensor up through the third approximation. Chapman, by a different method, arrived at the same results for the same three coefficients. The details of the Enskog-Chapman method are discussed in App. B and in Chapman and Cowling's book.⁹ Important additional reviews are presented in Ferziger and Kaper,¹⁰ Jancel and Kahan,¹¹ and in Wu's book.⁸ Further contributions on the subject are conveyed in the book of Hirschfelder, Curtiss and Bird.¹³ By the method of Enskog and Chapman, the Boltzmann equation is expanded, in powers of a scaling quantity for the number density, to a set of nonhomogeneous linear integral equations which are solved by way of the hydrodynamic equations for the macroscopic quantities of density, velocity, and temperature. In effect, the initial value problem of the Boltzmann equation is thus replaced by a series of spatial derivatives of various orders.

In an important paper,¹² Grad introduces a different method, which consists in expanding the distribution function in a complete set of orthogonal functions (either Hermite or Sonine polynomials, depending on the coordinate geometry) in the velocity, with coefficients that are functions of the spatial coordinates and time. When one substitutes this expansion in the Boltzmann equation and forms various velocity moments, he obtains for the moment equations an infinite set of first order linear differential equations for the coefficients $a_m(\underline{r}, t)$. In solving this system, one truncates such as to be left with 13 moments. Because no functional dependence of $f(\underline{r}, \underline{v}, t)$ on n , $\underline{v}$, and T is assumed, this method is in principle not restricted, like the Enskog-Chapman method, to small deviations from local equilibrium. However, since the coefficients $a_m(\underline{r}, t)$ are linear combinations of the moments of the distribution function, it follows that the set of equations for a_m is one of order m , which one would obtain otherwise by operating with the dynamic functions of order m on the Boltzmann equation. Once the expansion for the collision operators J_m is substituted, one has obtained an infinite set of differential equations for the coefficients

a_m , the latter being time derivatives (representing the distribution function). The description of a gas by thirteen variables* (10 in the 2-D case, 4 in the 1-D case) is characteristic for Grad's method, as distinguished from the Enskog-Chapman method, the latter operating with only the five parameters $n, \underline{v}, T$. Grad's method is more effective, therefore, in obtaining good accuracy, though one encounters difficulties in obtaining a solution of the m equations, because the left side contains all the Hermitian coefficients up to, and including, the order $m + 1$, whereas the r.h.s. contains the m coefficients in the expansion of J_m .

A detailed account of the Enskog-Chapman method is presented in App. B, where the hydrodynamic equations of the Lorentz gas are derived, and its transport coefficients calculated. The form of the streaming operator in the multicomponent plasma with magnetic field is developed in the section immediately following.

Let us agree that characters with the diacritical $\sim$ denote tensors, their order** being determined by the physical nature. Velocities, for example, are first order tensors (i.e., vectors) denoted $\underline{v}, \underline{V}$.

3.3 The Multicomponent Plasma

3.3.1. Multicomponent Averages

Let $f_i(\underline{r}, \underline{v}, t)$ be the distribution function of the i -th component, $\underline{v}_i$ the actual velocity of an i -th component particle, a bar denote the average value of a dynamic function, and $d\underline{v}$ a differential

*One counts 20 equations for the first through third moment, but he reduces the set to 13 by dropping the linearly nearly dependent ones; hence the name "13-moment method".

**Order is sometimes called valence, or rank, there being no convention.

volume in velocity space $dv = dv_x dv_y dv_z$. One may thus define for the i -th component the following quantities.

- a) Number density..... $n_i = \int f_i dv_i$;
- b) Average velocity..... $\bar{v}_i = \frac{1}{n_i} \int v_i f_i dv_i$;
- c) Mass average velocity..... $v_o = \frac{1}{\rho} \sum_i n_i m_i \bar{v}_i$
- d) Overall density of the gas
at a given point..... $\rho = \sum_i \rho_i = \sum_i n_i m_i$
- e) Peculiar velocity of Chapman $\underline{v}_i = v_i - v_o$
and Cowling where one has..... $\sum_i \rho_i \underline{v}_i = 0$
- f) Diffusion velocity w.r.t.
the mass average velocity of
the gas..... $\bar{\underline{v}}_i = \bar{v}_i - v_o = \frac{1}{n} \int (v_i - v_o) f dv$;
- g) Specific thermal velocity..... $\underline{v}'_i = v_i - \bar{\underline{v}}_i = v_i + v_o - \bar{v}_i$
reduced to $\bar{\underline{v}}_i$ where one has..... $\underline{v}'_i = 0$
- h) Gas temperature..... $\frac{3}{2} kT = \frac{1}{2} m \overline{v'^2} = \frac{1}{n} \sum_i \int f_i \frac{m_i}{2} v_i'^2 dv_i$
- i) Total pressure tensor..... $P = \sum_i p'_i + \sum_i n_i m_i \underline{u}_i \underline{u}_i$
where..... $\underline{u}_i = \bar{\underline{v}}_i - v_o$
- j) Partial pressure tensor $P'_i = n_i m_i \overline{\underline{v}'_i \underline{v}'_i}$
- k) Thermal flux vector $q'_i = \frac{1}{2} \rho_i \overline{v_i'^2 \underline{v}_i}$

The quantities defined in j) and k) are calculated in App. B.

The definitions a) - f) are the customary ones, given in Chap. 7.2 a of Hirschfelder, Curtiss and Bird's book.¹³ The quantities a) - k) are discussed also in Chap. 6.2 of Jancel and Kahan's book.¹¹

3.3.2. Derivation of the Maxwell Equation for a Multicomponent Plasma in the Presence of an E.M. Field

Consider encounters with impact parameter b between particles of the species i and j . The probable number of collisions in $db dt$ between particles having $\underline{v}_i$ in $d\underline{v}_i$ and those having $\underline{v}_j$ in $d\underline{v}_j$, $b db$ being the scattering cross section, is given by the collision integral. The collisions change $\underline{v}$ and hence decrease or increase f to f' . We follow the formalism developed in Jancel and Kahan's book¹¹ (Chaps. 5

and 6) and write the Boltzmann equation for the i -th component in terms of the streaming operator $\partial/\partial t$ and the collision integral:

$$\frac{\partial f_i}{\partial t} = \sum_j \iint (f_i' f_j' - f_i f_j) g_{ij} b db d\mathbf{v}_i \equiv \sum_j \left(\frac{\delta f_i}{\delta t} \right)_j. \quad (3-1)$$

Maxwell's transport equation is a macroscopic equation, obtained by multiplying Eq. (1) by the dynamic function $\phi_i(\mathbf{r}, \mathbf{v}_i, t)$ and integrating over velocity space:

$$\int \phi_i \frac{\partial f_i}{\partial t} d\mathbf{v}_i = \int \phi_i \frac{\delta f_i}{\delta t} d\mathbf{v}_i, \quad (3-2)$$

where $\delta f_i / \delta t \equiv \sum_j (\delta f_i / \delta t)_j$ is a sum over all the species. We have from Liouville's equation, discussed in App. C (q.v.):

$$\int \phi_i \frac{\partial f_i}{\partial t} d\mathbf{v}_i = \int \phi_i \left(\frac{\delta f_i}{\delta t} + \mathbf{v}_i \cdot \nabla_i f_i + \frac{\mathbf{F}_i}{m_i} \cdot \nabla_{\mathbf{v}_i} f_i \right) d\mathbf{v}_i. \quad (3-3)$$

With the chain rule, we easily obtain the following identities:

$$\int \phi_i \frac{\partial f_i}{\partial t} d\mathbf{v}_i = \frac{\partial}{\partial t} \overline{n_i \phi_i} - n_i \overline{\frac{\partial \phi_i}{\partial t}} \quad (3-3a)$$

$$\int \phi_i v_{ix} \frac{\partial f_i}{\partial x} d\mathbf{v}_i = \frac{\partial}{\partial x} \overline{n_i \phi_i v_{ix}} - n_i v_{ix} \overline{\frac{\partial \phi_i}{\partial x}} \quad (3-3b)$$

$$\frac{\mathbf{F}_i}{m_i} \int \phi_i \frac{\partial f_i}{\partial v_{ix}} d\mathbf{v}_i = \iint [\phi_i f_i] \sum_{v_{ix}=-\infty}^{\infty} dv_{iy} dv_{iz} - \int f_i \frac{\partial f_i}{\partial v_{ix}} d\mathbf{v}_i = - n_i \overline{\frac{\partial \phi_i}{\partial v_{ix}}}. \quad (3-3c)$$

By collecting the three identities, we can write Maxwell's transport equation in the form

$$\frac{\partial n_i \bar{\phi}_i}{\partial t} = - \underline{v}_r \cdot (n_i \underline{\dot{x}}_i \underline{v}_i) + n_i \left[\frac{\partial \bar{\phi}_i}{\partial t} + \underline{v}_i \cdot \nabla_r \bar{\phi}_i + \frac{\underline{F}_i}{m_i} \cdot \nabla_{\underline{v}_i} \bar{\phi}_i + \delta \bar{\phi}_i \right], \quad (3-4)$$

where we used the abbreviation $\delta \bar{\phi}_i = 1/n_i \int \bar{\phi}_i \delta f_i / \delta t \, d\underline{v}_i$. Note that $\delta \bar{\phi}_i$ is just the abbreviation defined, not to be confused with the abbreviations $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ used elsewhere in this treatise.

In the end, we will need the heat flow equation, therefore the variables should be changed from the present $\underline{v}_i$ to a new $\underline{V}_i$ dependency, where we recall $\underline{V}_i(\underline{r}, t) = \underline{v}_i(t) - \underline{v}_0(\underline{r}, t)$. It follows that the time derivatives in Eq. (4) transform like

$$\frac{\partial \phi(\underline{v}_i, \underline{r}, t)}{\partial t} = \frac{\partial \phi(\underline{V}_i, \underline{r}, t)}{\partial t} + \left(\frac{\partial \phi(\underline{V}_i, \underline{r}, t)}{\partial \underline{V}_i} - \frac{\partial \phi(\underline{v}_i, \underline{r}, t)}{\partial \underline{v}_i} \right) \frac{\partial \underline{v}_i}{\partial t} - \frac{\partial \phi}{\partial \underline{v}_i} \frac{\partial \underline{v}_0}{\partial t},$$

where $\partial \underline{v}_i / \partial t = 0$ is needed so the middle term on the r.h.s. may vanish and not complement to the total derivatives on both sides of the equation, which is consistent with the text statement immediately following Eq. (8). Thus we replace

$$\begin{aligned} \frac{\partial \phi(\underline{v}_i, \underline{r}, t)}{\partial t} &\rightarrow \frac{\partial \phi_i(\underline{V}_i, \underline{r}, t)}{\partial t} - [\nabla_{\underline{V}_i} \phi_i(\underline{V}_i, \underline{r}, t)] \cdot \frac{\partial \underline{v}_0}{\partial t} \\ \nabla_r \phi_i(\underline{v}_i, \underline{r}, t) &\rightarrow \nabla_r \phi_i(\underline{V}_i, \underline{r}, t) - [\nabla_{\underline{V}_i} \phi_i(\underline{V}_i, \underline{r}, t)] \cdot \nabla_r \underline{v}_0. \end{aligned}$$

With these substitutions, Eq. (4) becomes

$$\begin{aligned}
\frac{\partial}{\partial t} n_i \overline{\phi_i} = & - \nabla_r \cdot \{ n_i [\phi_i (\underline{v}_0 + \underline{v}_i)] \} + n_i \left\{ \frac{\partial \phi_i}{\partial t} - (\nabla_{\underline{v}_i} \phi_i) \cdot \frac{\partial \underline{v}_0}{\partial t} + \underline{v}_0 \cdot \nabla_r \phi_i + \right. \\
& + \overline{\underline{v}_i \cdot \nabla_r \phi_i} - \overline{(\nabla_{\underline{v}_i} \phi_i) \cdot (\underline{v}_0 \cdot \nabla_r) \underline{v}_0} - \overline{\nabla_{\underline{v}_i} \phi_i \cdot (\underline{v}_i \cdot \nabla_r) \underline{v}_0} + \\
& \left. + \frac{\underline{F}_i}{m_i} \cdot \overline{\nabla_{\underline{v}_i} \phi_i} + \delta \overline{\phi_i} \right\} \quad (3-5)
\end{aligned}$$

Let us introduce a special notation for the substantial derivative or derivative following the motion, $D/Dt \equiv \partial/\partial t + \underline{v}_0 \cdot \nabla_r$, by which any ambiguity with the total derivative d/dt (i.e., one w.r.t. all the independent variables, $\underline{r}, \underline{v}, t$) will be avoided.

We recall* the tensor identity $\nabla_{\underline{v}} \phi \cdot (\underline{v} \cdot \nabla_r) \underline{v} \equiv (\nabla_{\underline{v}} \phi) \underline{v} : \underline{\nabla}_r \underline{v}$ so we may write Eq. (5) in the following form

$$\begin{aligned}
& * \begin{pmatrix} \frac{\partial \phi}{\partial v_x} \\ \frac{\partial \phi}{\partial v_y} \end{pmatrix} \cdot \begin{pmatrix} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \\ v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \end{pmatrix} \equiv \frac{\partial \phi}{\partial v_x} \left(v_x \frac{\partial v_x}{\partial x} + \frac{\partial \phi}{\partial v_y} v_y \frac{\partial v_x}{\partial y} \right) + \frac{\partial \phi}{\partial v_y} \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) \\
& \equiv v_x \left(\frac{\partial \phi}{\partial v_x} \frac{\partial v_x}{\partial x} + \frac{\partial \phi}{\partial v_y} \frac{\partial v_y}{\partial x} \right) + v_y \left(\frac{\partial \phi}{\partial v_x} \frac{\partial v_x}{\partial y} + \frac{\partial \phi}{\partial v_y} \frac{\partial v_y}{\partial y} \right) \equiv \\
& \equiv T_r \left\{ \begin{pmatrix} \frac{\partial \phi}{\partial v_x} v_x & \frac{\partial \phi}{\partial v_y} v_y \\ \frac{\partial \phi}{\partial v_y} v_x & \frac{\partial \phi}{\partial v_y} v_y \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{D(n_i \phi_i)}{Dt} + n_i \phi_i \nabla_r \cdot \underline{v}_0 + \nabla_r \cdot (n_i \phi_i \underline{v}_i - n_i \overline{\left\{ \frac{D\phi_i}{Dt} + \underline{v}_i \cdot \nabla_r \phi_i \right.}} \\
& \left. + \left(\frac{\underline{F}_i}{m_i} - \frac{D\underline{v}_0}{Dt} \right) \cdot \overline{\underline{v}_i \phi_i} - \overline{(\nabla_{\underline{v}_i} \phi_i) \underline{v}_i : \nabla_r \underline{v}_0} \right\} = n_i \overline{\delta \phi_i} . \quad (3-6)
\end{aligned}$$

Thus far, we have been using for the external force function the velocity independent form $\underline{F}_i(\underline{r}, t)$. This is because, originally, $\underline{F}_i$ was defined by the Liouville equation, where $\underline{F}_i \equiv m d\underline{v}_i/dt$. In the Lorentz force $\underline{Z} = e(\underline{E} + \underline{v} \times \underline{B})$, however, the electrodynamic term $\underline{v} \times \underline{B}$ is a velocity function. It follows that the $\underline{v} \times \underline{B}$ term cannot, like $\underline{F}_i$, be taken in front of the integral in the Maxwell transport equation. Let us proceed with the integration. Consider the streaming terms in the Boltzmann ($\mathcal{B}$) and Maxwell ($\mathcal{M}$) equations, i.e.,

$$\begin{aligned}
\mathcal{B} & \equiv \frac{\nabla_{\underline{v}_i} f_i}{\nabla_{\underline{v}_i} t} = \frac{\partial f_i}{\partial t} + \underline{v}_i \cdot \nabla_{\underline{r}} f_i + \frac{\underline{F}_i}{m_i} \cdot \nabla_{\underline{v}_i} f_i ; \\
\mathcal{M} & \equiv \int \phi_i \frac{\nabla_{\underline{v}_i} f_i}{\nabla_{\underline{v}_i} t} d\underline{v}_i = \int \phi_i \left(\frac{\partial f_i}{\partial t} + \underline{v}_i \cdot \nabla_{\underline{r}} f_i \right) d\underline{v}_i + \int \phi_i \frac{\underline{F}_i}{m_i} \cdot \nabla_{\underline{v}_i} f_i d\underline{v}_i .
\end{aligned}$$

Substitute for $\underline{F}_i/m_i$ the electrodynamic force $\underline{Z}_i = e/m \cdot \underline{A} \equiv \underline{Z}_i = e/m \cdot (\underline{v} \times \underline{B})$. The source term of $\mathcal{M}$ may thus be written

$$\int \phi_i \sum_{\underline{x}} \underline{A}_{\underline{x}} \frac{\partial f_i}{\partial v_{ix}} d\underline{v}_i ,$$

where the summation over the coordinate related three algebraic terms of $\underline{A} \cdot \nabla_{\underline{v}_i} f_i$ is denoted. For any of these three terms, we get

$$\int \phi_i A_x \frac{\partial f_i}{\partial v_{ix}} dv_i = \iint [\phi_i A_x f_i]_{v_{ix}=-\infty}^{\infty} dv_{iy} dv_{iz} - \int f_i \frac{\partial A_x \phi_i}{\partial v_{ix}} dv_i ,$$

where the first term on the r.h.s. vanishes at the boundaries ($f_i = 0$ for $v \rightarrow \infty$). Hence,

$$\int \phi_i A_x \frac{\partial f_i}{\partial v_{ix}} dv_i = - n_i A_x \frac{\partial \phi_i}{\partial v_{ix}} - n_i \phi_i \frac{\partial A_x}{\partial v_{ix}} ;$$

where, in turn, the last term is zero identically, by definition [i.e., $A_x = A_x(v_y, v_z, B_y, B_z)$ is independent of v_x]. We are thus left with

$$\frac{Z_{ie}}{m_i} \int \phi_i (\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} f_i dv_i = - n_i \frac{Z_{ie}}{m_i} (\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} \phi_i .$$

Hence, if the Lorentz force is the only external force, the Maxwell equation in the $\underline{v}_i$ -dependent form of Eq. (4) becomes

$$\begin{aligned} \frac{\partial n_i \phi_i}{\partial t} = & - \nabla_r \cdot (n_i \phi_i \underline{v}_i) \\ & + n_i \left[-\frac{\partial \phi_i}{\partial t} + \underline{v}_i \cdot \nabla_r \phi_i + \frac{Z_{ie} E}{m_i} \cdot \nabla_{\underline{v}_i} \phi_i + \frac{Z_{ie}}{m_i} (\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} \phi_i + \delta \phi_i \right] . \quad (3-7) \end{aligned}$$

Similarly, for $\underline{v}_i$ as an independent variable, where we replace in Eq. (6) the term $\underline{E}_i / m_i \cdot \nabla_{\underline{v}_i} \phi_i$ by the new expression

$$\frac{Z_i e n_i}{m_i} (\underline{E} + \underline{v}_0 \times \underline{B}) \cdot \overline{\nabla_{\underline{v}_i} \phi_i} - \frac{Z_i e n_i}{m_i} \overline{(\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} \phi_i} .$$

With this substitution, Eq. (6) becomes:

$$\begin{aligned} \frac{D n_i \overline{\phi_i}}{Dt} + n_i \overline{\phi_i} \nabla_r \cdot \underline{v}_0 &= - \nabla_r \cdot (n_i \overline{\phi_i} \underline{v}_i) + n_i \left\{ \frac{D \phi_i}{Dt} + \overline{\underline{v}_i \cdot \nabla_r \phi_i} + \right. \\ &+ \left\{ \frac{Z_i e}{m_i} (\underline{E} + \underline{v}_0 \times \underline{B}) - \frac{D \underline{v}_0}{Dt} \right\} \cdot \overline{\nabla_{\underline{v}_i} \phi_i} + \\ &+ \frac{Z_i e}{m_i} \overline{(\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} \phi_i} - \overline{(\nabla_{\underline{v}_i} \phi_i) \underline{v}_i \cdot \nabla_r \underline{v}_0} + \overline{\delta \phi_i} \left. \right\} . \end{aligned} \quad (3-8)$$

We can now substitute the privileged quantities, formally denoted ϕ_i , to obtain the moment equations. Because ϕ_i represents a set of velocity functions, the terms in $\partial \phi_i / \partial t$ vanish in Eqs. (7) and (8), the average peculiar velocity being zero and the rate of change in the average velocity being small, cf. the footnote in the section immediately following.

3.3.3. The Moment Equations

$\phi_i = m_i$ leads to the continuity equation. Because $\nabla_{\underline{v}_i} \phi_i = 0$ in this case, the Lorentz force is inconsequential and the continuity equation for the i -th constituent reads

$$\frac{\partial n_i}{\partial t} + \nabla_r \cdot (n_i \underline{v}_i) = \int d\underline{v} \left(\frac{\partial f}{\partial t} \right)_c . \quad (3-9)$$

The integral on the right is the rate of ionization of the positive and

negative charges. The rate is negative for the neutral atoms. Because $f_i m_i = f'_i m'_i$, in elastic collisions no collision integral appears.

