Diffusion into a Hollow Cylinder

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Contract No. W-7405 eng 26

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Date Published - October 1978

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CONTENTS

List of Figures	v
Acknowledgment	vii
Abstract	1
I. Introduction and Statement of the Problem	1
II. General Solution	5
III. Alternate Solution Useful for the Case $L \gg b$	7
IV. Alternate Solution Useful for the Case $b \gg L$	9
V. Reduction to Special Cases	13
Specialization of the General Solution	13
Specialization of the First Alternate Solution ($L \gg b$)	14
Specialization of the Second Alternate Solution ($b \gg L$)	15
VI. Examples	17
VII. Conclusion	23
VIII. Notation	25
IX. References	27

LIST OF FIGURES

Fig. 1. The hollow cylinder of length L , inner radius a , and outer radius b	2
Fig. 2. Fractional uptake as a function of time for four values of the length-to-radius ratio R	18
Fig. 3. Fractional uptake calculated from the general solution compared to that calculated from the first alternate solution with the longitudinal summation truncated	19
Fig. 4. Comparison between the general solution and the first alternate solution with both summations truncated	20
Fig. 5. Comparison between the general solution and the converged second alternate solution	21
Fig. 6. Comparison between the general solution and the truncated second alternate solution	22

Acknowledgment

The authors would like to thank G. L. Powell of the Y-12 Development Division for bringing this problem to their attention and for providing the necessary financial support.

Thanks also are due the secretarial staff of the Computing Applications Department for their assistance in the preparation of this report.

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J. S. Tolliver and M. Reeves

ABSTRACT

We have obtained a completely general solution to the problem of diffusion into a hollow cylinder. When one of the dimensions of the cylinder, either length or radius, is large compared to the other, one of the two infinite series of this solution converges slowly and is not suitable for practical application. For these situations two alternate solutions, utilizing rapidly converging series, have been obtained. Both the concentration distribution and the total uptake as functions of time are considered. A number of specializations are used to show that the solutions are reducible to the cases of a finite solid cylinder, an infinite solid cylinder, an infinite plate, and an infinite hollow cylinder. Finally several examples are given to show the efficacy of our solutions.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

Figure 1 shows a hollow cylinder of length L , inner radius a , and outer radius b . The cylinder is immersed in a gas of constant concentration c , and has diffusivity D . The concentration c obeys the equation of diffusion

$$\frac{\partial c(r,z,t)}{\partial t} = D \nabla^2 c(r,z,t) \quad (1)$$

on the region

$$a \leq r \leq b \text{ and } 0 \leq z \leq L$$

subject to the boundary conditions

$$c(a,z,t) = c(b,z,t) = c_0 \quad (2a)$$

$$c(r,0,t) = c(r,L,t) = c_0 \quad (2b)$$

and the initial condition

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Fig. 1. The hollow cylinder of length L , inner radius a , and outer radius b .

$$\phi(r,z,0) = 0 \quad (2c)$$

We shall first obtain a general solution to Eq. (1) by a treatment similar to that of Carslaw and Jaeger [1959; Section 7.10]. Then we shall employ a method using the Laplace transformation to obtain two alternate solutions which are more useful for cases where values of the length and the outer radius differ substantially.

II. GENERAL SOLUTION

In order to obtain a purely homogeneous problem, we first make a change of variables:

$$c(r,z,t) = u(r,z,t) + c_0 \quad (3a)$$

Here, u satisfies the same partial differential equation as c , namely

$$\frac{\partial u}{\partial t} = D \nabla^2 u \quad (3b)$$

but has the homogeneous boundary conditions

$$u(a,z,t) = u(b,z,t) = 0 \quad (4a)$$

and

$$u(r,0,t) = u(r,L,t) = 0 \quad (4b)$$

and the initial condition

$$u(r,z,0) = -c_0 \quad (4c)$$

Upon expressing the Laplacian in cylindrical coordinates, Eq. (3b) becomes

$$\frac{\partial u}{\partial t} = D \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] \quad (5)$$

