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CONSTRUCTIVE POLYNOMIAL
APPROXIMATION
IN SOBOLEV SPACES*

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ABSTRACT

The primary purpose of this paper is to present a simple, constructive proof of the so-called Bramble-Hilbert Lemma. This result is frequently used in establishing the approximation properties of finite element function spaces and several of the extensions studied here are directed to this type of application. Some examples of the uses of these results will be presented.

1. INTRODUCTION.

In this note we give a constructive piecewise polynomial approximation theory having applications to finite element Galerkin methods. The main contribution is a proof of a polynomial approximation lemma popularized by Bramble and Hilbert [1] in which we calculate the constant involved in the error estimate (the proof used in [1] is nonconstructive so that no information is obtained about the constant other than the fact that it is finite). The proof, as well as the result itself, is very closely related to the work of Sobolev [6] on imbedding theorems. Another constructive approach to piecewise polynomial approximation via multipoint Taylor formulae developed by Ciarlet and Wagschall [3] also had an early influence on the present work.

The original objectives of this work were three-fold. First, we simply wanted some idea of the size of the constant in the Bramble-Hilbert lemma. Second, we wanted to know how the constant would change as the underlying domain is perturbed. Finally, we sought a simple proof of the Bramble-Hilbert lemma that could serve as an introduction to polynomial approximation in Sobolev spaces with minimal recourse to complicated functional analysis. While achieving the first

two objectives, this note is primarily directed toward the last objective. For this reason we restrict ourselves to approximation by complete polynomials and carry through an application to Lagrange piecewise polynomial interpolation on a domain in R^2 having a curved boundary, including an explicit estimate for the constant in the error estimate. Although we have strived to give reasonable estimates of the constants involved in deriving approximation results, we have made no attempt to get the best possible constants, and indeed the reader will easily see minor improvements (at least) that can result from more precise assumptions on the domains involved.

In a related paper [4], we treat several generalizations of the results presented here. We shall not describe them in detail but shall briefly mention the topics studied. In addition to complete polynomial approximation, [4] also treats approximation by a generalized tensor product polynomial that satisfies a property of commutativity with differentiation similar to the one discussed in the next section. In the tensor product case, the commutativity plays a crucial role in estimating the effect of differentiation on the error, whereas in the complete polynomial case treated here it appears only as a useful tool. A further generalization in [4] in the type of polynomial approximation considered yields, for example, optimal estimates for approximation of harmonic functions by harmonic polynomials. In a different direction, approximation by complete polynomials of functions in fractional order Sobolev spaces is studied, and, finally, a simple observation is used to extend the type of domains for which all of the polynomial approximation results hold. In the present paper, we restrict to domains that are star-shaped with respect to every point in an open ball. However in [4], such results are seen to extend to any domain that is a finite union of domains each of which is star-shaped with respect to a ball separately. Thus domains satisfying a restricted cone property, as in [1], and even more general domains, may be

allowed. Furthermore, whereas this paper is concerned primarily with estimates in an L^2 setting, [4] treats the general L^p case. The derivations of error estimates in the cases presented here use more elementary techniques than those required in [4] for the general case.

2. NOTATION.

Let Ω be a bounded open set in $\mathbb{R}^n$ and let dx denote Lebesgue measure in $\mathbb{R}^n$. Denote by $L^p(\Omega)$ the Banach space of measurable functions f such that

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p} < \infty$$

for $1 \leq p < \infty$, with the usual modification when $p = \infty$. Let $\mathbb{N}$ denote the nonnegative integers. For any $\alpha \in \mathbb{N}^n$, let $D^\alpha u = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n} u$ denote the distributional derivative of $u \in \mathcal{D}'(\Omega)$, the space of Schwartz distributions on Ω . Recall that $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$ for $\alpha \in \mathbb{N}^n$. When u is such that $D^\alpha u$ may be identified with a square integrable function for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$, define

$$|u|_m = \|u\|_{m,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Similarly define

$$\|u\|_m = \|u\|_{m,\Omega} = \left(\sum_{j=0}^m |u|_j^2 \right)^{1/2},$$

and let $H^m(\Omega)$ denote the Hilbert space having $\|\cdot\|_m$ as norm. Let $C^\infty(\Omega)$ denote the set of infinitely differentiable functions on Ω and $C_0^\infty(\Omega)$ its subspace consisting of functions having compact support in Ω .