$\phi_i = m_i \underline{v}_i$ By substituting ϕ_i in each term in Eq. (7), the following expressions appear:

$$\frac{\partial}{\partial t} n_i \phi_i = n_i m_i \frac{\partial \underline{v}_i}{\partial t} - \nabla_r \cdot (n_i m_i \bar{\underline{v}}_i); \quad (\text{cf. Eq. 9})$$

$$\nabla_r \cdot (n_i \overline{\phi_i \underline{v}_i}) = \nabla_r \cdot [n_i m_i \overline{(\underline{v}_i + \underline{v}'_i)(\underline{v}_i + \underline{v}'_i)}] = \nabla_r \cdot (n_i m_i \bar{\underline{v}}_i \bar{\underline{v}}_i) + \nabla_r \cdot (n_i m_i \overline{\underline{v}_i \underline{v}'_i}) ;$$

$$\nabla_r \cdot (n_i m_i \underline{v}_i \underline{v}_i) = n_i m_i (\underline{v}_i \cdot \nabla_r) \bar{\underline{v}}_i + n_i m_i \bar{\underline{v}}_i (\nabla_r \cdot \bar{\underline{v}}_i) + \bar{\underline{v}}_i (\bar{\underline{v}}_i \cdot \nabla_r n_i m_i) ;$$

$$\nabla_r \cdot (n_i m_i \overline{\underline{v}_i \underline{v}'_i}) = \nabla_r \cdot \underline{p}'_i .$$

By collecting terms, we obtain from Eq. (7)

$$n_i m_i \left[\frac{\partial \bar{\underline{v}}_i}{\partial t} + (\bar{\underline{v}}_i \cdot \nabla_r) \bar{\underline{v}}_i \right] = - \nabla_r \cdot \underline{p}'_i + n_i Z_i e [\underline{E} + \bar{\underline{v}}_i \times \underline{B}] + n_i \sum_{j \neq i} \delta_{jm_i} \bar{\underline{v}}_i, \quad (3-10)$$

where the collision integral

$$n_i \sum_{j \neq i} \delta_{jm_i} \bar{\underline{v}}_i \equiv \sum_{j \neq i} \int m_i \bar{\underline{v}}_i \left(\frac{\partial f_i}{\partial t} \right)_j d\underline{v}_i ; \quad (3-11)$$

and where

$$\sum_i \sum_{j \neq i} n_i \delta_{jm_i} \bar{\underline{v}}_i = 0 . \quad (3-12)$$

For instance, in a plasma composed of the components $j \equiv (e)$ (i.e.,

electrons), and $i = (i)$ (i.e., ions), we have $n_e \delta_i(m_e \bar{v}_e) + n_i \delta_e(m_i \bar{v}_i) = 0$ in the stationary state.

$\phi_i = m_i/2 \overline{v_i'^2}$. We have $\nabla_{\underline{v}_i} \phi_i = m_i \underline{v}_i' = 0$ and obtain from Eq. (7) the heat balance equation, where the following term needs to be evaluated.

$(\underline{v}_i \times \underline{B}) \cdot \nabla_{\underline{v}_i} (m_i/2) \overline{v_i'^2} = m_i (\underline{v}_i \times \underline{B}) \cdot \underline{v}_i' + m_i (\underline{v}_i' \times \underline{B}) \cdot \underline{v}_i' = 0$, because $(\underline{v}_i \times \underline{B}) \equiv \underline{\tilde{v}}_i \times \underline{B} + \underline{v}_i' \times \underline{B}$. In integrating the first term, $\underline{\tilde{v}}_i \times \underline{B}$ may be treated as a constant, whereas the second term must be integrated in its entirety. The first term vanishes, because $\underline{v}_i'$ does, and the second term is zero by vector identity. We may thus write the heat balance equation:

$$\frac{1}{2} \frac{\partial(\rho_i \overline{v_i'^2})}{\partial t} + \nabla_r \cdot \left(\frac{1}{2} \rho_i \overline{v_i'^2 \underline{v}_i} \right) + \underline{v}_r \cdot \underline{q}_i' + \underline{p}_i' : \nabla_r \underline{v}_i = \sum_{j \neq i} n_i \delta_j \frac{m_i}{2} \overline{v_i'^2} \cdot \quad (3-13)$$

In the first two terms, we have $(\rho_i/2) \overline{v_i'^2} = (3/2) k T_i$. The heat flux

$$\underline{q}_i' = (\rho_i/2) \overline{v_i'^2 \underline{v}_i} = - \lambda \nabla_r T_i \quad ,$$

as shown in App. B. The last term on the left is easily recognized as $\underline{p}_i' : \nabla_r \underline{v}_i = p \nabla_r \cdot \underline{v}_i$, because $\underline{p}_i' = p \underline{I}$, where $\underline{I}$ is the unit tensor, p being the hydrostatic pressure.* For the collision term, the same argument applies as for the equation of motion. We may thus write

*It is shown in App. B (Eqs. 23 and 33) that the quantities on the l.h.s. of our transport Eqs. (9), (10), and (13) represent zeroth order averages, $\phi = \int \phi^{(0)} d\underline{v}$, when equated with the first order collision integrals in the first order transport equations.

$$\sum_{j \neq i} n_i \delta_j \frac{m_i}{2} \overline{V_i'^2} = \sum_{j \neq i} \int_i \frac{m_i}{2} V_i'^2 \left(\frac{\partial f_i}{\partial t} \right)_j d\mathbf{v}_i ,$$

where

$$\sum_i \sum_{j \neq i} n_i \delta_j \frac{m_i}{2} \overline{V_i'^2} = 0 .$$

If one develops the heat balance equation from $\phi_i = m_i \mathbf{V}_i' \cdot \mathbf{V}_i'$ as the dynamic function, then one obtains a moment equation with more than twelve terms, many of which are functions of the Lorentz force. Most of the terms cancel or they are tensor identities of zero. The remaining tensor equation reduces to Eq. (13) identically.

CHAPTER 4
SAHA'S EQUATION FOR A THREE COMPONENT NONISOTHERMAL PLASMA
(mks units)

Saha's equation results from extremizing the free energy with respect to the number density varied in an ionization process which changes

$$n_i, n_e, n_0 \text{ to } n_i + \delta n_i, n_e + \delta n_e, n_0 - \delta n_0 \text{ .}$$

Consider a sufficiently dilute gas* consisting of the three components: electrons, ions, and neutral gas. Each component is an ideal system in the Boltzmann approximation, in the absence of external fields, with chemical reactions among the three components, i.e.,

$$(Z + 1) + e = Z$$

$$\nu_{Z+1} = 1; \nu_e = 1; \nu_Z = -1 \text{ ,} \quad (4-1)$$

where the ν_i are the coefficients of the reaction (stoichiometric coefficients). We need the free energy for the component γ of the reaction, i.e.,

$$A_\gamma(T, v, n) = - n_\gamma k T_\gamma \left[\ln \frac{v}{n_\gamma h^3} (2\pi m_\gamma k T_\gamma)^{3/2} Z_{i\gamma} + 1 \right] \text{ ,} \quad (4-2)$$

which derives from the partition function

*We assume canonical distribution, hence not cryogenic temperature and not very high density.

$$Z_Y = Z_{TrY} Z_{IY} \quad (4-3)$$

where Z_{TrY} is the translational partition function

$$Z_{Tr} = v\Lambda^{-3} \quad (4-4)$$

related to the de Broglie wavelength Λ , and the internal partition function

$$Z_{IY} = \sum_i g_{iY} e^{-\epsilon_{iY}/kT_Y} \quad (4-5)$$

is the sum over the i internal energy levels, ϵ_i , determined by quantum mechanics with g_{iY} being the statistical weight of the degeneracy of the i^{th} level. The law of mass action is a statement of conservation of energy and requires that the Helmholtz free energy $A(t,v,n)$ be an invariant of the reaction. We calculate, therefore,

$$\frac{\partial A}{\partial n} = \sum_Y \frac{\partial A}{\partial n_Y} \frac{\partial n_Y}{\partial n}$$

which is obtained from Eqs. (2) and (5) for the constituent Y , i.e.,

$$\frac{\partial A}{\partial n_Y} = - \ln \left[\frac{v}{n_Y h^3} (2\pi m_Y T_Y)^{3/2} \sum_i g_{iY} e^{-\epsilon_{iY}/kT_Y} \right]. \quad (4-6)$$

For the sum over the constituents, in thermodynamic equilibrium, we obtain

$$\frac{\partial A}{\partial n} = - \ln \prod_{\gamma} [f(n_{\gamma}, m_{\gamma}, T_{\gamma})]^{\partial n_{\gamma} / \partial n}, \quad (4-7)$$

where the exponents can be expressed by the reaction coefficients, i.e., $\delta n_{\gamma} = \nu_{\gamma} \delta n$. We set Eq. (7) equal to zero and obtain

$$1 = \prod_{\gamma} \left[\frac{v}{n_{\gamma} h^3} (2\pi m_{\gamma} k T_{\gamma})^{3/2} \sum_i g_{i\gamma} e^{-\epsilon_{i\gamma}/kT_{\gamma}} \right]^{\nu_{\gamma}}. \quad (4-8)$$

To bring Eq. (8) in the conventional form of Guldberg and Waage,¹⁴ we take the pressure $p_{\gamma} = kn_{\gamma}T_{\gamma}/v$ to the l.h.s. and get

$$\prod_{\gamma} p_{\gamma}^{\nu_{\gamma} T_{\gamma}} = \prod_{\gamma} \left[\frac{kT_{\gamma}}{h^3} (2\pi m_{\gamma} k T_{\gamma})^{3/2} \sum_i g_{i\gamma} e^{-\epsilon_{i\gamma}/kT_{\gamma}} \right]^{\nu_{\gamma} T_{\gamma}}, \quad (4-9)$$

where each component has been raised to the power T_{γ} . If we now extend the sum and product indices i and γ only over electrons with p_e, m_e, T_e, ν_e ; neutral gas with p_z, m_z, T_z, ν_z ; and ions with $p_{z+1}, m_{z+1} = m_z = m_h, T_{z+1} = T_z = T_h, T^*$ for the excited state, and ν_{z+1} , we get after raising the entire equation to the power $-T_e^{-1}$,

$$p_e \left(\frac{p_{z+1}}{p_z} \right)^{T_h/T_e} = \left[\frac{kT_e}{h^3} \cdot g_{i\gamma} \cdot (2\pi m_e k T_e)^{3/2} \right] e^{-\frac{E_I}{kT_h} \frac{T^*}{T_e} - \frac{E_I}{kT_e}}, \quad (4-10)$$

where $E_I = E_{z+1} - E_z$ is the ionization potential. To write Eq. (1) in the form of the Saha equation, we introduce the degree of ionization

$$x = \frac{n_{Z+1}}{n_Z + n_{Z+1}} .$$

4-11)

In addition to the variation of the free internal energy of the constituent components of the reaction, one must consider the corresponding external variation due to Coulomb interactions, because changes in the Debye potentials are being induced by the reaction. It has been shown¹⁵ that this effect in terms of an equivalent, hypothetical, change in ionization potential, amounts to $E_I - \Delta E_I$ where $\Delta E_I = (z+1)e^2/r_d$ (esu). For the ionization of neutral atoms, $z = 0$, and ΔE_I may presently be neglected, because one obtains $\Delta E_I/E_I = 10^{-6}$ with the numbers from Table 1.

Observe that the system considered consists of one sub-system for each constituent. Each subsystem is an ensemble of particles with a canonical distribution, where temperature, volume, and number of particles in the system are the independent variables. The system is submerged in a reservoir whose thermodynamic variables are supposed to be known. For this reason, the reservoir provides the flow balance of particles.

It has been postulated¹⁶ that the total gaskinetic pressure is invariant. We tried to impose a different invariant of the reaction by assigning invariance to the partial pressure of the neutral particles: We believed that only the neutrals' pressure was maintained by the gas reservoir surrounding the arc. We found this assumption would lead to the nonphysical condition, where in the limit of vanishing spatial gradients in the energy flow balance, an identity in $\partial T_0/\partial t$ results. Hence, we take the total pressure to be the invariant of the reaction. By writing Eq. (10) for the three-component nonisothermal plasma, and not allowing for excited states, we get

$$p_e \frac{p_i^{\tau_i}}{p_o^{\tau_o}} = 2 \frac{g_i^{\tau_i}}{g_o^{\tau_o}} kT_e \left(\frac{2\pi m_e kT_e}{h^2} \right)^{3/2} \frac{(kT_i)^{\tau_i}}{(kT_o)^{\tau_o}} \left(\frac{2\pi m_i kT_i}{h^2} \right)^{3/2} \tau_i$$

$$\left(\frac{2\pi m_o kT_o}{h^2} \right)^{-3/2} \tau_o e^{-E_I/kT_e} , \quad (4-12)$$

where we use the abbreviations $\tau_i = T_i|T_e$; $\tau_o = T_o|T_e$. The invariance condition, in thermodynamic equilibrium, is given by

$$p_o^o = p_o + p_e + p_i = n_o kT_o + n_e kT_e + n_i kT_i = \frac{n_o^o}{\eta} \left[(1-x)kT_o + x(kT_e + kT_i) \right] \quad (4-13)$$

where η is the parameter fixed by the invariance condition and where the superscript o denotes an initial value. We get

$$\eta = \frac{T_o^o}{T_o} \left(1-x + x \frac{T_e + T_i}{T_o} \right) , \quad (4-14)$$

$$\frac{n_o}{n_o^o} = \frac{T_o^o}{T_o} \cdot \frac{1-x}{1-x + x \frac{T_e + T_i}{T_o}} , \quad (4-15)$$

and

$$\frac{n_e}{n_o^0} = \frac{n_i}{n_i^0} \frac{x}{1-x + x \frac{T_e + T_i}{T_o}}, \quad (4-16)$$

where we confined ourselves to considering neutral and singly charged particles.

The invariance condition is substituted in Eq. (12) by the relationship

$$n_o k T_o = (n_o^0 k T_o^0)^{1+\tau_i+\tau_o} \left(\frac{x}{1-x+x \frac{T_e+T_i}{T_o}} \right)^{1+\tau_i-\tau_o}, \quad (4-17)$$

where the first expression in parentheses is identified with p_o^0 , which is the invariant total pressure. With this substitution, one can write

$$\left(\frac{n}{n_o} \right)^{\tau_o} \left(\frac{p_o^0}{1+\tau_i+\tau_o \frac{n_o}{n}} \right)^{1+\tau_i-\tau_o} = g \left(\frac{2\pi}{h^2} \right)^{3/2(1+\tau_i-\tau_o)} \left(\frac{m_e m_i}{m_o^{\tau_o}} \right)^{3/2} e^{-E_I/kT_e} \frac{(kT_i)^{3/2 \tau_i}}{(kT_o)^{3/2 \tau_o}} (kT_e)^{5/2+\tau_i-\tau_o}. \quad (4-18)$$

Henceforth, we use the ionization ratio $X = n/n_o$ instead of the ionization degree defined earlier. The invariance condition requires a flow such that

$$n_o = \frac{p_o^o}{X(kT_e + kT_i) + kT_o} .$$

Equation (18) is Saha's equation extended to the three-component nonisothermal plasma. It establishes the thermodynamic equilibrium relationship among number densities of three species (electrons, single-charged ions, and neutrals), dependent of the temperatures. Because we use Saha's equation later as one of the relationships to close the system of hydrodynamic equations, we need the differential dependency among the variables of the Saha equation. After differentiating Eq. (18) and setting $T_e = T_i$, $n_e = n_i$, we can write the relationship in the following convenient form

$$\begin{pmatrix} \frac{\partial n_o}{\partial t} \\ \frac{\partial n}{\partial t} \end{pmatrix} = - \frac{n_o}{kT_o + 2XkT_e} \begin{pmatrix} 2(X + T_e \frac{\partial X}{\partial T_e}) & 1 + 2T_e \frac{\partial X}{\partial T_o} \\ 2X^2 - T_o \frac{\partial X}{\partial T_e} & X - 1_o \frac{\partial X}{\partial T_o} \end{pmatrix} \begin{pmatrix} \frac{\partial kT_e}{\partial t} \\ \frac{\partial kT_o}{\partial t} \end{pmatrix}, \quad (4-19)$$

where the derivatives of X are obtained from the following two relationships derived directly from Eq. (18)

$$\left(1 + \frac{2 - \tau_o}{2X + \tau_o}\right) \frac{kT_o}{X} \left(\frac{\partial X}{\partial kT_e}\right) = 5 - 2\tau_o \frac{X+1}{2X + \tau_o} + \frac{E_I}{kT_e} + \tau_o \ln K, \quad (4-20)$$

$$\left(1 + \frac{2-\tau_0}{2X+\tau_0}\right) \frac{kT_0}{X} \left(\frac{\partial X}{\partial kT_0}\right) = \frac{-3}{2} + \frac{2-\tau_0}{2X+\tau_0} - \ln K, \quad (4-21)$$

where

$$K = \left(\frac{2\pi m_0 kT_0}{h^2}\right)^{3/2} \frac{kT_e}{p_0^0} (2X + \tau_0) .$$

Finally, we determine the total statistical weight g for the quantum mechanical degeneracies of the atomic nitrogen system. From Herzberg,¹⁷ we find $g_0 = 4$, $g_i = 3$ (ortho), and one has of course $g_e = 2$, yielding

$$g = \frac{g_e g_i^{\tau_i}}{g_0^{\tau_0}} .$$

The ground state ionization potential is given as 15.6 eV. In our numerical work, we used $g = 2$ and $E_I = 16$ eV.

CHAPTER 5
ELASTIC COLLISIONS
(esu, exceptions noted)

We confine ourselves to those remarks on Rutherford scattering that may complement Chap. 3, and to the topic of encounters between electrons and neutral particles. As regards the latter, we refer to the work by Engelhardt et al.,¹⁸ where the effective momentum transfer collision frequency is given by

$$\nu_{eo}/n_0 = 2 \cdot 10^{-9} \text{ cm}^3/\text{s}, \text{ at } > 1.4 \text{ eV}.$$

A synopsis of Spitzer's treatise²⁶ on Rutherford scattering in the following form may be useful. Let the local equilibrium distribution in velocity of the test particles--i.e., the ones being scattered by the field particles^{*}--be given by the one-particle reduced distribution function

$$f^{(0)}(v) = \frac{n}{\pi^{3/2}} \frac{1}{\bar{v}^3} e^{-(v/\bar{v})^2}, \quad (5-1)$$

where n is the local number density per unit volume, and $\bar{v}$ the most probable test particle speed. We assume, by Eq. (1), spatial homogeneity. Interaction with the field particles is introduced by way of a new phase space in which the spatial independence of the distribution function, Eq. (1), is augmented to a new dependence, which is one of the impact parameter. In other words, the impact parameter

*"Field particles" have been named in analogy with the astrophysics term "field star"; they are stationary, whereas the test particles are to test effects of their own motion.

assumes the place of the spatial variables; the field particles themselves are not represented by their distribution function. We simply have constructed a phase space in the variables (b,v) . We can thus define a special Debye cell in (b,v) space, i.e., a cylindrical shell with length v and cross section $2\pi b db$. The number of field particles found in the shell is thus $2\pi b db$. The number of field particles found in the shell is thus $2\pi b v n_i db$, where n_i is the number density per unit phase space volume, of the field particles.* The integral over the cylinder diverges for $b \rightarrow \infty$. Spitzer sets an upper and a lower limit corresponding to the Debye radius and the short range interaction radius, thus defining the parameter Λ .

Time is still a complementary independent variable, needed to describe the evolution of the interaction. Hence, one may calculate the characteristic times of the evolution. Spitzer obtains, cf. Eq. (8-25) in Rose and Clark,

$$\tau_{ei} = \frac{(2m_e)^{1/2} W_e}{4\pi e^4 n_i} \frac{1}{\ln \Lambda}, \quad (5-2)$$

where W_e is the total kinetic energy (corresponding to the total velocity square average). Because the drift speed is small, compared with the random speed, Spitzer equates

$$W_e \approx \frac{3}{2} kT_e. \quad (5-3)$$

Hence, the collision frequency is defined by

*Our derivation of this classical cross section offers a useful alternative to the one ordinarily given.

$$v_{ei} = 1.379 \cdot 10^{-12} \frac{n_i Z^2 \ell n \Lambda}{\left(\frac{kT_e}{e}\right)^{3/2}} \text{ (MKS) } , \quad (5-4)$$

where

$$\Lambda = 1.539 \cdot 10^{13} \frac{\left(\frac{kT_e}{e}\right)^{3/2}}{n_e^{1/2}} \text{ (MKS) } . \quad (5-5)$$

We emphasized our identification of the independent variables in Spitzer's interaction phase space because the same end result may be obtained by the following alternative argument inviting a possible misconception.