Equation (5) may be solved by separation of variables with the solution being

$$u(r,z,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} U_0(\epsilon_n r) \sin\left(\frac{(2m-1)\pi z}{L}\right) e^{-D\alpha_{mn}^2 t} \quad (6a)$$

where

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$$U_0(\epsilon_n r) = J_0(\epsilon_n r) Y_0(\epsilon_n a) - J_0(\epsilon_n a) Y_0(\epsilon_n r) \quad (6b)$$

$$\alpha_{mn}^2 = \left(\frac{\gamma_n}{b} \right)^2 + \left(\frac{(2m-1)\pi}{L} \right)^2 \quad (6c)$$

$$\epsilon_n = \gamma_n / b \quad (6d)$$

and the γ_n are the roots of U_0 . The normalization constant, found by applying the initial condition, Eq. (4c), is given by

$$A_{mn} = \frac{4c_0}{2m-1} \frac{J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \quad (6e)$$

Thus the solution to Eq. (1) is

$$c(r, z, t) = c_0 \left\{ 1 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{2m-1} \frac{J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} U_0(\epsilon_n r) \sin\left(\frac{(2m-1)\pi z}{L}\right) e^{-D\alpha_{mn}^2 t} \right\} \quad (7)$$

To find the uptake as a function of time, we integrate $c(r, z, t)$ over the volume:

$$M(t) = \int_0^{2\pi} \int_0^1 \int_a^b c(r, z, t) r dr dz dv$$

Evaluating the integral, we find

$$\frac{M(t)}{M(\infty)} = 1 - \frac{32}{\pi^2(b^2 - a^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-D\alpha_{mn}^2 t}}{(2m-1)^2 \epsilon_n^2} \frac{J_0(\epsilon_n a) - J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \quad (8)$$

Here, $M(t)$ is the amount of gas having diffused into the cylindrical shell after time t , and $M(\infty)$ is the corresponding quantity after an infinite time.

III. ALTERNATE SOLUTION USEFUL FOR THE CASE $L \gg b$

If the length L is large, the exponent in Eq. (6a) is small due to the longitudinal component of Eq. (6c). This small exponent leads to slow convergence, a difficulty similar to that arising from small values of time in the exponent. Such difficulties may be alleviated by application of a Laplace transformation technique as follows. We begin with Eq. (3b) above and consider a solution having the form

$$u(r,z,t) = U_0(\epsilon_n r) e^{-D(\epsilon_n^2 t)} [1 + v(z,t)] \quad (9)$$

Quantity v represents the variation of the solution from that of an infinite hollow cylinder. The differential equation for v is found to be

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial z^2} \quad (10a)$$

with boundary conditions

$$v(0,t) = v(L,t) = 1 \quad (10b)$$

and initial condition

$$v(z,0) = 0 \quad (10c)$$

Applying the Laplace transformation to Eqs. (10), we have the subsidiary equation

$$\frac{d^2 \bar{v}}{dz^2} - k^2 \bar{v} = 0 \quad (11a)$$

and the boundary conditions

$$\bar{v}(0) = \bar{v}(L) = 1/\lambda \quad (11b)$$

where

$$k = \sqrt{\lambda/D} \quad (11c)$$

The solution to Eq. (11a) is

$$v = A \cosh(kz) + B \sinh(kz) \quad (12a)$$

Imposing the boundary conditions and expanding the hyperbolic functions in a manner appropriate for large values of k yields the solution.

$$v = \frac{1}{\lambda} \left[\sum_{m=0}^{\infty} (-1)^m e^{-k(mL+z)} + \sum_{m=0}^{\infty} (-1)^m e^{-k(m+1)L-z} \right] \quad (12b)$$

Taking the inverse Laplace transform and collecting constants, we have

$$c(r,z,t) = c_0 \left\{ 1 - \left[\sum_{n=1}^{\infty} A'_n U_0(\epsilon_n r) e^{-D\epsilon_n^2 t} \right] \left[1 - \sum_{m=0}^{\infty} (-1)^m \operatorname{erfc}\left(\frac{mL+z}{2\sqrt{Dt}}\right) \right. \right. \\ \left. \left. - \sum_{m=0}^{\infty} (-1)^m \operatorname{erfc}\left(\frac{(m+1)L-z}{2\sqrt{Dt}}\right) \right] \right\} \quad (13a)$$

where A'_n is found from the initial condition:

$$A'_n = \frac{\pi J_0(\epsilon_n a)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \quad (13b)$$

Integrating to find the fractional uptake, we have

$$\frac{M(t)}{M(\infty)} = 1 - \frac{4}{b^2 - a^2} \left\{ \sum_{n=1}^{\infty} \frac{e^{-D\epsilon_n^2 t}}{\epsilon_n^2} \frac{J_0(\epsilon_n a) - J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \right\} \\ \times \left\{ 1 - \frac{4\sqrt{Dt}}{L} \sum_{m=0}^{\infty} (-1)^m \left[\operatorname{erfc}\left(\frac{mL}{2\sqrt{Dt}}\right) - \operatorname{erfc}\left(\frac{(m+1)L}{2\sqrt{Dt}}\right) \right] \right\} \quad (14)$$

It is apparent from the definition of ϵ_n , Eq. (6d), and the properties of the complimentary error function that Eqs. (13a) and (14) will converge rapidly when the dimensions of the cylinder and the time values are such that

$$b^2 \ll Dt \quad \text{and} \quad Dt \ll L^2 \quad (15)$$

Equations (13) and (14) are therefore the desired solutions.