Given a nonnegative integer m , let $\mathcal{P}_m$ denote the space of polynomials of degree less than m in n variables.

A subset Ω of $\mathbb{R}^n$ is said to be star-shaped with respect to the point x if $y \in \Omega$ implies $x + \theta(y-x) \in \Omega$ for all $\theta \in [0,1]$.

Given a finite set S , let $\#S$ denote the number of elements in S .

Let ω_n denote the measure of the unit sphere in $\mathbb{R}^n$ (i.e., the unit $(n-1)$ -sphere).

3. SOBOLEV REPRESENTATION.

We begin by recalling a constructive polynomial projection operator used by Sobolev [6] in proving imbedding theorems. Let B be an open ball in $\mathbb{R}^n$, and let $\varphi \in C_0^\infty(B)$ be such that $\int_B \varphi(x) dx = 1$. For m any positive integer and $u \in C^\infty(B)$, define

$$Q^m u(x) = \int_B \varphi(y) \sum_{|\alpha| < m} D^\alpha u(y) \frac{(x-y)^\alpha}{\alpha!} dy. \quad (3.1)$$

Note that this is simply the Taylor polynomial approximation

$$T^m u(x) = \sum_{|\alpha| < m} D^\alpha u(y) \frac{(x-y)^\alpha}{\alpha!} \quad (3.2)$$

averaged over B with φ as a weighting function and that Q^m is a projection onto the space of polynomials of degree less than m , P_m . Since the Taylor polynomial T^m satisfies the commutativity relation $D^\alpha T^m u = T^{m-|\alpha|} D^\alpha u$ for $|\alpha| < m$ (as is easily proved by induction), the same relation follows for Q^m by simply differentiating under the integral, namely,

$$D^\alpha Q^m u = Q^{m-|\alpha|} D^\alpha u \quad (3.3)$$

for any $u \in C^\infty(B)$ and $|\alpha| < m$.

Integrating by parts in (3.1), we find that Q^m may also be written as

$$Q^m u(x) = \int_B u(y) \left(\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_y^\alpha [\varphi(y) (x-y)^{\alpha/\alpha!}] \right) dy \quad (3.4)$$

$$= \sum_{|\alpha| \leq m} x^\alpha \int_B u(y) \psi_\alpha(y) dy,$$

where $\psi_\alpha \in C_0^\infty(B)$. Thus Q^m may be extended to be defined on $\mathcal{D}'(B)$, and since differentiation is a continuous operation on $\mathcal{D}'$, (3.3) is seen to hold for all $u \in \mathcal{D}'(B)$. It also follows easily from (3.4) that Q^m is a Lipschitz continuous mapping of $L^1(B)$ into the Banach space $\mathcal{P}_m$.

Of interest in the sequel will be the error $u - Q^m u$. Since $Q^m u$ is a polynomial and hence defined on all of $\mathbb{R}^n$, it is natural to ask how well $Q^m u$ agrees with u on any domain containing B where u may be defined. Let Ω be a bounded open set in $\mathbb{R}^n$ that is star-shaped with respect to every point in B . For any positive integer m and $u \in C^\infty(\Omega)$, define

$$R^m u(x) = m \sum_{|\alpha|=m} \int_\Omega D^\alpha u(x) k_\alpha(x, y) dy, \quad (3.5)$$

where

$$k_\alpha(x, y) \equiv (1/\alpha!) (x-y)^\alpha k(x, y)$$

and

$$k(x, y) \equiv \int_0^1 s^{-n-1} \varphi(x+s^{-1}(y-x)) ds. \quad (3.6)$$

Notice that the support of $k(x, \cdot)$, and hence the region of integration in (3.6), is contained in the convex hull of $\{x\} \cup B$, which is compactly contained in Ω . Further observe that, if $d \equiv \text{diam}(\Omega)$,

$$\begin{aligned} |k(x, y)| &= \left| \int_0^1 s^{-n-1} \varphi(x+s^{-1}(y-x)) ds \right| \\ &= \left| \int_{|x-y|/d}^1 s^{-n-1} \varphi(x+s^{-1}(y-x)) ds \right| \\ &\leq \gamma |x-y|^{-n} \end{aligned}$$

for all $x, y \in \Omega$, where $\gamma \equiv (\sup_{x \in B} \varphi(x)) \cdot d^n/n$. Thus each