The momentum change of the test particles has been expressed in a relationship¹⁹ that includes not only the spatial length of an interaction volume in phase space, but also the impact parameter, both being independent variables. Hence, one excessive independency has entered into the argument. The spatial length is then used to define a collision cross section. The correct relaxation time, i.e., Eq. (2), is obtained by correlating the two over the random kinetic energy. A relationship is in use²⁰ for the resistivity, which relates the correct cross section erroneously by the drift velocity, to the collision frequency, the latter being wrong. Refer to App. B-5 for a more rigorous treatise.

CHAPTER 6
CYLINDRICAL ARC ENVELOPE FUNCTION
(mks units)

We will see that the Lagrangian of the charge labeled i in an external EM field is given by

$$L = -\frac{1}{2} m_i v_i^2 - q_i \phi + q_i \vec{A} \cdot \vec{v}_i ,$$

where ϕ and $\vec{A}$ are the external electric and magnetic potentials, respectively, evaluated at the position of the particle with mass m_i , velocity $\vec{v}_i$, and charge q_i . We have thus written the Lagrangian for one of the charges of the system--under a given motion of the other charges. From this, one can easily find the Lagrangian of the whole system. By taking the sum over the system, one obtains

$$L = \frac{1}{2} \int_V (\vec{E} \cdot \vec{D} - \vec{H} \cdot \vec{B}) dt - \sum_i q_i (\phi_i - \vec{v}_i \cdot \vec{A}_i) + \frac{1}{2} \sum m_i v_i^2 , \quad (6-1)$$

where

$$\vec{E} = -\nabla\phi = \frac{\partial}{\partial t} \vec{A} \quad (6-2)$$

and

$$\vec{B} = \nabla \times \vec{A} , \quad (6-3)$$

where B and D are the flux densities, and H and E the field gradients (sometimes called intensities) in the magnetic and electric field,

respectively. The three terms on the r.h.s. of Eq. (1) are the Lagrangians, from left to right, of the field, the field-particle interaction, and the particles.

To operate with Euler's derivative $(\partial/\partial q - d/dt \partial/\partial \dot{q})$ on the Lagrangian, three choices of generalized coordinates $(q, \dot{q})$ are possible, i.e.,

$$\phi, \dot{\phi}$$

$$\vec{A}, \dot{\vec{A}}$$

$$x, \dot{x} = v$$

Two of Maxwell's equations, i.e.,

$$\nabla \cdot \vec{D} = \rho \quad (\text{from } \phi, \dot{\phi}) \quad , \quad (6-4)$$

$$\nabla \times \vec{H} - \frac{\partial}{\partial t} \vec{D} = \vec{J} \quad (\text{from } \vec{A}, \dot{\vec{A}}) \quad , \quad (6-5)$$

are thus obtained; the other two having been used to define the potentials in Eqs. (2) and (3). Let us turn to the kinetic coordinates $x_\gamma, \dot{x}_\gamma \equiv v_\gamma$ to obtain the desired kinetic equations. Cylindrical coordinates are suitable for the general geometry of the arc chamber and contacts. We assume azimuthal symmetry and thus obtain two expressions for each of which d'Alembert's principle obtains. Hence they vanish independently, i.e.,

$$\frac{\partial L}{\partial z} - \frac{d}{dt} \frac{\partial L}{\partial v_z} - Q = 0 \quad , \quad (6-6)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial v_r} - Q = 0 \quad . \quad (6-7)$$

The terms Q are required by d'Alembert's principle for nonconservative systems with frictional forces present. We set $Q = 0$, perhaps in analogy with zero on the r.h.s. of the Boltzmann equation leading to the conservation equations of fluid dynamics.

We assume that the current i is impressed so the current density be position-independent within the arc column, i.e.,

$$j \equiv j_z = i/\pi r^2 \quad (6-8)$$

and the electrostatic gradient

$$\frac{d}{dz} \phi = j_z/\sigma = i/\pi \sigma r^2 \quad . \quad (6-9)$$

We calculate the radius and length of the cylindrical column shown cross-hatched in Fig. 1d and neglect the disk-like (shaded) ends. Because the magnetic field at radii exceeding the electrodes' inner radius R is an invariant of the arc column's linear dimensions, that field is not being considered. We set

$$\phi = \frac{d\phi}{dz} \ell = i\ell/\pi \sigma r^2 \quad , \quad (6-10)$$

$$B_\theta = B(\rho = \bar{r}) = \frac{\mu i}{2\pi \bar{r}} \left(\frac{\bar{r}}{r}\right)^2 = - \frac{\partial A}{\partial \bar{r}} \quad ,$$

from which we get

$$A(\rho = \bar{r}) = \frac{\mu_i}{4\pi} i \left(\frac{\bar{r}}{r} \right)^2 . \quad (6-11)$$

By substituting this in Eq. (1), we get the Lagrangian in cylindrical coordinates, i.e.,

$$L = \frac{i\mu}{4\pi} \left\{ \frac{\epsilon}{\mu} \frac{4}{\sigma^2} i\ell \left(\frac{R^2}{r^4} \right) - \frac{i\ell}{4} - i\ell \ln \frac{R}{r} + \frac{4\ell}{n^2 \sigma \mu} \frac{1}{r^2} \sum q_i - \sum q_i \dot{\varphi}_i \frac{r_i^2}{r^2} \right\} . \quad (6-12)$$

To be consistent with the assumption of a position independent current density within the arc, we require $r_i/r \equiv \eta = \text{constant}$. We thus get the two conjugate derivatives

$$\frac{\partial L}{\partial r} = \frac{i\mu}{4\pi} \left[i\ell \left(-\frac{16\epsilon R^2}{\mu \sigma^2 r^5} + \frac{1}{r} \right) - \frac{8\ell}{\mu^2 \sigma \mu r^3} \sum q_i \right] \quad (6-13)$$

$$\frac{\partial L}{\partial \ell} = \frac{i\mu}{4\pi} \left[i \left(\frac{\epsilon}{\mu} \frac{4}{\sigma^2} \frac{R^2}{r^4} - \frac{1}{4} - \ln \frac{R}{r} \right) + \frac{\sum q_i}{n^2 \sigma \mu r^2} \right] . \quad (6-14)$$

Furthermore,

$$\frac{\partial L}{\partial \dot{\varphi}_i} = \frac{i\mu}{4\pi} n^2 q_i ;$$

hence

$$\int \frac{d}{dt} \frac{\partial L}{\partial \dot{v}_i} = \frac{\mu n^2}{4\pi} \frac{d}{dt} (i \int q_i) \quad . \quad (6-15)$$

The vector notation on the l.h.s. of Eq. (15) is to indicate that derivatives with respect to the velocity components v_r and v_z are meant, there being two equations in all. Because the signs of the products $q_i v_{ri}$ and $q_i v_{zi}$ are intrinsically positive (positive charge going right or negative charge going left), we may substitute in Eq. (15) $\int q_i = e(n_e + n_i)$ to obtain

$$- \frac{d}{dt} \frac{\partial L}{\partial v_z} = - \frac{d}{dt} \frac{\partial L}{\partial v_r} = - \frac{\mu n^2 e}{4\pi} \frac{d}{dt} i(n_e + n_i) \quad . \quad (6-16)$$

By summing Eqs. (13), (16), and (14), (16), respectively, we obtain the zero variations w.r.t. $(r, \dot{r})$ and $(\ell, \dot{\ell})$:

$$0 = i\ell \left(\frac{1}{r} - \frac{16\epsilon R^2}{\mu \sigma^2 r^5} \right) - \frac{8\ell e(n_e + n_i)}{n^2 \sigma \mu r^2} - \frac{n^2 e}{i} \frac{d}{dt} (n_e + n_i) i \quad , \quad (6-17)$$

$$0 = i \left(\frac{\epsilon}{\mu} \frac{4}{\sigma^2} \frac{R^2}{r^4} - \frac{1}{4} - \ell \ln \frac{R}{r} \right) + \frac{e(n_e + n_i)}{n^2 \sigma \mu r^2} - \frac{n^2 e}{i} \frac{d}{dt} (n_e + n_i) i \quad , \quad (6-18)$$

which are two equations in the four variables r, ℓ, i , and $(n_e + n_i)$. Equation (18) is an ODE containing only r, i , and $(n_e + n_i)$ as variables. It may be rewritten for $i = I = \text{const}$ in a more convenient form, i.e.,

$$\alpha(n_e + n_i) - \beta \cdot \frac{d}{dt} (n_e + n_i) = -\gamma \quad . \quad (6-19)$$

Euler's equation returns the function defining the arc envelope for which the Lagrangian is stationary. Because we use cylindrical coordinates, the minimum is represented by $r = \text{const}$, $l = \text{const}$. The Euler-Lagrange formalism requires a constant current density profile in Eq. (8) to yield the observed right cylindrical arc envelope.

CHAPTER 7
CURRENT EQUATION AND SUMMARY OF EQUATIONS
(mks units)

Let us rewrite the hydrodynamic Eqs. (3-9), (3-10), and (3-13) in a form suitable for numerical evaluation in cylindrical coordinates. It is easily verified the equations are identical in cgs and mks units, the factor of transformation from one to the other being 1 or $\sqrt{4\pi\epsilon_0}$ depending on the equation and not the individual term. As regards conservation of momentum, it may be written either as a momentum or current equation. We use the latter. Let e denote the unit positive charge; the electron and ion currents are then given by* $j_e = e n \bar{v}_e$, $j_i = e n \bar{v}_i$, with the total current

$$j = \sum_i n_i e \bar{v}_i = \sum_i n e (v_o + \bar{v}_i) = \sum_i \sigma v_o + j_i - j_e = \sum_i \sigma v_o + j \quad , \quad (7-1)$$

where σv_o is the electronic convection current due to a collective motion of the ionized gas, and j is the conduction current of the electrons' and ions' relative motion. There being no chance of confusion, we use the subscript i as a general label for the species, as well as for ions specifically when e is used for the electrons, and o for the atoms. We can neglect $n \bar{v}_i$, as discussed later in this chapter. The mass conservation Eq. (3-9) becomes therefore

$$\frac{\partial n_i}{\partial t} + \bar{v}_i \cdot \nabla n_i = \frac{\delta n_i}{\delta t} \quad , \quad (7-2)$$

* v_e and v_i are defined to point in opposite directions; hence, because e is defined as a positive charge, $j = en(v_i - v_e)$.

where the r.h.s. is to account for the gains and losses from the inelastic collisions discussed in Chap. 4.

7.1 Current Equation

Let us proceed to the equation of currents obtained from the momentum conservation Eq. (3-10). We move the pressure gradient term to the l.h.s. and proceed as follows:

- a) Simplify the l.h.s., reading $n_i m_i \left[\partial \bar{\mathbf{v}}_i / \partial t + (\bar{\mathbf{v}}_i \cdot \nabla) \bar{\mathbf{v}}_i \right] + \nabla \cdot \mathbf{p}'_i$;
- b) Simplify the set of all streaming terms;
- c) Write the collision terms;
- d) Combine the streaming and collision terms to the new form.

We recall the definition of Chapman and Cowling's peculiar velocity $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{v}_0$, i.e.,

$$\mathbf{v}_i = \bar{\mathbf{v}}_i + \mathbf{v}_0 = \bar{\mathbf{v}}'_i + \bar{\mathbf{v}}_i$$

and proceed with step (a). It is easy to show^{*} that

$$m n (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} + \nabla \cdot \mathbf{p}' = m \nabla \cdot (n \bar{\mathbf{v}} \bar{\mathbf{v}}) + m \bar{\mathbf{v}} \frac{\partial n}{\partial t}, \quad (7-3)$$

hence, Eq. (10) in Chap. 3 becomes

$$m_i \frac{\partial n_i \bar{\mathbf{v}}_i}{\partial t} = - m_i \nabla \cdot (n_i \bar{\mathbf{v}}_i \bar{\mathbf{v}}_i) + n_i e_i (\mathbf{E} + \bar{\mathbf{v}}_i \times \mathbf{B}) + n_i \sum_{j \neq i} \delta_{ij} \bar{\boldsymbol{\theta}}_i, \quad (7-4)$$

$$\begin{aligned} \star m \bar{\mathbf{v}} \cdot (n \bar{\mathbf{v}} \bar{\mathbf{v}}) &= m \nabla \cdot \left[n (\bar{\mathbf{v}}' + \bar{\mathbf{v}}) (\bar{\mathbf{v}}' + \bar{\mathbf{v}}) \right] = m \nabla \cdot \bar{\mathbf{v}}' \bar{\mathbf{v}}' + m \bar{\mathbf{v}} \cdot (n \bar{\mathbf{v}} \bar{\mathbf{v}}) = \\ &= \nabla \cdot \mathbf{p}' + m n (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} + m n \bar{\mathbf{v}} (\nabla \cdot \bar{\mathbf{v}}) + m \bar{\mathbf{v}} (\bar{\mathbf{v}} \cdot \nabla n), \end{aligned}$$

where $m n \bar{\mathbf{v}} (\nabla \cdot \bar{\mathbf{v}}) + m \bar{\mathbf{v}} (\bar{\mathbf{v}} \cdot \nabla n) = m \bar{\mathbf{v}} (\bar{\mathbf{v}} \cdot \nabla n) = - m \bar{\mathbf{v}} \frac{\partial n}{\partial t}$,

and where we substituted the continuity Eq. (2).

where we abbreviated $m\bar{v} \equiv \bar{g}$. After multiplying Eq. (4) by e_i , one can show* that the first term on the r.h.s. in Eq. (4) becomes

$$e\bar{v} \cdot (n\bar{v}\bar{v}) = \nabla \cdot \frac{e}{m} p + \nabla \cdot (j\bar{v}_0) + \nabla \cdot (\bar{v}_0 j) ,$$

where the last two terms may be neglected, because $\bar{v}_0$ and j are small, compared with the peculiar velocity and its e-multiple; more so since the terms are quadratic by the products of $\bar{v}_0$, j . The electrokinetic pressure tensor may be replaced by the scalar pressure p_i , cf., the text near the end of App. B-4. It is justified to neglect p_i/m_i , which is small when compared with p_e/m_e . The set of all streaming terms may thus be written

$$\frac{\partial j}{\partial t} = - \nabla \frac{e_i}{m_i} p_i + \frac{e_i^2}{m_i} n_i \bar{E} + \frac{e_i^2}{m_i} n_i \bar{v}_0 \times \bar{B} + \frac{e_i}{m_i} j \times \bar{B} . \quad (7-5)$$

The signs follow from $\text{sgn}(e_i E)$, Eq. (5) being derived from the momentum equation multiplied by e_i . If Eq. (5) is explicitly written for electrons and ions, then we obtain, by subtracting and adding that pair of equations, these two new relationships:

$$(i)-(e): \frac{\partial j}{\partial t} = \nabla \frac{e}{m_e} p_e + e^2 n \left(\frac{1}{m_i} + \frac{1}{m_e} \right) (\bar{E} + \bar{v}_0 \times \bar{B}) + \frac{e}{m_e} j \times \bar{B} \quad (7-6)$$

* $e m \nabla \cdot (n \bar{v} \bar{v}) = e m \nabla \cdot [n (\bar{V} + \bar{v}_0) (\bar{V} + \bar{v}_0)] = \bar{V} \cdot e/m \cdot p + \nabla \cdot (j \bar{v}_0) + \nabla \cdot (\bar{v}_0 j) + \nabla \cdot (\sigma \bar{v}_0 \bar{v}_0)$, where the first term on the r.h.s. is the electrokinetic pressure tensor.

$$(i)+(e): \quad m_i \frac{\partial \mathbf{j}_i}{\partial t} + m_e \frac{\partial \mathbf{j}_e}{\partial t} = 0 = - \nabla p_e + e \mathbf{j}_e \times \mathbf{B} \quad . \quad (7-7)$$

The collision terms are still neglected in Eqs. (6) and (7), thus keeping only the streaming terms. The l.h.s. in Eq. (7) is zero identically, because $m_e n_e \bar{\mathbf{v}}_e + m_i n_i \bar{\mathbf{v}}_i = 0$, and of $\bar{\mathbf{v}}_e$ and of $\bar{\mathbf{v}}_i$ being defined to point in opposite directions.

Let us proceed to step (c) in our outline and write the collision terms.* We have

$$\text{for the electrons: } n_e \delta_{i\bar{e}} \bar{g}_{\bar{e}} + n_e \delta_{e\bar{o}} \bar{g}_{\bar{o}} + m_e \bar{\mathbf{v}}_e \frac{\delta n_e}{\delta t} ; \quad (7-8)$$

$$\text{for the ions: } n_e \delta_{e\bar{i}} \bar{g}_{\bar{i}} + n_i \delta_{o\bar{i}} \bar{g}_{\bar{i}} + m_i \bar{\mathbf{v}}_i \frac{\delta n_i}{\delta t} ; \quad (7-9)$$

$$\text{and for the atoms: } n_o \delta_{e\bar{o}} \bar{g}_{\bar{o}} + n_o \delta_{i\bar{o}} \bar{g}_{\bar{o}} + m_o \bar{\mathbf{v}}_o \frac{\delta n_o}{\delta t} ; \quad (7-10)$$

where $n_e = n_i = n$, and $x = n/(n + n_o)$. The first two terms in each of the expressions (8)-(10) represent elastic interactions, the third terms the contributions from inelastic encounters. Because the sum of the elastic collision terms vanishes when taken over all three expressions (8)-(10), the rate-of-change operator δ_k must have the property

$$n_{\bar{\ell}} \delta_{k\bar{\ell}} \bar{g}_{\bar{\ell}} = - n_k \delta_{\bar{\ell}k} \bar{g}_{\bar{\ell}} .$$

The operator is defined by

*Jancel and Kahan's book,¹¹ Eqs. (6-99)-(6-104), p. 302-303.

$$n_k \delta_{\ell} \bar{\theta}_k = \int \bar{\theta}_k \left(\frac{\delta f_k}{\delta t} \right)_{\ell} d\mathbf{v}_k = 2\pi \iiint \bar{\theta}_k (f'_k f'_{\ell} - f_k f_{\ell}) g_{k\ell} b db d\mathbf{v}_k d\mathbf{v}_{\ell} ,$$

from which the antisymmetry just discussed follows, $g_{kl} = -g_{lk}$ being the relative velocity. Hence the explicit form for our three component system:

$$n_e \delta_{i\bar{e}} \bar{\theta} = -n_i \delta_{e\bar{i}} \bar{\theta} = -n_e m_e v_{ei} (\bar{v}_e - \bar{v}_i) = -n_e m_e v_{ei} (\bar{v}_e - \bar{v}_i) = v_{ei} \frac{m_e}{e} \mathbf{j} \quad (7-11)$$

$$n_e \delta_{o\bar{e}} \bar{\theta} = -n_o \delta_{e\bar{o}} \bar{\theta} = -v_{eo} n_e m_e (\bar{v}_e - \bar{v}_o) = -v_{eo} n_e m_e (\bar{v}_e - \bar{v}_i + \frac{\bar{v}_i}{x}) =$$

$$v_{eo} \frac{m_e}{e} \mathbf{j} - v_{eo} \frac{m_e}{e} \frac{\mathbf{j}}{x} \quad (7-12)$$

$$n_i \delta_{o\bar{i}} \bar{\theta} = -n_o \delta_{i\bar{o}} \bar{\theta} = -v_{io} n_i m_i (\bar{v}_i - \bar{v}_o) = -v_{io} n_i m_i \frac{\bar{v}_i}{x} = -v_{io} \frac{m_i}{e} \frac{\mathbf{j}}{x} \quad (7-13)$$

Following the same procedure of adding and subtracting from one another the streaming expressions for the ions and electrons, we derive new expressions for the collision terms. For electrons minus ions, we get

$$en \delta_{e\bar{i}} \bar{\theta} - en \delta_{i\bar{e}} \bar{\theta} + en \delta_{o\bar{i}} \bar{\theta} - en \delta_{o\bar{e}} \bar{\theta} + (em_e \bar{v}_e - em_i \bar{v}_i) \frac{\delta n}{\delta t} ,$$

or, explicitly, by neglecting m_e/m_i against 1:

$$= (2v_{ei} + v_{eo} + \frac{v_{io}}{x} + \frac{2}{n} \frac{\delta n}{\delta t}) m_e \mathbf{j} \quad (7-14)$$

Similarly, by adding electrons and ions, i.e.,

$$en\delta_{e\sim i}\bar{\theta}_i + en\delta_{i\sim e}\bar{\theta}_e + en\delta_{o\sim i}\bar{\theta}_i + en\delta_{o\sim e}\bar{\theta}_e + (em\bar{v}_e + em\bar{v}_i) \frac{\delta n}{\delta t} ,$$

and neglecting m_e/m_i against 1, we finally have

$$+ v_{eo} m_e \underline{j} . \quad (7-15)$$

We can thus proceed with the step (d) in our outline, combining the Eqs. (7) and (15):

$$0 = - \nabla \frac{e}{m_e} p_e + \frac{e}{m_e} \underline{j}_e \times \underline{B} + v_{eo} \underline{j} \approx - \nabla \frac{e}{m_e} p_e - \frac{e}{m_e} \underline{j} \times \underline{B} + v_{eo} \underline{j} . \quad (7-16)$$

Similarly, with Eqs. (6) and (14), where we substitute Eq. (16) to eliminate the hydrostatic pressure gradient. We omit the inelastic collision term, because the inelastic events are dominated by thermal ionization, in our case. Hence,

$$\frac{\partial \underline{j}}{\partial t} = \frac{e^2 n}{m_e} (\underline{E} + \underline{v}_o \times \underline{B}) - \left(2v_{ei} + \frac{v_{io}}{X} \right) m_e \underline{j} . \quad (7-17)$$

Under the stationary condition $\partial \underline{j} / \partial t = 0$, we have Ohm's law in Eq. (17). By having substituted Eq. (16), we put in Eq. (17) all the information from momentum conservation of electron and ion encounters. Convection current may be neglected in Eq. (17). This applies in an arc with cylindrical or slightly conical envelope, if we assume rotational symmetry (i.e., $\partial / \partial \phi = 0$) in addition to the radial component being small, $v_{or} \approx 0$. From classical E.M. theory, we have

$\nabla \cdot \mathbf{v}_{\perp 0} = 0 \approx \mathbf{e} n \nabla \cdot \mathbf{v}_{\perp 0}$, causing $\frac{\partial}{\partial z} v_{0z}$ to vanish when $\frac{\partial}{\partial \phi} v_{0\phi}$ does, as postulated, and $v_{0r} \propto r$ as in a conical arc envelope.