IV. ALTERNATE SOLUTION USEFUL FOR THE CASE $b \gg L$

In contrast to the previous section, we begin here by putting the radial dependence in a form suitable for application of the Laplace transformation technique. Consider

$$u(r, z, t) = \sin(\xi_m z) e^{-D\xi_m^2 t} [1 - \phi(r, t)] \quad (16a)$$

where

$$\xi_m = \frac{(2m-1)\pi}{L} \quad (16b)$$

Here, ϕ satisfies

$$\frac{\partial \phi}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) \quad (17a)$$

with boundary conditions

$$\phi(a, t) = \phi(b, t) = 1 \quad (17b)$$

and initial condition

$$\phi(r, 0) = 0 \quad (17c)$$

Solving in Laplace transform space, we find

$$\phi(r) = \frac{1}{\lambda} \frac{[I_0(kr)K_0(kb) - I_0(kb)K_0(kr)] + [I_0(ka)K_0(kr) - I_0(kr)K_0(ka)]}{I_0(ka)K_0(kb) - I_0(kb)K_0(ka)} \quad (18)$$

where

$$k = \sqrt{\lambda/D}$$

To zeroth order, the asymptotic expansions of the modified Bessel functions, I_0 and K_0 , yield

$$\phi(r) = \frac{1}{\lambda} \left\{ \sqrt{\frac{a}{r}} \sum_{n=0}^{\infty} \left[e^{-k[(2n+1)\rho - (b-r)]} - e^{-k[(2n+1)\rho + (b-r)]} \right] \right. \\ \left. \sqrt{\frac{b}{r}} \sum_{n=0}^{\infty} \left[e^{-k[(2n+1)\rho - (a-r)]} - e^{-k[(2n+1)\rho + (a-r)]} \right] \right\} \quad (19a)$$

where

$$\rho = b - a \quad (19b)$$

The summations in the above expression result from an additional expansion of the denominator of Eq. (18). Inverting $\phi(r)$, applying the initial condition, and collecting terms yields

$$\begin{aligned} \alpha(r,t) = & \sqrt{\frac{a}{r}} \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{\beta_n - (b-r)}{2\sqrt{Dt}}\right) - \operatorname{erfc}\left(\frac{\beta_n + (b-r)}{2\sqrt{Dt}}\right) \right] \\ & + \sqrt{\frac{b}{r}} \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{\beta_n + (r-a)}{2\sqrt{Dt}}\right) - \operatorname{erfc}\left(\frac{\beta_n - (r-a)}{2\sqrt{Dt}}\right) \right] \end{aligned} \quad (20a)$$

where

$$\beta_n = (2n+1)\rho \quad (20b)$$

To obtain an approximation which is correct to zeroth order, we take only the terms which have the smallest arguments for the complimentary error functions. (Function $\operatorname{erfc}(x)$ decreases rapidly as argument x increases.) Thus,

$$c \approx c_0 \left\{ 1 - \frac{4}{\pi} \left[\sum_{m=1}^{\infty} \frac{\sin(\xi_m r)}{(2m-1)} e^{-D\xi_m^2 t} \right] \left[1 - \sqrt{\frac{a}{r}} \operatorname{erfc}\left(\frac{r-a}{2\sqrt{Dt}}\right) - \sqrt{\frac{b}{r}} \operatorname{erfc}\left(\frac{b-r}{2\sqrt{Dt}}\right) \right] \right\} \quad (21)$$

We integrate to find

$$\frac{M(t)}{M(\infty)} \approx 1 - \frac{16}{\pi^2(b^2 - a^2)} \left\{ \sum_{m=1}^{\infty} \frac{e^{-D\xi_m^2 t}}{(2m-1)^2} \right\} \left\{ \frac{1}{2}(b^2 - a^2) + 2\sqrt{Dt} \left[\frac{1}{\sqrt{\pi}} - \operatorname{ierfc}\left(\frac{b-a}{2\sqrt{Dt}}\right) \right] \right\} \quad (22)$$