$k_\alpha(x, \cdot)$ is a well defined $L^1(\Omega)$ function if $|\alpha| > 0$. Therefore R^m is indeed defined on $C^\infty(\Omega)$.

THEOREM 3.1. Let Ω be a bounded open set in $\mathbb{R}^n$ that is star-shaped with respect to every point in an open ball B . Let $\varphi \in C_0^\infty(B)$ be such that $\int_B \varphi(x) dx = 1$, and let Q^m and R^m be the associated operators defined by (3.1) and (3.5) respectively. Then for any positive integer m and any $u \in C^\infty(\Omega)$,

$$u(x) = Q^m u(x) + R^m u(x)$$

for all $x \in \Omega$.

Remark. The proof of this theorem may be found, eg., in Sobolev's book [6]. More recently, related representation theorems used in the spirit of Sobolev's original work can be found in [2]. For completeness, we include its simple derivation here.

Proof of Theorem 3.1. For $x \in \Omega$ and $y \in B$, Taylor's theorem with integral remainder implies that

$$\begin{aligned} u(x) &= \sum_{|\alpha| < m} \frac{(x-y)^\alpha}{\alpha!} D^\alpha u(y) \\ &+ m \sum_{|\alpha|=m} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 s^{m-1} D^\alpha u(x+s(y-x)) ds. \end{aligned}$$

Multiply by $\varphi(y)$ and integrate with respect to y to obtain

$$u(x) = Q^m u(x) + m \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_B \varphi(y) (x-y)^\alpha \int_0^1 s^{m-1} D^\alpha u(x+s(y-x)) ds dy.$$

Now use the change of variables $z = x + s(y-x)$ and Fubini's theorem to see that

$$\begin{aligned} &\int \varphi(y) (x-y)^\alpha \int_0^1 s^{m-1} D^\alpha u(x+s(y-x)) ds dy \\ &= \int_0^1 \int \varphi(y) (x-y)^\alpha s^{m-1} D^\alpha u(x+s(y-x)) dy ds \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \int \varphi(x+s^{-1}(z-x)) (x-z)^\alpha s^{-1} D^\alpha u(z) s^{-n} dz ds \\
 &= \int (x-z)^\alpha D^\alpha u(z) \left[\int_0^1 \varphi(x+s^{-1}(z-x)) s^{-n-1} ds \right] dz \\
 &= \alpha! \int D^\alpha u(z) k_\alpha(x, z) dz. //
 \end{aligned}$$

4. THE BRAMBLE-HILBERT LEMMA.

THEOREM 4.1. Let Ω be a bounded open set in $\mathbb{R}^n$ of diameter d that is star-shaped with respect to every point in an open ball B of diameter ρd . Let m be a positive integer and $u \in H^m(\Omega)$. Then

$$\inf_{P \in \mathcal{P}_m} \left(\sum_{j=0}^m d^{2j} |u-P|_j^2 \right)^{1/2} \leq C_1 d^m |u|_m,$$

where

$$C_1 = (1 + (2/\rho)^{2n} v_{m,n}^2)^{1/2}$$

and

$$v_{m,n} = \left(\sum_{j=0}^{m-1} \#\{\alpha \in \mathbb{N}^n : |\alpha|=j\} \sum_{|\beta|=m-j} (\beta!)^{-2} \right)^{1/2}.$$