7.2 Energy Flow Equation

Let us proceed to the energy flow equation. In the text following Eq. (3-13) (i.e., the energy flow equation) and near the end of App. B-4, we discussed the reduction of the double dot term. We showed that $\mathbf{p} : \nabla \tilde{\mathbf{v}} = \mathbf{p} \nabla \cdot \tilde{\mathbf{v}} = nkT \tilde{\mathbf{I}} \nabla \cdot \tilde{\mathbf{v}}$. At the same occasions we discussed the heat flow vector $\mathbf{q}$, cf. the text following Eq. (62) in App. B. In conclusion, we can write, omitting the species subscripts,

$$\nabla \cdot \mathbf{q} = - \nabla \cdot \frac{\lambda}{m} \nabla kT = - \frac{\lambda}{k} \nabla^2 kT = - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}} + P_{\text{rad}} + P_{\text{turb}} ,$$

where the heat source $\lambda/k \nabla^2 kT$ has intuitively been decomposed into a term $\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}$ for the ohmic heating, one for the radiation loss, and one for the heat loss by turbulent flow. The only term deserving further discussion is $\nabla \cdot (kT \mathbf{n} \cdot \tilde{\mathbf{v}})$. For charged species, it follows from $\nabla \cdot \tilde{\mathbf{J}} = 0$ that $\nabla \cdot (kT \mathbf{n} \cdot \tilde{\mathbf{v}}) = \tilde{\mathbf{J}} \cdot \nabla kT$. To sum up, the heat flow equation may be written in the form

$$\frac{3}{2} \frac{\partial nkT}{\partial t} + \frac{3}{2} \frac{\tilde{\mathbf{J}}}{e} \cdot \nabla kT - (\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}} - P_{\text{rad}} - P_{\text{turb}}) - \frac{kT}{en} \tilde{\mathbf{J}} \cdot \nabla n = \frac{3}{2} \sum \frac{\delta nkT}{\delta t} . \quad (7-18)$$

The collision term in Eq. (18) includes the contributions from elastic collisions and ionization, the sum over all species being zero. Hence,

$$\begin{aligned} \frac{\partial kT_i}{\partial t} = \frac{1}{n} \left[- \frac{\tilde{\mathbf{J}}_i}{e} \cdot \nabla kT + \frac{2}{3} (\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}_i - P_{i\text{rad}} - P_{i\text{turb}}) + \frac{2}{3} \frac{kT_i}{en} \tilde{\mathbf{J}}_i \cdot \nabla n_i - \right. \\ \left. \left(\frac{2}{3} E_i + kT_i \right) \frac{\delta n_i}{\delta t} - \sum_{j \neq i} v_{ij} (kT_i - kT_j) \right] , \end{aligned} \quad (7-19)$$

where E_I is the ionization energy, and $\delta n/\delta t$ the source term in the continuity equation. The terms J/e are positive for ions, negative for electrons, consistent with the equation's derivation. We assume ∇kT to be small.

7.3 Summary of Equations

Writing Eqs. (17) and (19) in matrix form for the individual species, we get

$$H \cdot \begin{pmatrix} \frac{\partial J}{\partial t} \\ \frac{\partial v_{oz}}{\partial t} \\ \frac{\partial kT_e}{\partial t} \\ \frac{\partial kT_o}{\partial t} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}; \quad f_1 \equiv \frac{en}{m_e} eE - v_{io} \left(1 + \frac{n_o}{n}\right) (J + env_{oz}) - 2v_{ei}J \quad (7-20)$$

$$f_2 \equiv -v_{io} \frac{n}{n_o} \left[v_{oz} - \frac{nJ}{n} \left(1 + \frac{n_o}{n}\right) \right] + v_{eo} \frac{n}{n_o} \left[v_{oz} - \frac{J\eta}{en} \right] \quad (7-21)$$

$$f_3 \equiv \frac{2}{3} \frac{EJ}{n} - \frac{2}{3} \frac{P_e}{n} - v_{eo} (kT_e - kT_o) \quad (7-22)$$

$$f_4 \equiv -\frac{2}{3} \frac{P_o}{n_o} - \frac{n}{n_o} v_{eo} (kT_o - kT_e) - \frac{n}{n_o} v_{io} (kT_o - kT_i) \quad (7-23)$$

where $\eta \equiv m_e/m_i$ and where all time derivatives have been moved to the l.h.s., yielding the matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_{11} \frac{v_{oz}}{n_o} & a_{12} \frac{v_{oz}}{n_o} \\ 0 & 0 & 1+a_{21} \frac{kT_e + \frac{2}{3} E_i}{n} & a_{22} \frac{kT_e + \frac{2}{3} E_i}{n} \\ 0 & 0 & a_{11} \frac{kT_o}{n_o} & 1 + a_{12} \frac{kT_o}{n_o} \end{pmatrix} \quad (7-24)$$

The matrix elements of a_{ij} are defined in Eq. (4-19) where, for example

$$a_{12} \equiv -n_o \frac{1 + 2T_e \frac{\partial X}{\partial T_o}}{T_o + 2XT_e}$$

and where the derivatives of X are given explicitly by Eqs. (4-20) and (4-21). We have defined $J = -J_e + J_i$. We assume $T_e = T_i$ as before.

To gain an order of magnitude estimate of the collision frequencies, let us use the values from Table I, from which the following numbers derive.

T	kK	16	20
n	m^{-3}	$1.3 \cdot 10^{23}$	$2.6 \cdot 10^{24}$
n_o	m^{-3}	$1.8 \cdot 10^{23}$	$4.0 \cdot 10^{24}$
$\ln \Lambda$		4.2	3.0
v_{ei}	Hz	$4.7 \cdot 10^{11}$	$4.9 \cdot 10^{12}$
v_{eo}	Hz	$3.6 \cdot 10^8$	$8.0 \cdot 10^9$
v_{io}	Hz	$2.2 \cdot 10^6$	$5.0 \cdot 10^7$

It may be concluded, therefore, that $v_{io} \ll v_{eo} \ll v_{ei}$. Hence, because $0 < n/n_o \leq 50$, in general, we may omit Eq. (21) from the system. Physically, the relative smallness of the coefficients in Eq. (21), compared to those of the companion equations, means that the deviation of v_o from the velocity of the surrounding gas is small.

Likewise, we neglect v_{io} henceforth.

The matrix elements of a_{ij} are defined in Eq. (4-19) where, for example

$$a_{12} = -n_0 \frac{1 + 2T_e \frac{\partial X}{\partial T_0}}{T_0 + 2XT_e}$$

and where the derivatives of X are given explicitly by Eqs. (4-20) and (4-21). We have defined $J = -J_e + J_i$. We assume $T_e = T_i$ as before.

To gain an order of magnitude estimate of the collision frequencies, let us use the values from Table I, from which the following numbers derive.

T	kK	16	20
n	m^{-3}	$1.3 \cdot 10^{23}$	$2.6 \cdot 10^{24}$
n_0	m^{-3}	$1.8 \cdot 10^{23}$	$4.0 \cdot 10^{24}$
$\ln \Lambda$		4.2	3.0
v_{ei}	Hz	$4.7 \cdot 10^{11}$	$4.9 \cdot 10^{12}$
v_{e0}	Hz	$3.6 \cdot 10^8$	$8.0 \cdot 10^9$
v_{i0}	Hz	$2.2 \cdot 10^6$	$5.0 \cdot 10^7$

It may be concluded, therefore, that $v_{i0} \ll v_{e0} \ll v_{ei}$. Hence, because $0 < n/n_0 \leq 50$, in general, we may omit Eq. (21) from the system. Physically, the relative smallness of the coefficients in Eq. (21), compared to those of the companion equations, means that the deviation of v_0 from the velocity of the surrounding gas is small.

Likewise, we neglect v_{i0} henceforth.

To summarize, we need the following functions, to be substituted in our system of Eqs. (20), (22), and (23).

To summarize, we need the following functions, to be substituted in our system of Eqs. (20), (22), and (23).

$$v_{eo} = 2 \cdot 10^{-15} n_o, \text{ as given in Chap. 5;}$$

$$v_{io} = (m_e/m_i)^{1/2} \cdot v_{eo} \approx 0$$

$$v_{ei} = v_{ei}(n, kT_e) \text{ from Chap. 5;}$$

$$X = X(kT_e, kT_o) \text{ from Chap. 4;}$$

$$n_o = n_o(X, kT_e, kT_o) \text{ from Chap. 4;}$$

$$n = n_o X \text{ as in Chap. 4;}$$

$$P_e, P_o \text{ as given in Chap. 10 as function of } T_e.$$

One of the features of our system of ODE's is the large magnitude of its coefficients. A very fast rate of change is prompted even by small deviations from the stationary set, as a consequence. For example, when verifying the stability, we observed that most* small deviations from the stationary condition were corrected on a time scale of less than v_{ei}^{-1} . This observation is only of numerical significance; physically, this violates one of our premises discussed in Chap. 2 if one were to consider time scales on the order of v_{ei}^{-1} or less. Numerically, the system behaves like an asymptotically stable one, thus implying the existence of at least one Liapunov function. The search for one is simple, because the entropy as well as the free energy are known to be good candidates. At the same time, one may close the system by that same function. In Chap. 9 we discuss the Helmholtz free energy in the Debye space's electrostatic field and on the macroscopic scale, for that purpose.

*except for small excursions of kT_o whose characteristic times are v_{eo}^{-1} .

CHAPTER 8
STATIONARY CONDITIONS
(mks units)

The stationary subset of Eqs. (7-20), (7-22), and (7-23) yields the relationships

$$J = e \left[\frac{n}{2m_e v_{ei}} (P_e + P_o) \right]^{1/2} ; \quad (8-1)$$

$$E = \frac{1}{e} \left[\frac{2m_e v_{ei}}{n} (P_e + P_o) \right]^{1/2} . \quad (8-2)$$

We recognize the Spitzer resistivity in the ratio E/J of Eqs. (1) and (2).

From the discussion in Chap. 6, we recall that J is approximately constant over the arc cross section, $J \approx I/r^2\pi$, where r is the arc radius, $r \approx \text{const.}$ The loss function $P_e + P_o$ in gray body radiation is radius and temperature dependent, the radiated power being proportional to the surface area in addition to its T dependency. We write the Eqs. (1) and (2) thus as functions of T_e and r , replacing J . This leaves us with two equations in the unknowns E , T_e , and r . The system may be closed by the stationary condition in the free energy within a Debye space or, similarly, in macroscopic space, as mentioned in the chapter immediately preceding.

CHAPTER 9
LAGRANGIAN FUNCTION
(mks units)

Similar to the variation of the sum over quantum states in Chap. 4, we now minimize the variation of the potential energy in the classical system of particle-particle and particle-field interactions. We follow this procedure in our search for a Liapunov function, i.e., a function with a minimum at the equilibrium point. Longmire, in his book²¹ (q.v.), gave inspiration for our formalism in Sec. 9.1.

9.1 Interaction Free Energy (particle-particle) (esu)

The work done against the interaction of the ion (electron) α against the Coulomb field of all the others in a Debye sphere is given by the derivative of the interaction free energy F_{ei}

$$\frac{\partial F_{ei}}{\partial e_{\alpha}} = \psi_{\alpha} .$$

The average potential ψ_r around the α th ion (electron) is given by the Maxwell-Boltzmann statistics and Poisson's equation for the Coulomb field, i.e.,

$$\begin{aligned} \nabla^2 \psi(r) &= -\rho(r) = -\frac{1}{v} \sum_{\beta} e_{\beta} e^{-e_{\beta} \psi(r)/kT} \\ &= -\sum_i e_i n_i \exp \{ -e_i \psi(r)/kT \} . \end{aligned}$$

Debye and Hückel assume $e\psi(r) \ll kT$ (thus keeping only the first term in the expansion of $\exp$) and we know that the 0-order term vanishes because of charge neutrality. Hence,

$$\nabla^2 \psi(r) = \left(\frac{1}{kT} \sum_i e_i^2 n_i \right) \psi(r) \quad . \quad (9-1)$$

The integral of Eq. (1) is given by

$$\psi(r) = \frac{A}{r} \exp(-\kappa r) \quad , \quad (9-2)$$

where κ^2 is the coefficient in the r.h.s. of Eq. (1). The continuity condition of the electric displacement at $r = a$ determines the constant A, i.e.,

$$-\left. \frac{\partial \psi}{\partial r} \right|_{r=a} = \frac{A}{r^2} (1 + \kappa r) \exp(-\kappa r) \Big|_{r=a} = \frac{e_\alpha}{a^2} \quad .$$

Hence, Eq. (2) may be rewritten with the boundary condition considered, i.e.,

$$\psi(r) = \frac{e_\alpha^2}{r} \frac{\exp[-\kappa(r-a)]}{1 + \kappa a} \quad ,$$

which yields $\psi \equiv \psi(a) = e_\alpha^2 / a(1 + \kappa a)$, where $\psi(a) \delta e_\alpha$ is the work done by increasing the charge of the α th ion by δe_α , and moving the charge against the Coulomb fields of the rest of the charges present in the Debye space. The interaction free energy is then,

$$F_{ei} = \sum_\alpha \int_0^1 \psi_\alpha(\lambda) e_\alpha d\lambda \quad ,$$

where $\psi_\alpha(\lambda)$ is the value ψ_α when all charges $e_\alpha, e_\beta, \dots$ are replaced by $\lambda e_\alpha, \lambda e_\beta, \dots$. Hence,

$$F_{ei} = - \sum_{\alpha} n_{\alpha} e_{\alpha}^2 \kappa \int_0^1 \frac{\lambda^2 d\lambda}{1 + \lambda \kappa a}, \quad (9-3)$$

as a comparison with Eq. (1) will verify. By integrating Eq. (3) and expanding for $\kappa a \ll 1$, one obtains

$$F_{ei} = - \sum_{\alpha} \frac{n_{\alpha} e_{\alpha}^2}{3} \kappa g(\kappa a),$$

with $g(x) = 1 - 3/4x + 3/5x^2 - + \dots$. By neglecting all but the 0-order term in Eq. (4), an expression for the free energy per unit volume is obtained, i.e.,

$$F_{ei} = - \frac{1}{3} \left(\frac{1}{kT} \right)^{1/2} \left(\sum n_{\alpha} e_{\alpha}^2 \right)^{3/2}. \quad (9-4)$$

We have to normalize this with respect to the gas kinetic partial pressure which is invariant under elastic encounters, i.e.,

$$\frac{F_{ei}}{nkT} = \frac{2}{3} \pi^{1/2} \left(\frac{n^{1/3} e^2}{kT} \right)^{3/2}. \quad (9-5)$$

Questioning the validity of the assumption just made, concerning $\kappa a \ll 1$, we obtain Eq. (5) again from the following statistical argument.

Let N be the actual number of electrons in a Debye volume, and a be their average number, i.e.,

$$a = n\delta v.$$

The probability that there are N electrons and M ions is thus given by

$$P_G(N, M) = \frac{a^{N+M}}{N!M!} e^{-2a} . \quad (9-6)$$

Equation (6) shows that $M-N = 0$ on the average, as it must, for overall charge neutrality. The average square difference, i.e.,

$$\overline{(m-n)^2} = \overline{m^2} + \overline{n^2} - 2 \overline{mn} ,$$

which is a formula for computing dispersion. Evidently, one may write

$$\overline{M^2} = \overline{N^2} = e^{-a} \sum_{n=0}^{\infty} \frac{N^2 a}{N!} = e^{-a} a \frac{d}{da} a \frac{d}{da} \sum_{n=0}^{\infty} \frac{a^n}{N!} = a(1 + a)$$

and

$$\overline{MN} = \overline{M \cdot N} = a^2 .$$

Therefore, the mean square excess of ions over electrons in a Debye volume is given by

$$\overline{(M-N)^2} = 2a .$$

The mean potential in a Debye cell is then at least

$$F(N,M) = \frac{e^2}{2} \frac{\overline{(M-N)^2}}{r} \quad (9-7)$$

Equation (7) has been written as if all the charges were concentrated on the surface of a spherical Debye cell, which yields the minimum potential energy. If one now averages $1/r$ against $r^2 dr$ over the Debye volume (as a parochial correction) one gains a factor $3/2$, i.e.,

$$F(N,M) = \frac{3}{4} e^2 \cdot \frac{2a}{r} \quad (9-8)$$

If we now insert in Eq. (8) the Debye radius for r , equate $a = n \cdot (4/3) \pi r^3$, and normalize again over the partial pressure, then we get back Eq. (5) identically. Equation (5) exhibits a maximum w.r.t. the temperature, as plotted in Fig. 2. That maximum is an instable equilibrium.

9.2 Field Free Energy (particle-external field)

The potential energy of the interaction between the electrons and the external field is given by

$$L^F = (e\phi - e\vec{A} \cdot \vec{v})/kT, \quad (9-9)$$

where the first term is the energy supplied by ohmic heating (ϕ being the external electrostatic potential), which is converted to ionization energy or dissipated. Therefore, we consider only the second term in Eq. (9). For $J = \text{const}$ across the arc, one obtains

$$e\vec{A} \cdot \vec{v} = \frac{eJ}{en} \int_0^r B_z dr = \frac{1}{n} \left(\frac{I}{2\pi r} \right)^2, \quad (9-10)$$

where I is the total current, $I = \pi r^2 J$. If we normalize Eq. (10) w.r.t. kT , as in Eq. (9), we obtain

$$e\vec{A} \cdot \vec{v} = \frac{1}{n} \left(\frac{I}{2\pi r} \right)^2. \quad (9-11)$$

Our normalization w.r.t. kT is the same as that w.r.t. nkT , which we used in Eq. (5), if we consider that F_{ei} is an energy per unit volume. We recognize in Eq. (11) the external β^{-1} as defined in plasma physics, $L^F = \beta^{-1}$.

In Chap. 13, we will find the points where the two components of the macroscopic free energy, i.e., Eq. (5) in 9.1 and Eq. (11) will be stationary.

In Fig. 2 are plotted L' and L^F from Eqs. (5) and (9), where it can be seen that for the pressure range 1 to 25 bar the instable maxima in L' are within 0.5 kK of the stable minima in L^F . For the stationary arc, we have $T_e = T_0 = T$, for which the minimum of the field free energy is given by the condition

$$\frac{dL^F}{dkT} = \frac{d}{dkT} \frac{\mu_0}{nkT} \left(\frac{I}{2\pi r} \right)^2 = 0. \quad (9-12)$$

We discuss in Chap. 13 the solution of Eq. (12), presently confining ourselves to the observation that the naive assertion $\beta = 1$, found in several publications on the electric arc, is wrong.

Fig. 2 Interaction free energy and field free energy vs temperature for $T_e = T_0 = T$ and $\gamma T_0 = 0$.

CHAPTER 10
RADIATIVE ENERGY TRANSPORT
(esu)

An electric arc may be considered²² to radiate like an opaque body with an absorption coefficient $\Gamma \approx 0.14$ at 24 kK; 0.13 at 20 kK; 0.09 at 15 kK. Following are the balances in Planck's law.

If a body is assumed to be in equilibrium at temperature T with radiation inside an enclosure at this temperature, then the radiation power $h\omega^3/c^2 f(n) \cos \theta$ is incident on an unit area of this body, per unit frequency and unit solid angle about the normal κ to its surface. A fraction $a(\kappa)$ is absorbed, the rest of the radiation power reflected. The reverse process occurs with equal probability. The emitted power per unit area, in the frequency range ω to $\omega + d\omega$ is thus

$$P_e(\omega)d\omega = a(\omega) \frac{2\pi h\omega^3}{c^2} f(\kappa)d\omega. \quad (10-1)$$

Equilibrium considerations require that the mean power of the oscillators at temperature T be given by a canonical distribution. The total power emitted per unit area of surface will be

$$(P_e)_{\text{tot}} = \int_0^\infty P_e(\omega)d\omega = \left(\frac{\pi^2}{60} \frac{k^4}{c^2 h}\right) aT^4 \equiv \alpha T^4, \quad (10-2)$$

where the coefficient of aT^4 is $5.7 \cdot 10^{-5}$ [erg s⁻¹ cm⁻² K⁻⁴], which is the Stefan-Boltzmann constant. We set $a = \Gamma$ to obtain the opaque body radiation power, so $a = 1$ for black body radiation. Physically, $0 < \Gamma < 1$ means the black body property obtains over some parts of the frequency spectrum, transparency over the rest.