It is apparent from the definition of ξ_m in Eq. (16b) that the zeroth order expression, Eq. (22), will be valid whenever

$$L^2 \ll Dt \quad \text{and} \quad Dt \ll (b-a)^2 \quad (23)$$

The validity of Eq. (21), however, depends on the radii considered independently:

$$L^2 \ll Dt, \quad Dt \ll b^2 \quad \text{and} \quad Dt \ll a^2 \quad (24)$$

In cases where only the first two inequalities of Eq. (24) are satisfied, concentrations calculated from Eq. (21) will be incorrect only in the region $r = a$.

V. REDUCTION TO SPECIAL CASES

In this section we show how our solutions from the previous three sections may be specialized to the cases of a finite solid cylinder, an infinite solid cylinder, an infinite plate, and an infinite hollow cylinder.

Specialization of the General Solution.

Beginning with the fractional uptake for a finite hollow cylinder, Eq. (8), we reduce it to that for a finite solid cylinder by allowing the inner radius a to approach zero. The term

$$\frac{J_0(\epsilon_n a) - J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \rightarrow \frac{1 - \frac{a}{b}}{1 + \frac{a}{b}} \rightarrow 1 \quad (25)$$

In this limit, then,

$$\frac{M(t)}{M(\infty)} = 1 - \frac{32}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\epsilon_n^2 b^2} \frac{e^{-D \epsilon_n^2 t}}{(2m-1)^2} \quad (26)$$

for a finite solid cylinder.

We may further specialize Eq. (26) to the expression for an infinite solid cylinder by allowing L to approach infinity. From Eq. (6c), we observe that the term α_m^2 depends on L such that

$$\alpha_m^2 = \left(\frac{\gamma_n}{b} \right)^2 + \left(\frac{2m-1}{L} \right)^2 \rightarrow \frac{\gamma_n^2}{b^2} = \epsilon_n^2 \quad (27)$$

Thus,

$$\frac{M(t)}{M(\infty)} = 1 - \frac{32}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \sum_{n=1}^{\infty} \frac{e^{-D \epsilon_n^2 t}}{\epsilon_n^2 b^2}$$

or

$$\frac{M(t)}{M(\infty)} = 1 - \sum_{n=1}^{\infty} \frac{4}{\epsilon_n^2 b^2} e^{-D \epsilon_n^2 t} \quad (28)$$

for an infinite solid cylinder. Here, the summation over m has been replaced by its closed form expression

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

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from Dwight [1961: Eq. (48.12)]. Equation (28) agrees with Crank's expression for an infinite solid cylinder [Crank, 1975: Eq. (5.23)].

The next specialization begins again with Eq. (8) for a finite hollow cylinder and allows b to approach infinity, a to approach zero, and L to remain finite, with the result being the fractional uptake for an infinite slab. From Eqs. (6c) and (16b).

$$\alpha_{mn}^2 = \left(\frac{\gamma_n}{b}\right)^2 + \left(\frac{(2m-1)\pi}{L}\right)^2 \rightarrow \left(\frac{(2m-1)\pi}{L}\right)^2 = \xi_m^2$$

so that

$$\frac{M(t)}{M(\infty)} = 1 - \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\epsilon_n^2 b^2} \sum_{m=1}^{\infty} \frac{e^{-D\xi_m^2 t}}{(2m-1)^2}$$

or

$$\frac{M(t)}{M(\infty)} = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{e^{-D\xi_m^2 t}}{(2m-1)^2} \quad (29a)$$

provided

$$\sum_{n=1}^{\infty} \frac{1}{\epsilon_n^2 b^2} = \frac{1}{4} \quad (29b)$$

We have verified Eq. (29b) by numerical means only. Equation (29a) agrees with Crank's expression for an infinite slab [Crank, 1975: Eq. (4.23)].

Specialization of the First Alternate Solution ($L \gg b$).

Here, it is a simple task to reduce the fractional uptake for a long hollow cylinder, Eq. (14), to that of an infinite hollow cylinder by letting L be not only large, but in fact approaching infinity. Unfortunately, we do not have a readily available expression from Crank or others with which to compare our results. We nevertheless quote the expression so obtained with the comment that an identical expression may be obtained from Eq. (8), the general solution, by letting L approach infinity:

$$\frac{M(t)}{M(\infty)} = 1 - \frac{4}{(b^2 - a^2)} \left[\sum_{n=1}^{\infty} \frac{e^{-D\epsilon_n^2 t}}{\epsilon_n^2} \frac{J_0(\epsilon_n a) J_0(\epsilon_n b)}{J_0(\epsilon_n a) + J_0(\epsilon_n b)} \right] \quad (30)$$

for an infinite hollow cylinder.