Proof. Because of density, it suffices to assume that

$u \in C^\infty(\Omega) \cap H^m(\Omega)$. Let $\varphi \in C_0^\infty(B)$ be such that $\int_B \varphi(x) dx = 1$

and let Q^m be the associated polynomial projector constructed in the previous section. Since $|Q^m u|_m = 0$, it suffices to show that

$$\left(\sum_{j=0}^{m-1} d^{2j} |u-Q^m u|_j^2 \right)^{1/2} \leq c d^m |u|_m \quad (4.1)$$

with $c = (2/\rho)^n v_{m,n}$. By the commutativity result (3.3) and Theorem 3.1,

$$\begin{aligned}
 |u-Q^m u|_j^2 &= \sum_{|\alpha|=j} \|D^\alpha (u-Q^m u)\|_0^2 \\
 &= \sum_{|\alpha|=j} \|D^\alpha u - Q^{m-j} D^\alpha u\|_0^2 \\
 &= \sum_{|\alpha|=j} \|R^{m-j} D^\alpha u\|_0^2
 \end{aligned}$$

Now for each α such that $|\alpha| = j$,

$$\begin{aligned} \|R^{m-j} D^\alpha u\|_0^2 &= \int_{\Omega} ((m-j) \sum_{|\beta|=m-j} \int_{\Omega} D^{\alpha+\beta} u(y) k_{\beta}(x,y) dy)^2 dx \\ &\leq \int_{\Omega} ((m-j) \sum_{|\beta|=m-j} \frac{1}{\beta!} \int_{\Omega} |D^{\alpha+\beta} u(y)| |x-y|^{m-j-n} dy)^2 dx \\ &\leq \gamma^2 (m-j)^2 \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right) \\ &\quad \times \sum_{|\beta|=m} \int_{\Omega} \left(\int_{\Omega} |D^{\beta} u(y)| |x-y|^{m-j-n} dy \right)^2 dx \\ &\leq \gamma^2 (m-j)^2 \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right) \\ &\quad \times \sum_{|\beta|=m} \int_{\Omega} \left(\int_{\Omega} |D^{\beta} u(y)|^2 |x-y|^{m-j-n} dy \right) \left(\int_{\Omega} |x-y|^{m-j-n} dy \right) dx, \end{aligned}$$

where we used Schwarz's inequality at the last step. Note that

$$\begin{aligned} \int_{\Omega} |x-y|^{m-j-n} dy &\leq \omega_n \int_0^d r^{m-j-1} dr \\ &= \omega_n d^{m-j} / (m-j), \end{aligned} \quad (4.2)$$

where ω_n is the measure of the unit $(n-1)$ -sphere. Thus

$$\begin{aligned} \|R^{m-j} D^\alpha u\|_0^2 &\leq \gamma^2 (m-j) \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right) \omega_n d^{m-j} \\ &\quad \times \sum_{|\beta|=m} \int_{\Omega} \left(\int_{\Omega} |D^{\beta} u(y)|^2 |x-y|^{m-j-n} dy \right) dx. \end{aligned}$$

Using Fubini's theorem and (4.2) with the roles of x and y reversed, we obtain

$$\begin{aligned} \|R^{m-j} D^\alpha u\|_0^2 &\leq \gamma^2 (m-j) \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right) \omega_n d^{m-j} \\ &\quad \times \sum_{|\beta|=m} \int_{\Omega} |D^{\alpha} u(y)|^2 \left(\int_{\Omega} |x-y|^{m-j-n} dx \right) dy \\ &\leq \gamma^2 \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right) \omega_n^2 d^{2(m-j)} |u|_m^2. \end{aligned}$$

Summing appropriately, we obtain (4.1) with $c = \gamma \omega_n^{1/2} d^{m-j}$.

Recall that φ was arbitrary, except that $\varphi \in C_0^\infty(B)$ and

$\int_B \varphi(x) dx = 1$, and that $\gamma = (\sup_{x \in B} |\varphi(x)|) d^n/n$. We can choose

φ arbitrarily close to a constant multiple of the characteristic function of B , and hence $\sup_{x \in B} |\varphi(x)|$ may be chosen arbitrarily close to $1/\text{meas } B = (\frac{1}{n}(\frac{1}{2}\rho d)^n \omega_n)^{-1} = n 2^n (\rho d)^{-n} \omega_n^{-1}$. Thus γ may be chosen to be $(2/\rho)^n \omega_n^{-1}$. //