CHAPTER 11
HEAT CONDUCTIVITY
(mks)

We estimate the heat conductivity of the monoatomic gas, using the hard core potential, and the heat conductivity due to electron thermal motion.

11.1 Atomic Nitrogen (esu)

According to the Enskog-Chapman theory, the transport coefficients of a dense gas (cf. Chap. 9.3 in Hirschfelder, et al.'s book¹³) are given as functions of the dimensionless parameter $y = p\bar{V}/RT - 1$, where $\bar{V}$ is the mol volume, and R the universal gas constant. We get $y < 0$ and therefore the low density limit applies, i.e.

$$\lambda^* = \frac{75}{64\sqrt{\pi}} \frac{\sqrt{T^*}}{\Omega(2,2)^*(T^*)} \quad (\text{Hirschfelder 9.1-4})$$

where λ^* is the reduced conductivity and T^* the reduced temperature. The reduced quantities are defined as follows

$$T^* \equiv kT/\epsilon; \quad \lambda^* \equiv \frac{\lambda \sigma^2}{\sqrt{m\epsilon}} \frac{m}{k}.$$

For air, we have a well depth $\epsilon/k = 87[\text{K}]$, and a hard core radius $\sigma = 3.3[\text{\AA}]$, so for our temperature range of 15 to 25 kK we get $\Omega(2,2)^* = 1.0$ and $\lambda^* = 8.7$ to 11.2, respectively. The corresponding conductivity values are $\lambda = 2.5 \times 10^4$ to 3.2×10^4 [$\text{erg s}^{-1} \text{ cm}^{-1} \text{ K}^{-1}$] = $2.5 \dots 3.2 [\text{mW/cmK}]$ which are small, by an order of magnitude, compared with the electron heat conductivity.

11.2 Thermal Conductivity of the Lorentz Gas (esu)

The effective coefficient of electron-ion thermal conductivity

$$\lambda = \frac{(kT_e)^{5/2} k}{m_e^{1/2} e^4 \ln \Lambda}, \quad (11-1)$$

which yields 0.3 W/Kcm for the Coulomb logarithm $\ln \Lambda = 4$ and $T_e = 20$ kK, with Spitzer's coefficient (cf. App. B-4).

11.3 Enhancement of Thermal Conduction by Magnetohydrodynamic Turbulence

A pressure balance may imply stability, such as that derived from

$$\nabla p_0 = \frac{1}{\mu_0} (\nabla \times \underline{B}) \times \underline{B}. \quad (11-2)$$

For arcs of the type we are discussing, the magnetic pressure is much smaller than the gas pressure. We calculate, in Sec. 9.2, the temperature for asymptotic stability. However, that calculation neglects the conditions at the boundary, that is at the arc surface. Actually, the plasma occupies only the finite region confined by a vacuum magnetic field. Our discussion, thus far confined to the inside of the plasma region, will, in the text immediately following, be extended to the interface. We do this by the simplified assumption of a fluid film, that is a Lorentz gas, at the interface, conducting the total current. Hence, recall the force balance (that is d'Alembert's principle applied at the fluid's outer surface), the continuity equation, the adiabatic condition, and Faraday's law, yielding the set

$$\rho \dot{\tilde{v}} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \tilde{b}) \times \tilde{b} \quad (11-3)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \tilde{v}) = 0 \quad (11-4)$$

$$\frac{d}{dt} (\rho \tilde{v}^{-\gamma}) = 0 \quad (11-5)$$

$$\dot{\tilde{b}} = \nabla \times (\tilde{v} \times \tilde{b}) , \quad (11-6)$$

where $\gamma \equiv c_v/c_p = 5/3$. In the usual manner, we get the second order ODE by differentiation and successive substitutions,

$$\rho \ddot{\tilde{v}} = \nabla (\tilde{v} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \tilde{v}) + \frac{1}{\mu_0} \{ \nabla \times [\nabla \times (\tilde{v} \times \tilde{b}_0)] \times \tilde{b}_0 \} + \frac{1}{\mu_0} (\nabla \times \tilde{b}_0) \times [\nabla \times (\tilde{v} \times \tilde{b})] , \quad (11-7)$$

where p_0 and $\tilde{b}_0$ are defined at the outside surface of the current sheet, at which the terms enclosed in brackets[] are zero, similarly as ∇p_0 . Thus, only one term is left on the r.h.s. Integrating Eq. (7) yields, in our single-term form,

$$\rho_0 \ddot{\tilde{\xi}} = \nabla \gamma p_0 \cdot \nabla \cdot \tilde{\xi} = \gamma p_0 \nabla^2 \tilde{\xi} , \quad (11-8)$$

where $\tilde{\xi} = \tilde{x} - \tilde{x}_0$ measures the departure, in configuration space, from the equilibrium at $\tilde{x} = \tilde{x}_0$. One can separate $\tilde{\xi}(\tilde{x}_0, t)$ into a product of a space and a time-dependent part, $\tilde{\xi} = T(t) \cdot \tilde{\xi}(x_0)$, which yields $\ddot{T} = -\omega_k^2 T$, and

$$\rho_0 \tilde{\xi} \omega^2 = \gamma p_0 \nabla^2 \cdot \tilde{\xi} . \quad (11-9)$$

In cylindrical coordinates, Eq. (9), with

$$f_z(r, \phi, z) = f_z(r) \exp i(m\phi + kz)$$

leads, for the z-component, f_z to Bessel's equation

$$\frac{d^2 f_z}{dr^2} + \frac{1}{r} \frac{df_z}{dr} + \left(\alpha^2 - \frac{m^2}{r^2} \right) f_z = 0, \quad (11-10)$$

where

$$\alpha^2 = \frac{\omega^2 \rho}{\gamma p} - k^2, \quad \text{and} \quad k^2 = \left(\frac{\omega}{U_0} \right)^2 \mu_0 \rho.$$

Observe that

$$\frac{1}{\gamma p} < \frac{\mu_0}{B^2},$$

because our $\beta \leq 0.5$.

With the b.c. $r = r_0 + \xi_r = r_0 + A/ik \exp i(m\phi + kz) J'_m(\alpha r_0)$, and for the pinch instability with $m = 0$, descriptively referred to as the sausage instability, the integral of Eq. (10) becomes²⁹

$$- \frac{\omega^2 r_0 \rho}{2p} \frac{J_0(\alpha r_0)}{J'_0(\alpha r_0)} = 1. \quad (11-11)$$

The solution values, ω^2 , of Eq. (11) are negative, that is $\omega^2 < 0$, indicating an instability, as expected. Using some numbers from Chap. 13, near the extremes shown in Figs. 4 and 5, we get

T(kK)	n(cm ⁻³)	r ₀ (cm)	I(A)	ω (rad/s)
21	2.4·10 ¹⁸	0.18	2000	12·10 ⁶
18	0.34·10 ¹⁸	0.28	2000	14·10 ⁶

The characteristic time for the sausage instability is therefore $\tau_s \approx 0.5 \mu s$.

Lindemuth's two-dimensional calculations³⁰ show the sausage instability causes short, $\tau \ll \tau_s$, bursts of swirling motions of the plasma, starting at the boundary layer at intervals of $\sim \tau_s$ and convecting hot plasma to the surface. Over time intervals $T \gg \tau_s$, one observes a plasma with vanishing radial temperature gradient.

From the fact of τ_s being the longest relaxation time derives great significance in the dynamic arc: Phenomena evolving on the time scale of 1 ... 3 μs , or shorter, are treated inadequately in a one-dimensional formalism such as ours. Some heuristic approach is then suggested.

The plasma confined by a thin sheath of surface current, that is the external $\beta = 1$ condition, provides one of several possible preconditions for the $m = 0$ instability. That condition allows a simple estimate of the characteristic time for the sausage instability, as shown in Eqs. (3)-(11). Our stability criterion must be that for a plasma with $\beta > 1$, however. Hence, the Kadomtsev condition³⁰ obtains, that is

$$-\frac{r}{p} \frac{dp}{dr} \left(\frac{6 + 5\beta}{20} \right) < 1$$

which, if satisfied, insures stability, where β is the local $n \cdot kT / (1/2 H B)$. Because $-\Delta p/p \ll 1$, in our plasma, stability is provided by the

wall confinement from the external gas pressure. As a precondition for the sausage instability still to develop, we expect $-\Delta p/p$ to become large when the local gas pressure p approaches some small limit, as a result of overcooling by the turbulent removal of heat.

CHAPTER 12
TURBULENCE
(mks or esu)

The Navier-Stokes equation reduces to a form of the continuity equation if the axial variation of the drift velocity's radial component $\partial v_r / \partial z = 0$. Thiel¹ verified experimentally for a double-jet configuration typical for the electrodes of an air blast breaker that this condition is practically satisfied. Therefore, $|v_z| \propto z/L$ for $|z| < L$, and $|v_z|$ attains the speed of sound at $z = L$. According to Thiel's work, the turbulence in the discussed configuration is isotropic, i.e., $\int d\Omega \rho \vec{w} = 0$, and, furthermore, that $\int_0^\infty dt \rho \vec{w} = 0$, if $\vec{w}$ is defined as the turbulent and $\vec{v}$ the drift component in the velocity. It follows that the mean enthalpy flow reduces to

$$\vec{q}_T(r) = - \left(\rho w \frac{\ell_T}{3} \frac{\partial h}{\partial T} \right) \nabla T , \quad (12-1)$$

where ℓ_T is the mixing distance over which turbulence transfers momentum in the radial direction. Thiel defines an equivalent turbulence-induced thermal conductivity κ_T in the form of the expression in parentheses in Eq. (1) to obtain

$$\vec{q}_T = \kappa_T \nabla T . \quad (12-2)$$

Radiative and turbulent energy transport generate a pronounced temperature gradient in the radial direction at the outer fringes of the arc, thus providing a flat temperature profile in the interior. One can therefore write Eq. (2) in the form

$$q_T = \frac{k_T}{\ell_T} (T - T_{\text{ext}}) , \quad (12-3)$$

where T_{ext} is the external cooling medium's temperature. For the application in our equations, k_T/ℓ_T is a single parameter to fit observation.

CHAPTER 13
CLOSURE AND STUDY OF THE STEADY STATE
(mks units)

Considering the loss mechanisms discussed in Chaps. 10-12, we infer opaque body radiation and turbulence to be the most significant loss mechanisms. The heat flow per unit area at the arc surface, in this approximation, is given by

$$P_o + P_e = \frac{2}{r} (\alpha T^4 + \gamma T) \quad (13-1)$$

in the notations of Chaps. 10 and 12. By substituting $J = I/\pi r^2$ and Eq. (1), in Eq. (8-1) we get

$$r^3 = (m_e/\pi^2 e^2) \cdot v_{ei} I^2 / [n(\alpha T_e^4 + \gamma T_o)] , \quad (13-2)$$

where I is the current through the external circuit, and $v_{ei} = v_{ei}(kT_e)$. T_e and T_o are determined by the relationships discussed in Chap. 9 and the text immediately following.

By substituting in Eq. (9-12) the Eqs. (13-1), (13-2), (5-4), and (5-5), one obtains a relationship for the minimum condition of the field free energy, corresponding to the minima in the plots for L^F in Fig. 2, i.e.,

$$(a_{12} + a_{21}) \left(\frac{1}{\ln \Lambda} - 3 \right) \frac{kT}{n} + 2 \frac{1 + 4 \frac{\alpha}{\gamma} T^3}{a + \frac{\alpha}{\gamma} T^3} - \frac{3}{\ln \Lambda} = 0 . \quad (13-3)$$

We solve Eq. (13-3) numerically to find the stationary temperature. Observe that T in Eq. (13-3) is a dependent of the transfer coefficient γ for the turbulent heat transport, discussed in Chap. 12. Our system of equations is thus closed and we can calculate

- the arc radius, from Eq. (13-2);
- the voltage gradient, from Eq. (8-2); and
- the current density, from Eq. (8-1).

The two former quantities are plotted vs γ in Fig. 3, for an opaque body absorption coefficient, $\Gamma = 0.12$ (cf. Chap. 10).

In Fig. 4, we compare our results with the observations reported by W. Hermann et al.²³ Our basis for the comparison is:

- a) The set of static pressure points, P_{stat} , reported by Hermann et al., shown in Fig. 4a;
- b) Our assumed linear increase in turbulent heat dissipation, increasing from 0.2 to 0.8 MW/K m² over the arc length $z = 0 \dots 10$ cm;
- c) $\Gamma = 0.5$, which we infer from a paper of Thiel.²⁸

We obtain the values of T , r , and E plotted in Figs. 4b, c, and d, where the agreement between calculation and observation is best for the temperature, the difference between the two being less than 12%. Observation and calculation disagree by a factor of 3 in arc radius and voltage gradient. This disagreement is caused mainly by our choice of $\gamma(\Gamma)$ and, to a lesser degree, by that of Γ . In using Γ as a fitting parameter and calculating γ , with Thiel's formula given below, from Hermann et al.'s "outer cold gas velocity," v , and pressure, p , profiles, we obtain the agreement within a factor of less than 2 shown in Fig. 5, where $\Gamma = 0.017$. Thiel's formula is given by

$$\gamma = \frac{v}{c_s(T)} G(T) ,$$

where $c_s(T)$ is the adiabatic speed of sound, and $G(T)$ the turbulent heat flow potential plotted in Thiel's¹ Fig. 32, p. 107.

Fig. 3 Arc radius and voltage gradient vs turbulent heat transfer coefficient for $\Gamma = 0.12$.

Fig. 4 Computed and measured (Hermann et al.)²³ are profiles vs the axial coordinate. The turbulent heat transfer coefficient, γ , is assumed to be a linear function of the cold gas velocity which, together with the pressure, P , profile, is that reported by Hermann et al.; opacity, P , as reported by Thiel.²⁸

Fig. 5 Computed and measured (Hermann et al.)²³ are profiles vs the axial coordinate. The turbulent heat transfer coefficient, γ , is calculated from Hermann et al.'s pressure and cold gas velocity profiles; opacity, κ , computed by fitting our "assumed" temperature profile.

The stationary condition for our Eqs. (9-9) and (9-10) defines an equilibrium. This is so, because the components $\hat{A}$, $\hat{Z}$, and net of L^F depend on the coordinate and not on its time derivative. Consequently, L^F is a potential energy for which a hydrostatic energy condition is expected to obtain. A system is asymptotically stable at the equilibrium, if all possible small perturbations $\delta r = r - r_0$ render the variation δL^F positive. Because Eq. (8-1) uniquely relates kI to $r = \sqrt{I/\pi J}$, for any given positive real value of the total current, I , the minimum in (L^F, kI) space corresponds to a minimum in the (L^F, r) space. Therefore, all possible small perturbations δr make the variation in potential energy,

$$\delta L^F = \left(\frac{dW}{dr} \right) \delta r,$$

positive. A Liapunov function may be defined in L^F , as an alternative proof of the asymptotic stability.

CHAPTER 14 SOME CONSIDERATIONS FOR THE TRANSIENT STATE

Let us estimate the characteristic times for the equilibrium temperature $T_0 \rightarrow T_c$ in Sec. 7.3. In using the abbreviated notation h_{ij} for the four elements in the lower right hand quadrant of the H matrix, we can write the characteristic equation of the ODE's (7-22), (7-23), assuming $Ed = \text{constant}$,

$$\begin{vmatrix} -h_{43} - \lambda & -v_{co}k \\ -h_{44} - \lambda & -v_{co}k \\ -h_{33} - \lambda & -v_{co}k \\ -h_{34} - \lambda & -v_{co}k \end{vmatrix} = 0 \quad .$$

Hence, $\lambda_1 = 0$, and

$$\lambda_2 = - \frac{h_{33} + h_{34} + h_{43} + h_{44}}{h_{34} - h_{43} - h_{33} - h_{44}} v_{co} \quad . \quad (14-1)$$

For the numbers in Table I, the fraction part in Eq. (1) is one, within an order of magnitude. Hence, $\lambda_2 \sim -v_{co}$, and the internal equilibrium is attained within the short times mentioned in Sec. 7.3.

During the transient state, $|I|$ may be large, resulting in significant changes of r and E over a short period of time. Consequently, one must take the boundary conditions into account.

The entropy and $\exp(-H)$, where H is Boltzmann's H -function, are additional Liapunov functions, which might, perhaps, contribute additional useful relationships when one considers the transient state.

CHAPTER 15

CONCLUSIONS

By the agreement between our calculated numbers and Herman et al.'s corresponding experimental data²³ of the arc axis temperature, we are led to infer a fairly flat temperature profile in the arc, most of the measured values being a few percent greater than our computed ones. In other words, if the calculated results predicted the correct average values (in the arc cross section), and the measurements were a correct indication of the temperature on the axis, then our premise of the temperature gradient being small would be confirmed. It is the premise for simplifying the equations immediately following Eq. (7-19), as mentioned in the text supervening that equation. We have identified magnetohydrodynamic turbulence as the mechanism for generating the flat temperature profile.

From the cylindrical arc envelope function developed in Chap. 6, we infer a constant current density. Conversely, a fairly flat current profile is implied when the deviations of the arc from a right cylindrical geometry are as small as observed by Hermann et al.²³ as well as by others. The observed arc shape was approximately that of a truncated cone with a half angle of 5 to 6%.

Historically, Steenbeck's intuitively stated minimum principle²⁵ has been discussed for several decades. It postulates an arc envelope described by a function $r = r(z)$ such that the arc voltage $\int E dz$ be minimized. Some observations have been known to confirm, other ones to disagree with this postulate. In departure, for the more fundamental reason to introduce an energy principle, we minimize the Helmholtz free energy on the microscopic and the macroscopic scales, and use the same functions to close the system of hydrodynamic equations.

The system of hydrodynamic equations (7-20)~(7-23) is derived in Chaps. 3, 7, and Apps. B-C. Our formalism is based on the Enskog-Chapman approximation to derive the hydrodynamic equations from the Boltzmann equation. The method is presented explicitly in basically simple mathematics, not exceeding vector and tensor calculus. We have

thus been able to clarify two false assumptions found in the literature. One concerns the collision frequency, and the other is the frequently encountered assertion that the external $\beta = 1$, which, if true, would be inconsistent with theory and observation.

Using the pressure and air velocity vs the axial coordinate data observed by Hermann et al. to test our equations, we calculate numbers for the voltage gradient, arc radius, and temperature, which seem to agree with the same investigators' experimental data, where we use two fitting parameters. These are the heat flow parameter for the turbulent transport, and the opacity. The former is a constant correlation factor between the air speed at the arc surface and the turbulent heat conductivity. Our numbers seem to indicate that these two are the dominant heat transport mechanisms.

Regarding the microscopic basis for our formalism, we have shown in Chap. 2 that the Boltzmann approximation is applicable to provide us with a microscopic picture of the arc plasma. We have shown in Chap. 13 that the stationary condition is asymptotically stable, with characteristic times of typically less than a few tens of nanoseconds.

We have thus defined the complete set of equations that obtain in the arc's length element. The radius and axial voltage gradient are given as functions of the pressure in the adjoining gas, the heat dissipation coefficients Γ and γ , and the total current. Because the inductance is a function of the arc radius r and length l , the latter is determined by the boundary conditions. In referring to Fig. 1.1 the inductance element is given by

$$dL = \mu_0 \frac{dz}{4\pi} \left(1 + 2 \ln \frac{R}{r} \right),$$

where r is given in Eq. (8-1) as function of $\gamma(z)$ and pressure $p = p(z)$. Hence, the total voltage over the arc length becomes

$$v = \int_0^{\ell} E[i, p(z), \gamma(z)] dz + \frac{di}{dt} \int_0^{\ell} dL + i \int_0^{\ell} \frac{dL}{dt}, \quad (15-2)$$

where the first term on the r.h.s. is the line integral of Eq. (8-2). For the simple configuration in Fig. 1d, it is straightforward to use Eq. (2) for closing the set of external circuit equations. Usually, the first term is the only significant one on the right, in Eq. (2). In computing that term, the solving of Eq. (13-3) for T is the most expensive part. Using Newton's method, our experience has been excellent if we provide the analytical derivatives, neglecting the terms with $\partial^2 X / \partial T^2$ for which we compensate by scaling the Newton correction step, ΔT , to $1.3 \Delta T$. The accounting time is thus approximately 5 ms per evaluation of Eq. (13-8), on a CDC-7600 computer.

We discuss, in Sec. 11.3, the $m = 0$ magnetohydrodynamic instability, probably caused by pressure reduction from turbulent cooling of the arc surface. That instability exhibits a characteristic time of approximately $0.5 \mu s$ and forces the radial convection of the heat dissipated by the turbulent gas.