Proceeding from Eq. (30), let us now allow a to approach zero and recover the fractional uptake for an infinite solid cylinder. From Eq. (25) the term involving the Bessel functions

approaches unity. The solution is, then,

$$\frac{M(t)}{M(\infty)} = 1 - \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} e^{-D n^2 t} \quad (31)$$

Equation (31) is identical to Eq. (28) which was deduced from the general solution, and both agree with Crank's expression for an infinite solid cylinder. Observing that Eqs. (30) and (31) may both be deduced from the first alternate solution as well as the general solution lends credence to the alternate formulation.

Specialization of the Second Alternate Solution ($b \gg L$)

Beginning with Eq. (22), we deduce the fractional uptake for an infinite slab. Setting a equal to zero produces

$$\frac{M(t)}{M(\infty)} = 1 - \frac{16}{\pi^2 b^2} \left\{ \sum_{m=1}^{\infty} \frac{e^{-D t m^2}}{(2m-1)^2} \right\} \left\{ \frac{1}{2} b^2 + 2\sqrt{Dt} b \left[\frac{1}{\sqrt{\pi}} \operatorname{erfc} \left(\frac{b}{2\sqrt{Dt}} \right) \right] \right\} \quad (32)$$

Letting b approach infinity reduces this expression to

$$\frac{M(t)}{M(\infty)} = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{e^{-D t m^2}}{(2m-1)^2} \quad (33)$$

for an infinite slab. Equation (33), deduced from the second alternate solution, is identical to Eq. (29a), which was obtained from the general solution. This fact serves to validate the formulation of the second alternate solution since both expressions agree with Crank's result.

As a final specialization we once again consider an infinite solid cylinder. Equation (21), which gives the concentration distribution, was formulated to be rapidly convergent whenever the radius b was large compared to L . It should be noted that the approximations inherent in this equation pertain to the radial variable only. Therefore, if we now allow L to approach infinity, we do not destroy the correctness of the formulation; at worst, we shall have a slowly converging series. We do, however, retain the condition $Dt \ll b^2$ so that our zeroth order approximations will remain valid. Using the fact that the exponential term in Eq. (21) approaches unity for small times and large values of L and setting a equal to zero yield

$$c(r,z,t) = c_0 \left\{ 1 - \left[\sum_{m=1}^{\infty} \frac{4}{\pi} \frac{\sin(\xi_m z)}{(2m-1)} \right] \left[1 - \sqrt{\frac{b}{r}} \operatorname{erfc} \left(\frac{b-r}{2\sqrt{Dt}} \right) \right] \right\} \quad (34)$$

The summation over m may be identified with the Fourier series expansion of a function which has the value unity on the region $0 \leq z \leq L$. Thus,

$$c(r,z,t) = c_0 \sqrt{\frac{b}{r}} \operatorname{erfc}\left(\frac{b-r}{2\sqrt{Dt}}\right) \quad (35)$$

Equation (35) is just the first term of the small time solution for an infinite solid cylinder given by Carslaw and Jaeger [1959; Eq. (3) of Section 13.3].

VI. EXAMPLES

In order to demonstrate some of the characteristics of our solutions, we plot fractional uptakes as functions of time for several hollow cylinders. Each cylinder has an inner radius $a = 0.125$ cm, an outer radius $b = 0.75$ cm, and a diffusivity $D = 1.0$ cm²/sec. The ratio of length to radius, $R = L/b$, is allowed to vary from cylinder to cylinder.

Figure 2 illustrates the effect of varying R . As R decreases, which implies that the diffusion from the ends of the cylinder becomes more important, the fractional uptake rises more rapidly in time. For the four R values indicated on the graph, the general solution, Eq. (8), and the first alternate solution, Eq. (14), are indistinguishable provided that sufficiently many terms are used in the summations. (In each case we used 100 terms in both the longitudinal summation over m and the radial sum over n . Such equivalence is to be expected since both Eq. (8) and Eq. (14) are exact solutions containing no approximations.