Remark. What we have proved is in fact sharper, namely that

$$\inf_{P \in \mathcal{P}_m} \max_{0 \leq j \leq m} \frac{1}{\mu_j} d^j |u - P|_j \leq (2/\rho)^n d^m |u|_m,$$

where

$$\mu_j = (\#\{\alpha \in \mathbb{N}^n: |\alpha| = j\})^{-1} \sum_{|\beta| = m-j} (\beta!)^{-2}^{1/2}$$

for $j = 0, 1, \dots, m-1$, and $\mu_m = (\rho/2)^n$. When j is small and m is large, the constant μ_j is quite small. For example, suppose $n = 2$. Then

$$\begin{aligned} \sum_{|\beta|=k} (\beta!)^{-2} &= \sum_{i=0}^k (i!(k-i!))^{-2} \\ &= (k!)^{-2} \sum_{i=0}^k \binom{k}{i}^2. \end{aligned}$$

Using Hölder's inequality, we find that

$$\sum_{i=0}^k \binom{k}{i}^2 \leq \left[\max_i \binom{k}{i} \right] \sum_{i=0}^k \binom{k}{i} \leq 2^{2k-1}.$$

Therefore

$$\sum_{|\beta|=k} (\beta!)^{-2} \leq 2^{2k-1} / (k!)^2. \quad (4.3)$$

Since $\#\{\alpha \in \mathbb{N}^2: |\alpha| = j\} = j+1$, we find

$$\mu_j \leq \sqrt{(j+1)/2} \cdot 2^{m-j} / (m-j)!.$$

Since $v_{m,2} = \left(\sum_{j=0}^{m-1} \mu_j^2 \right)^{1/2}$, we also find that

$$\begin{aligned} v_{m,2}^2 &\leq \sum_{i=1}^m (m-i+1) 2^{2i-1} / (i!)^2 \\ &\leq m \sum_{i=1}^m 2^i / i! \\ &\leq m e^2. \end{aligned}$$

Therefore, $v_{m,2} \leq \sqrt{m} e. //$

The following theorem is closely related to the previous one and in fact follows from it via Sobolev's inequality. It will be central to the estimation of interpolation error given in the next section, and hence we give an independent proof.

THEOREM 4.2. Let Ω be a bounded open set in R^n of diameter d that is star-shaped with respect to every point of an open ball B of diameter ρd . Let k be a nonnegative integer and let m be an integer such that $m > k + n/2$. Then for all $u \in H^m(\Omega)$ and $\alpha \in N^n$ such that $|\alpha| \leq k$,

$$\inf_{P \in P_m} \max_{|\alpha| \leq k} d^{|\alpha|} \|D^\alpha(u-P)\|_{L^\infty(\Omega)} \leq C_2 d^{m-n/2} |u|_m,$$

where

$$C_2 = m \omega_n^{-1/2} (2(m-k)-n)^{-1/2} \max_{0 \leq j \leq k} \left(\sum_{|\beta|=m-j} (\beta!)^{-2} \right)^{1/2} (2/\rho)^n.$$

Proof. It suffices to assume that $u \in C^\infty(\Omega) \cap H^m(\Omega)$ and to show that

$$\|D^\alpha(u-Q^m u)\|_{L^\infty(\Omega)} \leq C'_2 d^{m-n/2-|\alpha|} |u|_m$$

for all $|\alpha| \leq k$, where C'_2 may be made arbitrarily close to C_2 by choosing ρ appropriately as in Theorem 4.1. For $x \in \Omega$, we have