CHAPTER 16

SUMMARY

We show in Chap. 2 that the enormous electron and ion densities in an electric arc in air, at 10 to 50 atmospheres, imply a highly ionized plasma; and that the Boltzmann equation is the proper kinetic equation to be used. Our results agree with observation, even though we limit ourselves to considering a plasma with only three components, i.e., electrons, singly charged nitrogen ions, and nitrogen atoms. In Chap. 3, we derive, from the Boltzmann equation, the hydrodynamic equations for the three-component plasma, the equations right side being treated formally by simple functions of the collision frequencies among the species. The gas kinetic pressure exceeds the magnetic pressure by one to two orders of magnitude, the magnetic pressure gradient, therefore, being negligible. In App. B, the collision frequency and thermal conductivity of the fully ionized plasma are derived by integrating the right side of the Boltzmann equation, by the normal method of Enskog and Chapman. Our results are compared with Spitzer's corresponding equations quoted in Chap. 5. The equations agree, except for the numerical coefficients, in which we differ by 6% in the collision frequency λ_{ei} , and by a factor 2.6 in the thermal conductivity λ_{ei} .

Like all hydrodynamic equations, our set of equations is not closed and additional relationships are needed. Much of our work is devoted to obtaining closure. A few relationships are provided by the conditions of a minimum Helmholtz free energy, asymptotic stability of the equilibrium in each case provided. As the first of these asymptotically stable equilibria in our three-component plasma, we derive the Saha equation in Chap. 4, the gas kinetic total pressure being the invariant of the ionization process. In Chap. 9, we derive the free energy minimum in the Debye space (Sec. 9.1, q.v.) and show that minimum condition to be an instable equilibrium not useful for closure. In Sec. 9.2, we derive the free energy minimum for the particle-external field interactions, for which we can show asymptotic stability. Hence,

the Saha equation and the equation defining the locus of the minimum particle-external field free energy are available for closure. From these, we get 15 to 23 kK, typically, for the arc temperature, and 0.2 to 0.6 for the degree of ionization. We find the temperatures of the three species to be nearly identical, thus implying the arc to be stationary. In Chaps. 7 and 8, we show that the characteristic time for establishing the stationary condition is in a subnanosecond regime.

Additional relationships for closure are provided by the boundary conditions. The first one of these is derived in Chap. 6, where we show that, for a given total current, the Lagrangian of a set of moving charges, in the magnetic and electric collective self-fields, is stationary, if the arc envelope function is related to the current density profile. As a useful example, a constant current density profile, i.e., one independent of the radial coordinate, requires a cylindrical arc envelope. This is expected from observation also. In airblast breakers, the arc envelopes are cylindrical or slightly conical and the current density is nearly constant across the arc radius. A second relationship is provided by the Komtsev stability criterion at the arc surface, discussed in Sec. 11.3. The Komtsev condition indicates local instability for a low pressure or large pressure gradient. Because the pressure gradients are predominantly electromagnetic and small, compared with the p/r ratio of the gas kinetic pressure over the arc radius, magnetohydrodynamic stability is provided by the gas pressure. Thus, we have wall confinement, except when the pressure is being reduced locally by overcooling from the forced removal of heat in the turbulent interaction between the high velocity gas and the arc surface. Overcooling may thus provide the starting condition for the $m = 0$ sausage instability. We show, in Sec. 11.3, the sausage instability to evolve with a characteristic time of $0.5 \mu\text{s}$. Hence, because of mixing from the instability, a uniform radial distribution of current density and temperature can be expected, when averaged over a time long compared with the characteristic time of the magnetohydrodynamic instability.

By the energy and boundary conditions, our set of hydrodynamic equations has thus been closed, for given thermal transport functions at the arc surface. As to the surface transport functions, we determine, in Chaps. 10 and 12, radiation and turbulence to be the predominant mechanisms. We introduce, in Chap. 10, the opacity Γ as the characteristic coefficient for the radiative transport, and, in Chap. 12, the surface coefficient $\gamma = \kappa_T / \ell_T$ of the turbulent heat transfer, where ℓ_T is the mixing length, or turbulent layer thickness, over which the heat transfer is effected, κ_T being the turbulent thermal conductivity. We use Γ and γ as fitting parameters in Chap. 13, where we compute the axial profiles of the arc radius r , voltage gradient E , and temperature T , assuming the axial profile of the gas pressure p , and the two fitting parameters, Γ and γ . We obtain fair agreement with Hermann et al.'s reported data, as discussed in Chap. 13. We obtain agreement within a factor two, with Hermann et al.'s observed E , r , and T profiles, if we use the velocity and pressure profiles, reported likewise in their paper, for calculating the profile of the turbulent heat transfer, γ , with a formula reported by Thiel, yielding γ as a function of the gas velocity and plasma surface temperature. We can thus show, in Chap. 13, that the accuracy in calculating the electric arc characteristics is limited by the available accuracy of the pressure and gas velocity profiles.

We have an additional relationship from the stationary condition of the Lagrangian, discussed in the text just preceeding, and in Chap. 6, relating the arc length to the arc radius, in Eqs. (6,17) and (6-18). These equations may be useful for closing the equation of the external circuit, in addition to our discussed variables E , r , and T . The usefulness of that additional relationship derives from the numerical insignificance of the arc inductance in its interaction with the external circuit. We have not explored that global interaction between the arc and external circuit.

APPENDIX A
CONVERSION FROM CGS TO MKS UNITS

Definitions

$$F_1 = k_1 \frac{qq'}{r^2}; \quad E = k_1 \frac{q}{r^2};$$

Ampère: $\frac{dF_2}{dl} = 2k_2 \frac{II'}{d^2}; \quad k_3 B = 2k_2 \frac{I}{d}$

Faraday: $\Delta \times E = -k_3 \frac{\partial B}{\partial t}; \quad \frac{k_1}{k_2} = c^2$

	Conventional System:			Rational System:
	$D = \epsilon_0 E + 4\pi P$			$D = \epsilon_0 E + P$
	$H = \frac{1}{\mu_0} B - 4\pi M$			$H = \frac{1}{\mu_0} B - M$
	Gauss'	emu	esu	mks
k_1	1	c^2	1	$(4\pi\epsilon_0)^{-1}$
k_3	c^{-1}	1	1	1
ϵ_0	1	c^{-2}	1	$10^7/(4\pi c^2)$
μ_0	1	1	c^{-2}	$4\pi \cdot 10^{-7}$
$\nabla \times E = -k_3 \frac{\partial B}{\partial t}; \quad \nabla \cdot E = 4\pi k_1 \rho$ $k_3 \nabla \times B = 4\pi k_2 J + \frac{1}{c^2} \frac{\partial E}{\partial t}; \quad \nabla \cdot B = 0$ Simply use $4\pi k_1 \rho, k_1 q; 4\pi k_2 J, 2k_2 I; k_3 B$				

Gaussian	for mks, replace Gauss' by
(E, ϕ, V)	$\sqrt{4\pi\epsilon_0} (E, \phi, V)$
(ρ, q, J, I, P)	$(\rho, q, J, I, P) / \sqrt{4\pi\epsilon_0}$
$(R, L, 1/C)$	$4\pi\epsilon_0 (R, L, 1/C)$
(H)	$\sqrt{4\pi\mu_0} (H)$
(B)	$\sqrt{4\pi/\mu_0} (B)$
(D)	$\sqrt{4\pi/\epsilon_0} (D)$
c	$(\mu_0\epsilon_0)^{-1/2}$

cgs dimensions
$E, D, E, B:$ $g^{1/2}/cm^{1/2} s$
$I: g^{1/2} cm^{3/2}/s^2$
$L, C: cm$
$W: erg \equiv gcm^2/s^2$

APPENDIX B

ENSKOG-CHAPMAN METHOD FOR SOLVING THE BOLTZMANN AND MAXWELL TRANSPORT EQUATIONS

1. OUTLINE OF THE METHOD

It is sometimes convenient to write the Boltzmann equation in the simple notations of the streaming operator and the collision operator I , i.e.

$$\dot{f} = I(ff) \equiv \int d\mathbf{v}_1 \int d\Omega \sigma g [f'f'_1 - ff_1] \quad . \quad (B-1)$$

where

$$I f \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \mathbf{E} \cdot \bar{\nabla}_{\mathbf{v}} f \quad . \quad (B-2)$$

If the system is not far from thermodynamic equilibrium, one may solve Eq. (1) by the method of successive approximations in which the distribution function f is expanded in powers of the parameter ε , i.e.

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots, \quad (B-3)$$

and where ε measures the departure in density, from equilibrium. We simply redefine the velocity distribution by writing the Boltzmann equation (1) in the new form

$$\mathcal{L} f = \frac{1}{\varepsilon} I(ff) \quad . \quad (B-4)$$

Equation (4) has the appearance of the Boltzmann equation (1) for a velocity distribution of the form $f_{\text{formal}} = f/\epsilon$. We proceed, therefore, by inserting the expanded velocity distribution function (3) in Eq. (4) and equating the coefficients of the like powers of ϵ :

$$\epsilon^0: \quad 0 = I[f^{(0)} f^{(0)}]; \quad (\text{B-5})$$

$$\epsilon^1: \quad f^{(0)} = I[f^{(0)} f^{(1)}] + I[f^{(1)} f^{(0)}]; \quad (\text{B-6})$$

$$\epsilon^2: \quad f^{(1)} = I[f^{(1)} f^{(1)}] + I[f^{(0)} f^{(2)}] + I[f^{(2)} f^{(0)}]; \quad (\text{B-7})$$

and so forth. The solution of Eq. (5) is given by the equilibrium distribution

$$f^{(0)}(\underline{v}) = n^{(0)} \left(\frac{m}{2\pi kT} \right)^{2/3} e^{-m\mathbf{v}^2/2kT} . \quad (\text{B-8})$$

When the system is close to equilibrium, we define $n = \int f d\underline{v}$; $n\bar{v} = \int \underline{v} f d\underline{v}$; $n \frac{3}{2} kT = \int \frac{m}{2} \mathbf{v}^2 f d\underline{v}$; which are slowly varying functions of the arguments $(\underline{r}, t)$. The following important postulate, which is prerequisite to the resolving procedure we are discussing, was contributed by Enskog. "Time does not enter explicitly among the arguments of f and $\partial\phi/\partial t$, but only implicitly through $\bar{\phi}$ and its spatial derivatives, where $\phi = \int \phi f d\underline{v}$." When the system is not far from local equilibrium (i.e., Boltzmann-Maxwellian), then $f(\underline{r}, \underline{v}, t)$ varies with time only through the functional dependence on $n(\underline{r}, t), \bar{v}(\underline{r}, t), T(\underline{r}, t)$ postulated by Enskog, i.e., $f(\underline{r}, \underline{v} | n, \bar{v}, T)$; similarly for $f^{(m)}$ of all orders.* The procedure is, therefore, to insert the expansion $f^{(0)} +$

*The physics reason for this is the fact that $n, \bar{v}$, and T are the privileged quantities that are invariant under binary collisions. A few collision intervals after the initial instant of the encounter, the time evolution is determined by our reduced set of variables.

$\epsilon f^{(1)} + \dots$ into the Boltzmann equation, the hydrodynamic equations, and into the relationships defining the thermal velocity (i.e., Enskog's "peculiar velocity") $\underline{v} = \underline{v} - \bar{\underline{v}}$, the momentum flow tensor $\underline{p} = m \int \underline{v} \underline{v} f d\underline{v}$, and the heat flux vector $\underline{q} = \int m/2 v^2 \underline{v} f d\underline{v}$. Solving the Boltzmann equation to the order ϵ^0 , yields $f^{(0)}$. With this zeroth order distribution function, we calculate $\underline{v}$, $\underline{p}$, and $\underline{q}$ for the zeroth order, i.e., $v^{(0)}$, $p^{(0)}$, $q^{(0)}$, and, insert them into the hydrodynamic equations for $n, \underline{v}$, and T . With the latter quantities, we solve the Boltzmann equation to the order ϵ^1 , yielding $f^{(1)}$. In principle, this procedure may be continued to orders of arbitrary degree. Thus, we follow Enskog in defining

$$f(\underline{r}, \underline{v}, t) = f(\underline{r}, \underline{v} | \bar{\Phi}, \nabla_{\underline{r}} \bar{\Phi}, \dots) \quad (\text{B-9})$$

$$\frac{\partial}{\partial t} \bar{\Phi}(\underline{r}, t) = \Psi(\underline{r}, \bar{\Phi}, \nabla_{\underline{r}} \bar{\Phi}, \dots) \quad , \quad (\text{B-10})$$

where $\bar{\Phi}$ is any of the privileged dynamical (i.e., velocity-dependent) functions averaged over the velocity space (i.e., $n, \overline{m \underline{v}}, T$). Since a power series has been adopted for f , the moments of f , i.e., $\bar{\Phi}$ and consequently $\Psi = \dot{\bar{\Phi}}$, should now be written as a power series, as well:

$$\bar{\Phi} = \Phi^{(0)} + \epsilon \Phi^{(1)} + \dots \quad (\text{B-11})$$

$$\bar{\Psi} = \Psi^{(0)} + \epsilon \Psi^{(1)} + \dots \quad (\text{B-12})$$

The time derivative of the distribution function is obtained, in tensor formalism, by operating with the chain rule on Eq. (9), i.e.,

$$\frac{\partial f}{\partial t} = \Psi \cdot \nabla_{\underline{v}} f + (\nabla_{\underline{r}} \bar{\Psi}) : (\nabla_{\nabla_{\underline{r}}} \bar{\Phi} f) + \dots \quad , \quad (\text{B-13})$$

where the higher order spatial derivatives are implied by the continuation dots in Eq. (9). The order of the tensors on both sides of the multiplication operators in Eq. (13) increases term by term; so must the multiplication operators, to be consistent with the chain rule. The following linear differential operator may thus be defined:

$$\frac{\partial i}{\partial t} = \underline{\Psi}^{(1)} \cdot \nabla_{\underline{\Phi}} + (\nabla_{\underline{r}} \underline{\Psi}^{(1)}) : \nabla_{\underline{v}} \underline{\Phi} + \dots \quad (\text{B-14})$$

Following Enskog's procedure, we substitute Eqs. (3) and (12) in Eq. (13) and write the resulting expansion in terms of the differential operator (14), i.e.,

$$\frac{\partial f}{\partial t} = \frac{\partial_o f^{(0)}}{\partial t} + \epsilon \left(\frac{\partial_1 f^{(0)}}{\partial t} + \frac{\partial_o f^{(1)}}{\partial t} \right) + \epsilon^2 \left(\frac{\partial_2 f^{(0)}}{\partial t} + \frac{\partial_1 f^{(1)}}{\partial t} + \frac{\partial_o f^{(2)}}{\partial t} + \dots \right) . \quad (\text{B-15})$$

In this way, we have developed an expansion of the streaming term, $\mathfrak{A}f$, of the Boltzmann equation (4) that may be written in the simple symbolic form

$$\mathfrak{A}f = (\mathfrak{A}f)^{(0)} + \epsilon (\mathfrak{A}f)^{(1)} + \epsilon^2 (\mathfrak{A}f)^{(2)} + \dots \quad (\text{B-16})$$

Hence, by comparing Eqs. (2) and (15) to Eq. (16), the identity

$$(\mathfrak{A}f)^{(i)} = \frac{\partial_1 f^{(0)}}{\partial t} + \dots + \frac{\partial_o f^{(i)}}{\partial t} + \underline{\tilde{v}} \cdot \nabla_{\underline{r}} f^{(i)} + \underline{\tilde{F}} \cdot \nabla_{\underline{v}} f^{(i)} \quad (\text{B-17})$$

emerges for $i = 0, 1, 2, \dots$. Similarly, from Eqs. (5)-(6), we obtain for the collision integral, $I(ff) = \int d\mathbf{v}_1 \int d\Omega \sigma g [f'f'_1 - ff_1]$, the expansion

$$I(ff) = I[f^{(0)}f^{(0)}] + \varepsilon [I(f^{(0)}f^{(1)})] + I[f^{(1)}f^{(0)}] + \dots \quad (\text{B-18})$$

With both sides of the Boltzmann equation thus being expanded in powers of ε , we may write in detail the two lowest order integrals of Eq. (18) by using the forms $f = f^{(0)} [1 + \varepsilon\phi + \varepsilon^2\psi + \dots]$; $f^{(1)} = f^{(0)}\phi(r, \mathbf{v}|n, V, T)$; $f^{(2)} = f^{(0)}\psi(r, \mathbf{v}|n, V, T)$. Thus,

$$I[f^{(0)}f^{(0)}] \equiv I_{(0)}(1) = \iint d\mathbf{v}_1 d\Omega \sigma g [f^{(0)}(\mathbf{v}')f^{(0)}(\mathbf{v}'_1) - f^{(0)}(\mathbf{v})f^{(0)}(\mathbf{v}_1)] \quad (\text{B-19})$$

is the zeroth order integral, which vanishes as it should, agreeing with the result we obtained by comparing the coefficients [cf. Eq. (5)]. The first order (ε^1) integral becomes

$$I[f^{(0)}f^{(1)}] + I[f^{(1)}f^{(0)}] \equiv I_{(1)}(\phi) = \iint d\mathbf{v}_1 d\Omega \sigma g f^{(0)}(\mathbf{v})f^{(0)}(\mathbf{v}_1) \cdot \\ \cdot [\phi(\mathbf{v}') + \phi(\mathbf{v}'_1) - \phi(\mathbf{v}) - \phi(\mathbf{v}_1)] \quad , \quad (\text{B-20})$$

which is an integral over the unknown function ϕ . Thus, we may use a general notation, such as $\phi^{(r)}$, for the quantities $1, \phi, \psi, \dots$; and denote the integral (2) by $I_{(r)}[\phi^{(r)}]$. It will have been noticed that r is used here to denote the order, not radius vector.

To determine the resolving method for our integral equation (1), one must first determine the type of the integral equation. Hilbert has shown for the case of a gas of rigid elastic spherical molecules, and Enskog for the general case, that $I(\phi)$ is an integral operator with a

symmetrical kernel. To see this, we write the collision integral in terms of the collision cross section σ and the angle χ enclosed by the trajectories of the colliding particles, i.e.,

$$I(\phi) = \frac{1}{n^2} \iint f_1 f(\phi + \phi_1 - \phi' - \phi'_1) |\underline{v} - \underline{v}_1| \sigma \sin \chi d\chi d\underline{v}_1 \quad . \quad (B-21)$$

The first term in parentheses, ϕ , is independent of the variables of integration and the integral over it is written:

$$K_0(\underline{v}) \phi(\underline{v}) \quad ,$$

where $K_0(\underline{v}) = n^2 (m/2\pi kT)^3 \iint d\chi d\underline{v}_1 e^{-(m/2\pi kT)(V^2 + V_1^2)} \cdot |\underline{v} - \underline{v}_1| \sigma \sin \chi \quad .$
The part of the integral (21) over the second term, ϕ_1 , may be expressed as

$$\int K_1(\underline{v}, \underline{v}_1) \phi(\underline{v}_1) d\underline{v}_1 \quad ,$$

and so forth, for the terms ϕ' and ϕ'_1 , leading to the final result

$$n^2 I_{(r)}[\phi^{(r)}] = K_0(\underline{v}) - \int K(\underline{v}, \underline{v}_1) \phi^{(r)}(\underline{v}_1) d\underline{v}_1 \quad , \quad (B-22)$$

where $K(\underline{v}, \underline{v}_1)$ is a symmetrical function of its arguments, i.e., $K(\underline{v}, \underline{v}_1) = k(\underline{v}_1, \underline{v})$. A Fredholm equation is thus suggested, which we will discuss briefly below.