It has been indicated earlier that the first alternate solution will converge rapidly for cylinders having large length-to-radius ratios R provided the time is within the limits specified by Eq. (15). Figures 3 and 4 demonstrate this characteristic for three different values of R . Here, as in all figures to follow, the solid curves represent the converged general solution of Eq. (8) and may be considered correct. The dashed curves of Fig. 3 result from truncating the longitudinal sum of the first alternate solution to just one term ($m = 0$, only, in Eq. (14)). The number of terms in the radial summation is kept sufficiently large to insure convergence for this part of the solution. As expected from the second inequality of Eq. (15), the solutions diverge in the large-time regime with the divergence being less severe for large R . The dashed curves of Fig. 4 result from truncating both summations of the first alternate solution to one term ($n = 1$, only, and $m = 0$, only, in Eq. (14)). For large times the curves here show the same divergence from the general solution as in Fig. 3. As is to be expected from the first inequality of Eq. (15), they also diverge in the small-time regime, and furthermore, the severity of this divergence is apparently independent of R . The logarithmic vertical axis is included in Fig. 4 to show the approximately exponential nature of the converged regions of the curves for $R = 1$ and $R = 4$.

In contrast to the first alternate solution, the second alternate solution is most appropriate for cylinders having small length-to-radius ratios R (cf. Eq. (23)). Also by way of contrast, Eq. (22) is not general in its radial formulation. In effect, the radial summation has been truncated analytically through our approximations to the one term shown in Eq. (22). Figure 5 provides a comparison between the converged general solution (solid curves) and a converged second alternate solution (dashed curves). It shows that our approximations become more and more valid as the ratio R becomes small. As is also apparent from the graph, the second alternate curve closely resembles the general-solution curve for $R = 0.04$. The remaining longitudinal summation of Eq. (22) is truncated to one term to produce the dashed curves of Fig. 6. These may be compared with the solid general solution curves. Obviously there is a large error for the larger values of R . However, for R equal to 0.01, the agreement becomes quite good.

Fig. 2. Fractional uptake as a function of time for four values of the length-to-radius ratio R .

Fig. 3. Fractional uptake calculated from the general solution compared to that calculated from the first alternate solution with the longitudinal summation truncated.

Fig. 4. Comparison between the general solution and the first alternate solution with both summations truncated.

Fig. 5. Comparison between the general solution and the converged second alternate solution.

Fig. 6. Comparison between the general solution and the truncated second alternate solution.

VII. CONCLUSION

Three solutions to the problem of diffusion into a hollow cylinder have been derived. The first is a general solution and is correct for all values of the spatial dimensions and of the time. The second solution is equally general but more useful for the case of a long cylinder. The third, a zeroth order approximation in the radial variable, is rapidly convergent for the case of a cylinder with large radius. Each solution has been reduced to a number of different specializations for the purpose of validating our results by comparison with accepted work by Crank [1975] and Carslaw and Jaeger [1959]. Graphs have been presented to demonstrate the characteristics of the various solutions in several special cases.

VIII. NOTATION

A_{∞}	Normalization constant defined by Eq. (6e).
A_z	Normalization constant defined by Eq. (13b).
a	Inner radius of the hollow cylinder ... L.
b	Outer radius of the hollow cylinder ... L.
c	Concentration ... $M \cdot L^3$.
c_{∞}	Concentration of gas surrounding the cylinder ... $M \cdot L^3$.
D	Diffusivity of cylinder ... $L^2 \cdot T$.
k	Parameter defined by Eq. (11c).
L	Length of cylinder ... L.
$M(t)$	Uptake as a function of time ... M.
$M(t) / M(\infty)$	Fractional uptake.
R	Length-to-radius ratio L / b .
r	Radial coordinate ... L.
t	Time coordinate ... T.
u	Function related to concentration by Eq. (3a).
U_{∞}	A combination of the Bessel functions J_n and Y_n defined by Eq. (6b).
v	Function carrying z -dependence defined by Eq. (9).
$\bar{v}$	Laplace transform of v .
z	Longitudinal coordinate ... L.
α_{mn}	Parameter defined by Eq. (6c).
β_n	Parameter defined by Eq. (20b).
γ_n	Roots of U_{∞} .
ϵ_n	γ_n / b
λ	Laplace transform variable ... T^{-1} .

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- ξ Parameter defined by Eq. (16b).
- ρ Parameter defined by Eq. (19b).
- ϕ Function carrying radial-dependence defined by Eq. (16a).
- $\bar{\phi}$ Laplace transform of ϕ .

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