$$\begin{aligned} |D^\alpha(u-Q^m u)(x)| &= |(D^\alpha u - Q^{m-|\alpha|} D^\alpha u)(x)| \\ &= |R^{m-|\alpha|} D^\alpha u(x)| \\ &= |(m-|\alpha|) \sum_{|\beta|=m-|\alpha|} \frac{1}{\beta!} \int_\Omega D^{\alpha+\beta} u(y) k_\beta(x,y) dy| \\ &\leq (m-|\alpha|) \gamma \sum_{|\beta|=m-|\alpha|} \frac{1}{\beta!} \int_\Omega |D^{\alpha+\beta} u(y)| |x-y|^{m-|\alpha|-n} dy \\ &\leq (m-|\alpha|) \gamma \left(\sum_{|\beta|=m-|\alpha|} \frac{1}{\beta!} \|D^{\alpha+\beta} u\|_0 \right) \left(\int_\Omega |x-y|^{2(m-|\alpha|-n)} dy \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 &\leq (m-|\alpha|) \gamma \left(\sum_{|\beta|=m-|\alpha|} \frac{1}{\beta!} \|D^{\alpha+\beta} u\|_0 \right) \omega_n^{1/2} \left(\int_0^d r^{2(m-|\alpha|)-n-1} dr \right)^{1/2} \\
 &\leq (m-|\alpha|) \gamma \omega_n^{1/2} d^{m-|\alpha|-n/2} (2(m-|\alpha|)-n)^{-1/2} \\
 &\quad \times \left(\sum_{|\beta|=m-|\alpha|} \frac{1}{\beta!} \|D^{\alpha+\beta} u\|_0 \right) \\
 &\leq (m-|\alpha|) \gamma \omega_n^{1/2} d^{m-|\alpha|-n/2} (2(m-|\alpha|)-n)^{-1/2} \left(\sum_{|\beta|=m-|\alpha|} (\beta!)^{-2} \right)^{1/2} |u|_m.
 \end{aligned}$$

Letting γ tend to the value $(2/\rho)^n \omega_n^{-1}$ as in the proof of the previous theorem, the conclusion is reached. //

Remarks. First note that since $Q^m u$ was used to approximate u in the proofs of both Theorem 4.1 and 4.2, we can conclude that the stated approximations hold simultaneously, i.e.,

$$\begin{aligned}
 \inf_{P \in P_m} \left(\sum_{j=0}^m d^{2j} |u-P|_j^2 \right)^{1/2} + \max_{|\alpha| \leq k} d^{|\alpha|+n/2} \|D^\alpha (u-P)\|_{L^\infty(\Omega)} \\
 \leq (C_1 + C_2) d^m |u|_m.
 \end{aligned}$$

Second, observe that Sobolev's inequality follows almost immediately from the proof of Theorem 4.2. All that remains to be estimated, in view of the triangle inequality, is $\|D^\alpha Q^m u\|_{L^\infty(\Omega)}$. It follows from (3.4) that

$$\|D^\alpha Q^m u\|_{L^\infty(\Omega)} \leq \text{constant} \|u\|_{L^1(\Omega)},$$

where we shall not evaluate the constant, but the eager reader may easily do so. Thus we obtain

$$\max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} \leq \text{constant} (\|u\|_{L^1(\Omega)} + |u|_m).$$

This is essentially the proof used by Sobolev [6] for the imbedding theorems.

Finally, we direct the reader back to the bound (4.3) for the quantity $\sum_{|\beta|=j} (\beta!)^{-2}$ appearing in the definition of C_2 in the case $n = 2$. //

5. ESTIMATES FOR LAGRANGE INTERPOLATION.

Let $\mathcal{G}$ be a bounded open set in $\mathbb{R}^2$ with smooth boundary $\partial\mathcal{G}$. Let $\mathcal{T}$ be a family of triangulations T of $\mathcal{G}$ consisting of triangles T having straight interior edges but (possibly) one curved edge on $\partial\mathcal{G}$. Define $h(T) = \max\{\text{diam}(T) : T \in \mathcal{T}\}$ and let $b(T)$ be the diameter of the largest open ball with respect to which T is star-shaped. Suppose that $\mathcal{T}$ satisfies the following nondegeneracy condition with $\rho > 0$:

$$\min_{T \in \mathcal{T}} b(T) \geq \rho h(T) \quad \text{for all } T \in \mathcal{T}. \quad (5.1)$$