Let us first proceed with the topic of the successive approximations. To this end, it may be useful to summarize from Eqs. (4), (16), and (18) the formalism we developed thus far:

$$\varepsilon^0: \quad I[f^{(0)}_f(0)] = 0$$

$$\varepsilon^1: \quad I[f^{(0)}_f(1)] + I[f^{(1)}_f(0)] = (\mathfrak{A}f)^{(0)}; \quad (B-23)$$

$$\varepsilon^2: \quad I[f^{(0)}_f(2)] + I[f^{(2)}_f(0)] = (\mathfrak{A}f)^{(1)} - I[f^{(1)}_f(1)]; \quad (B-24)$$

$$\varepsilon^3: \quad I[f^{(0)}_f(3)] + I[f^{(3)}_f(0)] = (\mathfrak{A}f)^{(2)} - I[f^{(1)}_f(2)] - I[f^{(2)}_f(1)] .$$

For every power of ε , the right side involves functions that are known from the previous equations of lower order. The unknown function appears only on the left and appears there linearly. It follows, that the solution of the set of integral equations (23), (24), ... is such that the macroscopic variables are always determined from $f^{(0)}$ alone, i.e.,

$$\int \psi f^{(0)} d\mathbf{y} = \int \psi f d\mathbf{y} = \bar{\psi}(\mathbf{r}, t) \quad (B-25)$$

$$\int \psi f^{(r)} d\mathbf{y} = 0 \quad r = 1, 2, \dots \quad (B-26)$$

The unknowns are the functions $\phi^{(r)} \equiv (1, \phi, \psi, \dots)$ in the integrals (22) which should be substituted in the l.h.s. of Eqs. (23), (24), Instead of rewriting Eqs. (23) and (24) again, let it suffice to say that they are inhomogeneous Fredholm integral equations of the second kind, whose general form is $\phi(z) + \lambda \int K(z, z_0) \psi(z_0) dz_0 = \psi(z)$. Theory predicts a solution if and only if the right member is orthogonal to all solutions of the homogeneous equation. In attempting to solve the set of integral equations (23), (24), ... successively, we must satisfy, in each step, the orthogonality relationship

$$\int \psi (\mathfrak{A}f^{(r-1)} - I[f^{(1)}_f(r-1)] - \dots - I[f^{(r-1)}_f(1)]) d\mathbf{y} = 0 ,$$

where ψ is one of the summational invariants; these being the dynamical

functions that are the solutions of the right members of Eqs. (23), (24), It follows that we have

$$\int \psi \mathcal{L} f^{(r-1)} d\mathbf{v} = 0, \quad (\text{B-27})$$

which is the set of general conservation equations (ψ being the vector of the privileged functions), with the distribution function replaced by $f^{(r)}$ of the r -th order. We may summarize by saying that the only nontrivial solutions admitted by the integral equations (23), and (24), are the dynamic functions $\phi(\mathbf{v}) = 1, \mathbf{v}, v^2$ on considerations of conservation of momentum and energy in binary collisions, leading us to the conservation equations (27) for $r = 1, 2, 3, \dots$.

2. THE ZEROth ORDER APPROXIMATION

We find for ϵ^0 :

$$I_{(0)} = 0;$$

$$p = m \int \mathbf{v} \mathbf{v} f^{(0)} d\mathbf{v} = nkT \delta_{ij} = \text{scalar (hydrostatic) pressure } p \quad (\text{B-28})$$

$$q = \frac{m}{2} \int v^2 \mathbf{v} f^{(0)} d\mathbf{v} = 0. \quad (\text{B-29})$$

Similarly, for the conservation equations*

*The general conservation equations are the equation of continuity for $\psi = m$; the equation of motion for $\psi = m \mathbf{v}$; and the equation of energy for $\psi = m/2 v^2$ of a fluid, i.e.,

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \nabla_r \cdot \mathbf{\tilde{v}}$$

$$\frac{1}{\rho} \frac{d\mathbf{\tilde{v}}}{dt} = \sum_i \rho_i \mathbf{F}_i - \nabla_r \cdot \mathbf{p}$$

$$\rho \frac{dv^2}{dt} = - \nabla_r \cdot \mathbf{q} - p : \nabla_r \mathbf{\tilde{v}} + \sum_i \rho_i \mathbf{F}_i \cdot \mathbf{\tilde{v}}_i$$

$$\psi = 1: \quad \frac{dn}{dt} = -n \nabla \bar{v} \quad (\text{B-30})$$

$$\psi = m\bar{v}: \quad \frac{d\bar{v}}{dt} = \bar{F} - \frac{1}{\rho} \nabla_r p \quad (\text{B-31})$$

$$\psi = \frac{m}{2} \bar{v}^2 \quad \frac{dT}{dt} = -\frac{2}{3} T \nabla_r \bar{v} \quad . \quad (\text{B-32})$$

Equations (30)-(31) are Euler's hydrodynamic equations which are reversible in time, with adiabatic temperature changes, and with an equation of state which is the ideal gas law. The reversibility results simply from $I_{(0)} = 0$.

3. THE FIRST ORDER CORRECTION

If we substitute $f^{(1)} = f^{(0)} \phi$, as before, in Eq. (23); we get

$$n^2 I[\phi^{(1)}] = (f^{(0)}) \quad .$$

Because $f^{(0)}$ is given as a function of $\bar{v}$, not $\underline{v}$, we write the zeroth order streaming term

$$(\mathcal{K}f)^{(0)} = \frac{d_0 f^{(0)}}{dt} + \bar{v} \cdot \nabla_r f^{(0)} + \left(\bar{F} - \frac{d_0 \bar{v}}{dt} \right) \cdot \nabla_{\bar{v}} f^{(0)} - [\nabla_{\bar{v}} f^{(0)}] \bar{v} \cdot \nabla_r \bar{v}, \quad (\text{B-33})$$

where $\bar{v}$ is regarded now as independent of r ; and where

$$\frac{d_0}{dt} = \frac{\partial}{\partial t} + \bar{v} \cdot \nabla_r \quad . \quad (\text{B-34})$$

By writing Eq. (33) in terms of $\ln f^{(0)}$ instead of $f^{(0)}$, and substituting [from Eq. (8)], i.e.,

$$\ln f^{(0)} = \ln n - \frac{3}{2} \ln T - mV^2/2kT + \text{const}, \quad (\text{B-35})$$

we obtain

$$\frac{d_0 \ln f^{(0)}}{dt} = \frac{1}{n} \frac{d_0 n}{dt} + \left(\frac{mV^2}{2kT} - \frac{3}{2} \right) \frac{1}{T} \frac{d_0 T}{dt}. \quad (\text{B-36})$$

By substituting into Eq. (36) for $d_0 n/dt$ and $d_0 T/dt$ the corresponding right hand terms from the zeroth order conservation equations, we obtain

$$\frac{d_0 \ln f^{(0)}}{dt} = - \frac{mV^2}{2kT} \nabla \cdot \tilde{\mathbf{v}}. \quad (\text{B-37})$$

Regarding the spatial and velocity derivatives, we proceed in a similar fashion and obtain

$$\nabla_{\mathbf{r}} \ln f^{(0)} = \nabla_{\mathbf{r}} \ln n + \left(\frac{mV^2}{2kT} - \frac{3}{2} \right) \nabla \ln T; \quad (\text{B-38})$$

$$\nabla_{\mathbf{v}} \ln f^{(0)} = - \frac{\mathbf{m}}{kT} \mathbf{v}. \quad (\text{B-39})$$

With the substitutions defined by Eqs. (37)-(39), the first order Boltzmann equation becomes:

$$(\mathcal{A}f)^{(0)} = f^{(0)} \left\{ \left(\frac{m\tilde{v}^2}{2kT} - \frac{5}{2} \right) \tilde{v} \cdot \tilde{v} \ln T + \frac{m}{kT} \left(\tilde{v}\tilde{v} - \frac{1}{3} v^2 \underline{I} \right) : \tilde{v}\tilde{v} \right\} = -n^2 \underline{I} [\phi^{(1)}] \quad .$$

(B-40)

Equation (40) is one of our inhomogeneous Fredholm equations of the second kind for which the constraint $\int \psi f^{(0)} d\tilde{v} = 0$ provides uniqueness of the solution, as already discussed. This orthogonality condition is guaranteed exactly by the three zeroth order conservation equations. A unique solution exists therefore, and it appears from Eq. (40) that it is a linear combination of the terms $\tilde{v} \ln T$, $\tilde{v}\tilde{v}$, and a scalar function of $I(\phi^{(1)}) = 0$. We can thus postulate a solution function, substitute it in the original Eq. (40), and solve for the parameter functions. Let

$$\phi^{(1)} = -\frac{1}{n} \underline{A} \cdot \tilde{v} \ln T - \frac{1}{n} \underline{B} : \tilde{v}\tilde{v} + \underline{\alpha}^{(1)} \cdot \tilde{v} \quad , \quad (B-41)$$

where $\underline{A}$ and $\underline{B}$ are a vector and a tensor function, respectively, of $\tilde{v}$, and $\underline{\alpha}^{(1)}$ is a constant vector. The components $\tilde{v} \ln T$ and $\tilde{v}\tilde{v}$ are linearly independent. By substituting Eq. (41) in (40), using the form (21) for the collision integral, we can identify the functions $\underline{A}$ and $\underline{B}$ by the following integral equations.

$$nI(\underline{A}) = f^{(0)} \left(\frac{m\tilde{v}^2}{2kT} - \frac{5}{2} \right) \tilde{v} \quad (B-42)$$

$$nI(\underline{B}) = \frac{m}{kT} f^{(0)} \left(\tilde{v}\tilde{v} - \frac{1}{3} v^2 \underline{I} \right) \quad . \quad (B-43)$$

The pertinent orthogonality conditions may be written at once:

$$\int \psi f^{(0)} \left(\frac{mV^2}{2kT} - \frac{5}{2} \right) d\mathbf{v} = 0 \quad ; \quad (\text{B-44})$$

$$\int \psi f^{(0)} \left(\mathbf{v}\mathbf{v} - \frac{1}{3} v^2 \mathbf{I} \right) d\mathbf{v} = 0 \quad . \quad (\text{B-45})$$

Straightforward integration by parts shows that the integrals in Eqs. (44) and (45) indeed vanish identically, orthogonality thus being assured.

The vector constant α in Eq. (41) is fixed by the requirement (26) that

$$\int f^{(1)} d\mathbf{v} = \int f^{(1)} \mathbf{v} d\mathbf{v} = \int f^{(1)} \mathbf{v}^2 d\mathbf{v} = 0 \quad . \quad (\text{B-46})$$

Because α is a vector, the equalities (46) imply $\alpha \cdot \mathbf{v} \equiv c_1 + c_2 \mathbf{v} + c_3 v^2 = 0$; consequently, $c_1 = c_2 = c_3 = \alpha = 0$. By the same argument used in deriving Eq. (41) as a solution of Eq. (4), we can postulate the forms that $\mathbf{A}$ and $\mathbf{B}$ will have, to be solutions of the linear integral equations (42) and (43), i.e.,

$$\mathbf{A} = A(\mathbf{v}) \mathbf{v} \quad ; \quad (\text{B-47})$$

$$\mathbf{B} = B(\mathbf{v}) \left(\mathbf{v}\mathbf{v} - \frac{1}{3} v^2 \mathbf{I} \right) \quad , \quad (\text{B-48})$$

where A and B are scalar functions of n , T , and $\mathbf{v}$.

In pursuing our procedure of successive approximations, we will determine now the first order corrections to the momentum flow and heat flow tensors:

$$\mathbf{p}^{(1)} = \int m \mathbf{v} \mathbf{v} f^{(1)} d\mathbf{v} \quad ; \quad (\text{B-49})$$

$$q^{(1)} = \int \frac{m}{2} v^2 v_f^{(1)} d\mathbf{v} ; \quad (B-50)$$

where $f^{(1)} = -1/n f^{(0)} (\mathbf{A} \cdot \nabla \ln T + \mathbf{B} : \nabla \mathbf{\bar{v}})$ results from Eq. (41) with $\alpha = 0$. Since integrals over odd powers of $\mathbf{v}$ are zero, when integrated over the velocity space, the part in Eq. (49) contributed by $\mathbf{A}$ vanishes and we are left with

$$p^{(1)} = -\frac{m}{n} \int f^{(0)} B(V) \mathbf{v} \mathbf{v} (\mathbf{v} \mathbf{v} - \frac{1}{3} v^2 \mathbf{I}) : \nabla \mathbf{\bar{v}} d\mathbf{v} . \quad (B-51)$$

Since $\nabla \mathbf{\bar{v}}$ is a second rank tensor independent of $\mathbf{v}$, we may use the integral identity*

$$\begin{aligned} \int F(\mathbf{C}) \mathbf{C} \mathbf{C} [(\mathbf{C} \mathbf{C} - \frac{1}{3} C^2 \mathbf{I}) : \mathbf{w}] d\mathbf{C} &= \int F(\mathbf{C}) (\mathbf{C} \mathbf{C} - \frac{1}{3} C^2 \mathbf{I}) [(\mathbf{C} \mathbf{C} - \frac{1}{3} C^2 \mathbf{I}) : \mathbf{w}] d\mathbf{C} \\ &= \frac{1}{5} \int F(\mathbf{C}) (\mathbf{C} \mathbf{C} - \frac{1}{3} C^2 \mathbf{I}) : (\mathbf{C} \mathbf{C} - \frac{1}{3} C^2 \mathbf{I}) d\mathbf{C} \mathbf{\bar{S}} , \end{aligned}$$

where $\mathbf{\bar{S}}$ is the symmetric traceless (i.e., nondivergent symmetrical) part of $\mathbf{w}$, ($\mathbf{w} \equiv \partial/\partial \mathbf{r} \mathbf{c}_0$ in Chapman and Cowling's notation), i.e.,

$$\mathbf{\bar{S}} = \frac{1}{2} (\mathbf{w} + \mathbf{w}^T) - \frac{1}{3} (\mathbf{w} : \mathbf{I}) \mathbf{I} .$$

Thus,

*Chapman and Cowling Eq. (1.421-i,ii,iv),⁹ where $\mathbf{C} \mathbf{C} - 1/3 C^2 \mathbf{I}$ is the nondivergent tensor denoted $(\mathbf{C} \mathbf{C})^0$.

$$p^{(1)} = - \frac{m}{5n} \int f^{(0)} B(\underline{v}) (\underline{v}\underline{v} - \frac{1}{3} v^2 \underline{I}) : (\underline{v}\underline{v} - \frac{1}{3} v^2 \underline{I}) d\underline{v} \underline{S} , \quad (B-52)$$

where

$$S_{\alpha\beta} = \left(\frac{\partial \bar{v}_\beta}{\partial x_\alpha} + \frac{\partial \bar{v}_\alpha}{\partial x_\beta} \right) - \frac{1}{3} \nabla \cdot \bar{\underline{v}} \delta_{\alpha\beta} , \quad (B-53)$$

which is the rate of shear tensor.* The first order correction, $p^{(1)}$, to the pressure tensor is thus proportional to the rate of shear tensor. We may now substitute the original integral equation in Eq. (52) to obtain the simple expression

$$p^{(1)} = - \frac{1}{5} kT \int \underline{B} : \underline{I}(\underline{B}) d\underline{v} , \quad (B-54)$$

which is a bracket integral $[\underline{F}, \underline{G}] = \int \underline{G} : \underline{I}(\underline{F}) d\underline{v}$ and may be written in bracket form, i.e.,

$$p^{(1)} = - \frac{1}{5} kT [\underline{B}, \underline{B}] \underline{S}^{**} . \quad (B-55)$$

Since Newton's law defines the pressure tensor

*cf. tensor 1.33,2 in Chapman and Cowling p. 18, also called "rate of strain tensor".

**In Ferziger and Kaper,¹⁰ $\underline{S}$ has erroneously been omitted, cf. Eq. 5.4-35, p. 130.

$$\underline{p} = p \underline{I} - 2\eta \underline{S} , \quad (\text{B-56})$$

the coefficient of viscosity in the first order Enskog theory simply follows from Eqs. (55) and (56), i.e.,

$$\eta = \frac{1}{10} kT [\underline{B}, \underline{B}] . \quad (\text{B-57})$$

We use a similar procedure for calculating the heat flux vector whose definition is given by

$$\underline{q}^{(1)} = \int \frac{m}{2} v^2 \underline{v} f^{(1)} d\underline{v} . \quad (\text{B-58})$$

We use, again, the first order correction to the distribution function given by Eq. (41), with the constant $\alpha^{(1)} = 0$, which we just determined. Hence

$$\underline{q}^{(1)} = - \frac{m}{2n} \int f^{(0)} v^2 \underline{v} [\underline{A} \cdot \underline{v} \ln T + \underline{B} : \underline{v} \underline{v}] d\underline{v} . \quad (\text{B-59})$$

Because $\underline{B}$ is a function of an even power of $\underline{v}$, which is multiplied by $v^2 \underline{v}$, $\underline{B}$ does not contribute to the integral (59). We substitute $\underline{A}$ from Eq. (47) into Eq. (59) and make use of the integral identity

$$\int F(\underline{C}) \underline{C} \underline{C} d\underline{C} = \frac{1}{3} \int F(\underline{C}) C^2 d\underline{C} \underline{I} ,$$

by which Eq. (58) becomes

$$g^{(1)} = -\frac{m}{6n} \int f^{(0)} V^4 A(V) d\mathbf{v} \nabla \ln T . \quad (B-60)$$

Now, we add to Eq. (60) a zero-valued function of the expression $-5/2 V^2$, only to reduce Eq. (59) into one of the standard forms of the bracket integral. Such a zero function is the integral

$$f^{(0)} A(V) V^2 d\mathbf{v} = 0 ,$$

for reasons shown in the footnote.* Hence, we replace V^4 in Eq. (60) by the expression $(2 kT/m) V^2 (m V^2/2 kT - 5/2)$ to get

$$g^{(1)} = -\frac{kT}{3n} \int f^{(0)} V^2 A(V) \left(\frac{mV^2}{2kT} - \frac{5}{2} \right) d\mathbf{v} \nabla \ln T . \quad (B-61)$$

The scalar expression $V^2 A(V) (m V^2/2 kT - 5/2)$ in Eq. (60) may be

*Write the first order hydrodynamic equation $\int \psi f^{(1)} d\mathbf{v} = 0$ specifically for the case $\psi = m\mathbf{v}$ and with the general solution (distribution function $f^{(1)} = f^{(0)} \phi^{(1)}$) given in Eq. (41):

$$\frac{1}{n} \int f^{(0)} A(V) \mathbf{v} \cdot \nabla \ln T d\mathbf{v} + \frac{1}{n} \int f^{(0)} B(V) \mathbf{v} : \nabla \mathbf{v} d\mathbf{v} = \int f^{(0)} \mathbf{v} m c_2 \cdot \mathbf{v} d\mathbf{v} ,$$

where $A(V)$ and $B(V)$ are given in Eqs. (47) and (48). The second term on the left is zero, because

$$\int f^{(0)} B(V) \mathbf{v} \left(\mathbf{v} \cdot \mathbf{v} - \frac{1}{3} V^2 \mathbf{I} \right) : \nabla \mathbf{v} d\mathbf{v}$$

is a velocity integral over the velocity to an odd power. We are thus left with the following expression for the constant

$$c_2 = \frac{\frac{1}{3n} \int f^{(0)} A(V) V^2 d\mathbf{v}}{m \int f^{(0)} V^2 d\mathbf{v}} \nabla \ln T .$$

It was pointed out in the comments on Eq. (46) that $c_2 = 0$; hence, because $\int f^{(0)} V^2 d\mathbf{v} = n kT/m \neq 0$, we have $\int f^{(0)} A(V) V^2 d\mathbf{v} = 0$.

replaced by $\underline{\underline{V}} \cdot \underline{\underline{V}} A(\underline{\underline{V}})$ ($m V^2/2 kT - 5/2$) which, with Eqs. (47) and (42) becomes $\underline{\underline{A}} \cdot \underline{\underline{n}} \underline{\underline{I}}(\underline{\underline{A}})$. It follows that Eq. (61) may be written in the form

$$q^{(1)} = - \frac{kT}{3} \underline{\underline{V}} \cdot \underline{\underline{n}} T [\underline{\underline{A}}, \underline{\underline{A}}] \quad , \quad (B-62)$$

(cf. Eq. (7.4.1, Chapman and Cowling)).⁹ Thus, by Fourier's definition $q = - \lambda \underline{\underline{V}}_r T$, the thermal conductivity is given by

$$\lambda = - \frac{k}{3} [\underline{\underline{A}}, \underline{\underline{A}}] \quad . \quad (B-63)$$

We have thus identified $q = - \lambda \nabla T$, which is the heat flow vector in the first order Enskog-Chapman theory.

The first order fluid dynamic equations are the Navier-Stokes equations of hydrodynamics, which are obtained by substituting into the general conservation equations the first order expressions for the pressure tensor and the heat flux vector:

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \underline{\underline{V}} \cdot \underline{\underline{V}} \quad , \quad (B-64)$$

$$\rho \frac{d\underline{\underline{V}}}{dt} = \rho \underline{\underline{F}} - \nabla p + 2\eta \underline{\underline{V}} \cdot \underline{\underline{S}} \quad , \quad (B-65)$$

$$\rho \frac{dT}{dt} = - \frac{2m}{3k} (- \underline{\underline{V}} \cdot \lambda \underline{\underline{V}} T + p \underline{\underline{V}} \cdot \underline{\underline{V}} - 2\eta \underline{\underline{S}} : \underline{\underline{V}} \underline{\underline{V}}) \quad , \quad (B-66)$$

where the rate of shear tensor is given by Eq. (53), and its gradient $\underline{\underline{V}} \underline{\underline{S}}$ is, in vector notation,

$$2\eta \nabla \cdot \underline{\underline{S}} \equiv \eta \Delta \bar{v} + (\zeta + \frac{1}{3} \eta) \nabla^2 \bar{v} \quad (\text{B-67})$$

or, in component notation

$$(2\eta \nabla \cdot \underline{\underline{S}})_i = \frac{\partial}{\partial x_k} \left[\eta \left(\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial \bar{v}_\ell}{\partial x_\ell} \right) \right] + \frac{\partial}{\partial x_i} \left(\zeta \frac{\partial \bar{v}_\ell}{\partial x_\ell} \right) , \quad (\text{B-68})$$

where the constants η and ζ are called the coefficients of viscosity. For the incompressible fluid (i.e., for $\nabla \bar{v} = 0$), the last term vanishes. Recalling Boltzmann's H theorem, the Navier-Stokes equations are irreversible, because $I_{(1)} \neq 0$.

To summarize, the problem of solving the Boltzmann equation has been reduced to solving the bracket integrals $[\underline{A}, \underline{A}]$, and $[\underline{B}, \underline{B}]$, which will now be discussed.

4. CALCULATION OF THE QUANTITIES A AND B

We demonstrated that the properties of the gas in the first order approximation are defined by the vector $\underline{A}$ and tensor $\underline{B}$. These quantities are determined by expressing them in the terms of Sonine polynomials $S_m^{(r)}(x)$ defined by the expression

$$(1-s)^{-m-1} e^{-xs/2} = \sum_{n=0}^{\infty} S^n S_m^{(n)}(x) . \quad (\text{B-69})$$

N. J. Sonine (1880) defined these originally. They are related to the generalized Laguerre polynomials used by Schrödinger, both being orthonormal spherical functions. The Sonine polynomials are used in the kinetic theory of gases because of their rapid convergence for interaction potential functions of the type $1/r^S$. In the special case of the $U(r) \propto r^{-5}$ potential function used by Maxwell, the Sonine polynomials are eigenfunctions of the collision operator $I[\phi^{(1)}]$ and

hence the exact solution for $\underline{A}$ and $\underline{B}$. The expansion (69) may be written explicitly for $S_m^{(r)}$:

$$S_m^{(n)}(x) = \sum_{p=0}^n \frac{\Gamma(m+n+1)}{(n-p)!p!\Gamma(m+p)} (-x)^p. \quad (B-70)$$

Following are relationships which allow evaluating the bracket integrals.

$$[\underline{A}, \underline{A}] = \frac{\alpha_1^2}{a_{11}} + \frac{[A_2^{(2)}]^2}{A^{(1)}A^{(2)}} + \frac{[A_3^{(3)}]^2}{A^{(2)}A^{(3)}} + \dots;$$

$$[\underline{B}, \underline{B}] = \frac{\beta_1^2}{b_{11}} + \frac{[B_2^{(2)}]^2}{B^{(1)}B^{(2)}} + \frac{[B_3^{(3)}]^2}{B^{(2)}B^{(3)}}$$

(cf. Eq. 7.51.12, Chapman and Cowling)⁹

$$\alpha_r = -\frac{1}{n} \int f^{(0)} \left(\mathcal{V}^2 - \frac{5}{2} \right) \mathcal{V} \cdot s_{3/2}^{(r)} \mathcal{V} d\mathcal{V};$$

$$\beta_r = -\frac{1}{n} \int f^{(0)} (\mathcal{V}\mathcal{V})^0 : (\mathcal{V}\mathcal{V})^0 s_{5/2}^{(r-1)} d\mathcal{V};$$

$$\alpha_1 = \frac{15}{4}; \quad \alpha_2 = \alpha_3 = \alpha_4 = \dots = 0; \quad \beta_1 = -5; \quad \beta_2 = \beta_3 = \beta_4 = \dots = 0$$

(cf. Eqs. (7.51.5 and 6); 7.51.12, Chapman and Cowling)⁹

$$a_{rs} = [s_{3/2}^{(r)} (\mathcal{V}^2), s_{3/2}^{(s)} (\mathcal{V}^2)];$$

$$b_{rs} = [s_{5/2}^{(r-1)} (\mathcal{V}^2)(\mathcal{V}\mathcal{V})^0, s_{5/2}^{(s-1)} (\mathcal{V}^2)(\mathcal{V}\mathcal{V})^0];$$

(cf. Eq. 7.51.2 and 3, and Sect. 9.7, Chapman and Cowling)

$$\begin{aligned}
 A_2^{(2)} &= -\alpha_1 a_{21} & B_2^{(2)} &= -\beta_1 b_{21} \\
 A_3^{(2)} &= -\alpha_1 (a_{21} a_{32} - a_{31} a_{22}) & B_3^{(2)} &= -\beta_1 (b_{21} b_{32} - b_{31} b_{22}) \\
 A^{(1)} &= a_{11} \equiv \det^{(1)}(a_{ij}) & B^{(1)} &= b_{11} \equiv \det^{(1)}(b_{ij}) \\
 A^{(2)} &= a_{11} a_{22} - a_{12}^2 \equiv \det^{(2)}(a_{ij}) & B^{(2)} &= \det^{(2)}(b_{ij}) \\
 A^{(3)} &= \det^{(3)}(a_{ij}) & B^{(3)} &= \det^{(3)}(b_{ij})
 \end{aligned}$$

(cf. Sects. 7.51 and 9.7, Chapman and Cowling)

$$\begin{aligned}
 a_{11} &\approx 50\Omega^{(1,1)} - 40\Omega^{(1,2)} + 8\Omega^{(1,3)}; & b_{00} &\approx 8\Omega^{(2,2)} \\
 a_{12} &\approx \frac{175}{2}\Omega^{(1,1)} - 105\Omega^{(1,2)} + 38\Omega^{(1,3)} - 4\Omega^{(1,4)}; & b_{01} &\approx 28\Omega^{(2,2)} - 8\Omega^{(2,3)} \\
 a_{22} &\approx \frac{1225}{8}\Omega^{(1,1)} - 245\Omega^{(1,2)} + 133\Omega^{(1,3)} - \\
 &\quad - 28\Omega^{(1,4)} + 2\Omega^{(1,5)}; & b_{11} &\approx 98\Omega^{(2,2)} - 56\Omega^{(2,3)} + 8\Omega^{(2,4)}
 \end{aligned}$$

(cf. Ferziger and Kaper: derived from Tables 7.3-7.5 for $m_e/m_i \ll 1$; $\mu_i = m_i/(m_i + m_j) \approx 0$; $\mu_j \approx m_j/(m_i + m_j) \approx 1$; i.e., for two-component gas mixture, Sect. 7.2 Ferziger and Kaper)¹⁰

$$\Omega(l, \rho) = \frac{\pi^{1/2} A_l(s) (2kT)^{\frac{s-5}{2(s-1)}} K_{12}^{2/(s-1)} \Gamma(\rho + 2 - \frac{2}{s-1})}{2 m^{1/2}}$$

[cf. Eq. 10.3.10, Chapman and Cowling,⁹ Eq. IV-102 (corrected)* Jancel and Kahan,¹¹ where $A_\ell(s) = \int (1 - \cos^\ell \chi) u_0 du_0$ --cf. Eq. 10.3.9, Chapman Cowling]⁹ We have defined the interparticle force $F_{12} = K_{12}/r^s$. $A_\ell(s)$ is a number, and χ is the scattering angle, $\chi = 2 \arcsin[(1 + u_0^2)^{-1/2}]$, where u_0 is the dimensionless number $u_0 \equiv b(M_r g^2/K_{12})^{1/(1-s)}$, which is defined in binary collisions, where b is the impact parameter, g the relative speed, and M_r the reduced mass (cf. Sect. 9.1, Ferziger and Kaper,¹⁰ Chap. 7, Jancel and Kahan).¹¹

For the inverse square (i.e., Coulomb interaction) law, $s = 2$, we get $a_{12}/a_{11} = 1.327$, $a_{22}/a_{11} = 2.082$, and $a_{11} = 26.0 \pi^{1/2} A_1(2) (2 \text{ kT})^{-3/2} e^4/2m^{1/2}$, and therefore,

$$\lambda = \frac{2k^2 T}{3m} [\underline{A}, \underline{A}] = 7.46 \frac{k(kT)^{5/2}}{m^{1/2} e^4 A_1(2)} .$$

For the collision integral, we get

$$A_1(2) = \int_1^{b_0} (1 - \cos \chi) u_0 du_0 = \ln \left[1 + \left(\frac{b_0 g^2 M_r}{e^2} \right)^2 \right] , \quad (\text{B-71})$$

where $M_r g^2 = 3 \text{ kT}$, and $b_0 = (kT/4\pi n e^2)^{1/2}$ is the Debye length. Consequently, $A_1(2) = \ln \{ 1 + [3/2 (kT)^{3/2}/(\pi n)^{1/2} e^3] \}$, where we may neglect 1 by comparison to the expression in parentheses, thus reducing the expression to

$$A_1(2) \approx 2 \ln \left[\frac{3}{2} \frac{(kT)^{3/2}}{(\pi n)^{1/2} e^3} \right] = 2 \ln \Lambda .$$

* $(2 \text{ m})^{1/2}$ should be $2 \text{ m}^{1/2}$ as given above.

Λ is the Coulomb cutoff parameter in Spitzer's definition.

To summarize this, we obtain

$$\lambda = \frac{7.46}{2} k(kT)^{5/2} / (m^{1/2} e^4 \ln \Lambda) ,$$

which is an expression in agreement with Spitzer's formula, except for the numerical factor, which is $20(2/\pi)^{3/2} \approx 10.16$ in Spitzer's formula, and 3.73 in our's.

To obtain an estimate of the coefficient of viscosity, i.e., $\ell \approx s = 2$, we compute the bracket expression $[\underline{B}, \underline{B}]$ for a typical electron density n . Evidently, the integral $A_2(2) = \int_0^\infty (1 - \cos^2 \chi) u_0 \, du_0$ is straightforward to evaluate if we replace, again, the lower limit by the hard-core cutoff parameter $b_{00} = e^2/3 kT$ given in Chap. 2, and the upper limit by the Debye distance $b_0 = (kT/4\pi n e^2)^{1/2}$. We define the dimensionless number $u_0 = \cotan(\chi/2) = b (M_{rg}^2/e^2)$, where b is the impact parameter, with $b_{00} \leq b \leq 0$, and $M_{rg}^2 = 3 kT$. For electrons with $n = 2.6 \cdot 10^{19} \text{ cm}^{-3}$, we obtain $A_2(2) = 5.4$. By evaluating only the term B_1^2/b_{11} of the lowest order in the expansion for the bracket expression, we obtain $[\underline{B}, \underline{B}] \approx 1.6 \cdot 10^6 \text{ s/cm}^3$. We conclude that the viscosity term in Newton's law is negligible. One may see this when Eq. (56) is written in the dimensionless form

$$\frac{\underline{p}}{p} = I + \frac{[\underline{B}, \underline{B}]}{5n^2 e} \left(\frac{\partial J_\alpha}{\partial x_\beta} + \frac{\partial J_\beta}{\partial x_\alpha} \right) ,$$

where the second term on the r.h.s., with our numbers, becomes $7.5 \cdot 10^{-25} (\partial J_\alpha / \partial x_\beta + \partial J_\beta / \partial x_\alpha)$, thus justifying our conclusion that we may equate the momentum flow tensor with the pressure tensor as follows

$$p = \int p = \int n kT .$$

5. SCATTERING CROSS SECTION

It is useful to relate the integral (71)--i.e., the quantity $A_q(s)$ introduced by Maxwell--to the trajectory equations, because one may thus obtain the collision frequency in the first order theory. Suppose the polar angle θ be measured from the center of the stationary scatterer, as shown in Fig. 6a. Let M_r , g , and b denote, again, the reduced mass, relative velocity, and impact parameter, respectively. Conservation of angular momentum is then expressed by the relationship

$$\mu r^2 \dot{\theta} = \mu g b .$$

If $U(r)$ denotes the interaction potential, then the conservation of energy in the binary collision process is expressed by

$$\frac{1}{2} M_r (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r) = \frac{1}{2} \mu g^2 .$$

Autonomy from time is easily obtained, to form an equation for the path, i.e.,

$$\frac{1}{2} M_r \left(\frac{gb}{r^2} \right)^2 \left[\left(\frac{dr}{d\theta} \right)^2 + r^2 \right] = \frac{1}{2} M_r g^2 - U(r) .$$

Hence, we have the path integral

$$\theta = \int_r^{\infty} r^{-1} \left[\left(\frac{r}{b} \right)^2 - \frac{2}{M_r} U(r) \left(\frac{r}{gb} \right)^2 - 1 \right]^{-1/2} dr.$$

The angle $\theta = \theta_m$ for the point $r = r_m$ of closest approach is evidently obtained from the condition $dr/d\theta = 0$, i.e.,

$$\frac{dr}{d\theta} = 0 = r_m \left[\left(\frac{r_m}{b} \right)^2 - \frac{2U(r_m)}{M_r} \left(\frac{r_m}{gb} \right)^2 - 1 \right]^{1/2},$$

or,

$$0 = 1 - \frac{2U(r_m)}{M_r g^2} - \left(\frac{b}{r_m} \right)^2. \quad (B-72)$$

It follows that the scattering angle $\chi = \pi - 2\theta_m$, shown in Fig. 6a, is given by

$$\chi = \pi - 2 \int_{r_m}^{\infty} r^{-1} \left[\left(\frac{r}{b} \right)^2 - \frac{2U(r)}{M_r} \left(\frac{r}{gb} \right)^2 - 1 \right]^{-1/2} dr.$$

For an attractive Coulomb potential, one has

$$U(r) = - \frac{e_1 e_2}{r},$$

for which Eq. (72) becomes

Fig. 6

(a) View of an elastic collision in the laboratory, M at rest initially

(b) Asymptotic views of an elastic collision in the CM system:

$$m(v_o - v_{cm}) = Mv_{cm}$$

Loss of x-directed momentum component of
 $m = m(v_o - v_{cm})(1 - \cos\chi)$

$$1 - u_m^2 - 2 \left(\frac{u_m}{u_o} \right) = 0 ,$$

where

$$u_m \equiv \frac{b}{r_m} , \quad u_o \equiv -b \left(\frac{M_R b^2}{e_1 e_2} \right) ,$$

and

$$\chi = \pi - 2 \int_0^{u_m} \left[1 - u^2 - 2 \left(\frac{u}{u_o} \right) \right]^{-1/2} du ;$$

thus yielding

$$\cos \chi = \frac{u_o^2 - 1}{u_o^2 + 1} . \quad (B-73)$$

With the solid angle element being defined as $d\Omega = 2\pi \sin \chi d\chi$, we obtain an implicit relationship for the differential cross section, in the form of

$$dn = 2\pi N b db \equiv 2\pi N \sigma(\chi) \sin \chi d\chi . \quad (B-74)$$

From Eq. (74) follows directly, with Eq. (73), the Rutherford formula, i.e.,

$$\sigma(\chi) = \frac{b}{\sin\chi} \frac{db}{d\chi} = \frac{\frac{e^2 e^2}{l^2}}{4M_r^2 g^4 \sin^4 \frac{\chi}{2}} .$$

One may infer from Fig. 6b that the momentum transfer cross section is given by

$$Q^{(1)} = \frac{dn}{N} = 2\pi \int_0^\pi \sigma(\chi)(1-\cos\chi)\sin\chi d\chi = 2\pi \int_0^\infty (1-\cos\chi)b db , \quad (B-75)$$

because the quantity $M_r g \cdot (1-\cos\chi)$ represents the momentum loss in the initial direction of motion of a particle with the reduced mass M_r and velocity $v = g$.^{*} The cross sections, $\phi_{12}^{(\ell)}$, appearing in Chapman and Cowling's solution of the Boltzmann equation, are related as follows to our quantities $Q^{(1)}$ and $Q^{(2)}$:

$$\phi_{12}^{(\ell)} = \frac{g}{2\pi} Q^{(\ell)} ,$$

where the subscripts refer to the particles of type 1 and 2. Hence, we have

^{*}The v -directed momentum loss with respect to the center of mass velocity $\underline{V}_{cm}$ is obviously given by $m(\underline{v}-\underline{V}_{cm})(1-\cos\chi)$. It is permissible to assume the particle j with the mass M to be stationary--i.e., $\underline{V} = 0$. If it is agreed that $\underline{v}$ be the velocity, $\underline{g}_{ij}$, of the moving particle w.r.t. the stationary one, then the following relationship obtains, for the momenta in the binary collision, $M_r g = M_r \underline{V}_{cm} = mg = m\underline{v} = m\underline{v}' + M\underline{V}'$, where post-collisional velocities are primed, and the precollisional velocities unprimed. It follows that $m(\underline{v}-\underline{V}_{cm})(1-\cos\chi) = M_r g(1-\cos\chi)$.

$$\begin{aligned}\phi_{\{2\}}^{(2)} &= \int_0^\infty (1 - \cos^2 \chi) g b db = \left(\frac{e_1 e_2}{M_r} \right)^2 g^3 \int_0^\infty (1 - \cos^2 \chi) u_0 du_0 \\ &\equiv \left(\frac{e_1 e_2}{M_r} \right)^2 g^3 A_2(2) ,\end{aligned}$$

with the integral $A_2(s) \equiv \int_0^\infty (1 - \cos^2 \chi) u_0 du_0$ corresponding to Eq. (71); and s denoting the inverse power in the potential law.

The collision frequencies are obtained from the definition

$$\nu = g n \sigma ,$$

if one substitutes in it the proper expressions for the cross section, $\sigma = Q^{(1)}$, and the flux of test particles, $g n$. In considering $s = 2$ (i.e., the Coulomb interaction) and using the Debye length b_0 to define the upper limit, $u_{01} = b_0 g^2 \cdot (M_r / e_1 e_2)$ for the collision integral, one obtains the quantity*

$$A_1(2) \approx 2 \ln \left| \frac{3}{2} \frac{(kT)^{3/2}}{(\pi n)^{1/2} e^3} \right| .$$

Hence,

$$\nu_{ei} = \frac{2\pi n e^4 \ln \Lambda}{m_e^{1/2} (3kT_e)^{3/2}} , \quad (B-76)$$

*The approximation is defined in the text after Eq. (71). Evidently, because the approximation is poor for high densities and low temperatures, such values correspond to blanks in Spitzer's²⁶ Table 5.1, p. 128. Perhaps our equation given in the text after our Eq. (71) might be used.

where we have substituted $3 kT_e = M_r g^2$, and $m_e \approx m_e m_i / (m_e + m_i) = M_r$ for the electron-ion encounter. Spitzer's formula (Eq. 5-51, p. 135 in his book)²⁶ agrees with our Eq. (76), except for the numerical factor, which is 2π in our case and $8/3(2\pi)^{1/2}$ in Spitzer's; so the two results compare like 1:1.06, respectively. Rose and Clarke's collision frequency [Eq. (8.20), p. 165 as corrected by the factor 2 mentioned in the text on p. 166 (q.v.)]²⁷ is the same as our Eq. (76).

APPENDIX C

LIIOUVILLE'S EQUATION

The fluid dynamic equations for a system with external forces exerted by the EM field represent the fundamental set of balances that obtain in the electric arc. We derive them in a form in which they can be used in the arc model. The Liouville equation expresses the principle of conservation of phase space as an ensemble of particles evolves in time:

$$\frac{\partial}{\partial t} F(q,p;t) = \left[\sum_{n=1}^N \left\{ \frac{\partial H}{\partial q_n} \frac{\partial}{\partial p_n} - \frac{\partial H}{\partial p_n} \frac{\partial}{\partial q_n} \right\} \right] F, \quad (C-1)$$

where F is a dynamical function, q_n, p_n are the canonical coordinates of phase space, and H is the Hamiltonian. The summation is taken over the N degrees of freedom of the system. The r.h.s. in Eq. (1) is the Liouvillian of F , LF ; where L is the operator sum of three components, $L = L^0 + L' + L^F$, corresponding to $H = H^0 + H' + H^F$ for the free particle, the particle-field interaction, and the field.

For the free particle Hamiltonian $H^0 = p^2/2m$, we substitute Hamilton's equations of motion

$$\frac{\partial H}{\partial q_n} = -\dot{p}_n; \quad \frac{\partial H}{\partial p_n} = \dot{q}_n$$

in the Poisson Bracket of Liouville's Eq. (1). To obtain the kinetic equations we simply move the Liouvillian [i.e., the r.h.s. in Eq. (1)] over to the l.h.s. and equate it with source-sink terms, i.e.,

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\hbar}{p} \nabla_p f = G - L. \quad (C-2)$$

We assume constant mass, i.e., $\delta \vec{p} = m \delta \vec{v}$ and obtain for the laboratory (i.e., Euler's reference frame)

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = \text{r.h.s.}$$

where $\vec{F}/m = \dot{\vec{v}}$. The conservation equations are obtained from Eq. (3) by multiplying it with some velocity function $\phi(\vec{v})$ and integrating the equation over velocity space $\int d\vec{v}$. It is necessary to do the integration, because a PDE with respect to time and space and not w.r.t. velocity is desired. We thus define the mean values

$$\text{average number density} \quad n(\vec{r}; t) = \int d\vec{v} f(\vec{r}, \vec{v}; t)$$

$$\text{average dynamical function} \quad \bar{B}(\vec{r}; t) = \frac{1}{n} \int d\vec{v} B f(\vec{r}, \vec{v}; t) ,$$

where f is the reduced one-particle distribution function. It is reduced because the system consists of N distribution functions for N identical particles which may, for symmetry reasons, be described by the distribution function of just one, arbitrarily selected, particle. Because we have a mixture of species, we need a kinetic equation for each species. The equations are correlated by the r.h.s., where the correlation functions are the collision integrals.

Because no tensors appear in this appendix, we have used the diacritical mark $\rightarrow$ to denote the vectors.

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