Let $\tau = \{(x, y) : x, y \geq 0; x + y \leq 1\}$ and define Lagrange interpolation nodes for polynomials of degree less than m on τ to be the $\frac{1}{2} m(m+1)$ points in τ whose coordinates have values i/k for some integer i . For each $T \in \mathcal{T}$ and each $T \in \mathcal{T}$, define a reference domain T_R by letting T_R be the image of T under an affine mapping that maps the vertices of T onto the set $\{(0, 0), (1, 0), (0, 1)\}$ (there are six such affine maps -- choose one arbitrarily). If T has straight edges, then T_R (or its closure) equals τ , and we define Lagrange interpolation nodes for T via the inverse affine mapping $T \leftarrow T_R = \tau$. If T has a curved edge, we must modify this. For h_0 sufficiently small, depending on the curvature of $\partial\mathcal{G}$ and ρ , if $h(T) \leq h_0$ and $T \in \mathcal{T}$, then the interior nodes of τ lie in the interior of T_R . In this case, we define nodes for T by mapping the interior nodes of τ back to T , and for the edges of T we take the $k-1$ points on each edge that divide the edge into k arcs of equal length, plus the vertices of T .

As is well known, given any continuous function u defined on $\mathcal{G}$, there is a unique function $\mathcal{J}_T u \in H^1(\Omega)$ such that for all $T \in \mathcal{T}$, $\mathcal{J}_T u|_T$ is a polynomial of degree less than m that agrees with u at the Lagrange interpolation nodes for T defined above, at least provided $h(T) \leq h_0$ and h_0 is sufficiently small depending on $\partial\mathcal{G}$ and ρ . Furthermore

$$\sup_{x \in T} |\mathcal{J}_T u(x)| \leq K \sup_{x \in T} |u(x)| \quad \text{for all } T \in \mathcal{T}, \quad (5.2)$$

where K is easily computed. For example, for piecewise quadratic interpolation, $K \leq 7$ for $h(T) \leq h_0$ and h_0 sufficiently small depending on $\partial\mathcal{G}$ and ρ . To see this, consider the Lagrange interpolation basis functions for T . Each is bounded by 1, and there are six of them, so if T has only straight sides, we may take $K \leq 6$. Since the nodes for a T having a curved side are an $O(h^2)$ perturbation from nodes for a straight-edged triangle, we conclude $K \leq 7$ is sufficient for h small.

We can estimate the error $u - \mathcal{J}_T u$ as follows. Let $T \in \mathcal{T}$. Then for any $P \in \mathcal{P}_m$

$$\begin{aligned} \|u - \mathcal{J}_T u\|_{L^2(T)} &\leq \|u - P\|_{L^2(T)} + \|\mathcal{J}_T(u - P)\|_{L^2(T)} \\ &\leq \sqrt{\text{meas}(T)} \left(\|u - P\|_{L^\infty(T)} + \|\mathcal{J}_T(u - P)\|_{L^\infty(T)} \right) \\ &\leq \sqrt{\text{meas}(T)} (1+K) \|u - P\|_{L^\infty(T)}, \end{aligned}$$

where we have used the fact that $\mathcal{J}_T P = P$, Hölder's inequality, and the bound (5.2). Since P was arbitrary and $\sqrt{\text{meas}(T)} \leq \sqrt{\pi/4} h(T)$, Theorem 4.2 implies that

$$\|u - \mathcal{J}_T u\|_{L^2(T)} \leq Ch(T)^m |u|_{m,T},$$

where

$$C \leq (m/\sqrt{m-1}) \left(\sum_{|\beta|=m} (\beta!)^{-2} \right)^{1/2} \rho^{-2} (1+K).$$

Squaring and summing over $T \in \mathcal{T}$, we find

$$\|u - \mathcal{J}_T u\|_0 \leq Ch(T)^m |u|_m,$$

where the norms refer to the domain $\mathcal{G}$. In the case of piecewise quadratic interpolation ($m=3$), we thus have

$$\|u - \mathcal{J}_T u\|_0 \leq 10\rho^{-2} h(T)^3 |u|_3.$$

Remarks. Estimates similar to those above were derived in [5] using a much more complicated technique and without any estimate for the constant involved. Such estimates can also be derived when $\partial\Omega$ is piecewise smooth; when $\partial\Omega$ is polygonal and each $T \in \mathcal{T}$ has only straight sides, the restriction " $h(T) \leq h_0$ " may be removed.

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