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SAUSAGE MODE OF A PINCHED CHARGED
PARTICLE BEAM

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SAUSAGE MODE OF A PINCHED CHARGED PARTICLE BEAM*

by

Edward P. Lee

ABSTRACT

The axisymmetric oscillations of a self-pinched charged particle beam are analyzed using a dispersion relation derived from a $3/2$ dimensional model. This calculation includes the effects of rounded profiles, finite conductivity, a steady return current, and phase mix damping among particle orbits. However, only the lowest order radial mode of distortion is treated, and this is done in an approximate fashion.

In the absence of a steady return current, it is found that a beam with no phase mix damping (flat profile) is unstable, and there is no linear bound on growth for a pulse of finite length. When damping is incorporated into the model, the beam exhibits either bounded growth (for finite pulse length) or purely damped oscillations, depending on the degree of profile rounding. In particular, a Gaussian profile beam is stable while a parabolic profile beam is unstable.

The effect of a steady return current is to weaken stability, with severe non-oscillatory growth predicted when the degree of current neutralization exceeds a threshold in the range (50 - 80%) depending on extent of current profile overlap. This mode enhancement is the consequence of positive free energy in the transverse degrees of freedom, which can drive expansion even when the condition for equilibrium is satisfied. However, due to the finite value of electrical conductivity the mode remains convective in character.

I. INTRODUCTION

The axisymmetric oscillations (sausage mode) of a self-pinched beam have been previously studied, both analytically and with particle simulations. Because of the wide range of models employed and conclusions drawn, it is convenient to summarize this work in tabular form (Table I). It is generally concluded that in the absence of return current, there will be unstable growth of the oscillations if the beam profile can be approximated as flat. Rounded

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TABLE 1. I_p = return current, σ = electrical conductivity, k = mean betatron frequency, $J_b(J_p)$ = beam (return) current density.

Calculation	Model	Conclusion
H. Lewis ¹ 1963 Analytical	$I_p = 0$ Flat profile Self similar expansion	Simple dispersion relation showing unbounded growth
S. Weinberg ² 1967 Analytical	$I_p = 0$ Flat profile Helical orbits	Unstable with no linear bound on growth in a finite length pulse
K. Watson ³ 1971	Common sense	Spread betatron frequency eliminates the sausage instability because it only occurs at high frequency.
E. Lee ⁴ 1973 Analytical	$I_p = 0$ spread k ($= \Delta k$) 1-d model	Unstable but bounded growth for small Δk . Stable for $\Delta k \geq 50\%$ k .
E. Lee and J. Leary ⁵ 1974 Simulation	$I_p = 0$ Flat profile Monoenergetic or helical orbits.	Unstable growth in agreement with Weinberg's calculation ⁽²⁾
E. Lee and J. Leary ⁵ 1974 Simulation	$I_p = 0$ Bennett profile Isothermal distribution.	Stable oscillations (Damped mode)
E. Lee and R. Cooper ⁶ 1976 Analytical	$I_p = 0$ J_b rounded $J_p = -(1 - f_m)J_b$ Constant f_m	Stable oscillations at frequency $= \sqrt{2} k$
M. Lampe and H. S. Uhm ⁷ 1980 Analytical	$I_p \neq 0$ Flat profile Monoenergetic	Unstable at zero frequency when $-I_p/I_b > 1/2$
F. Chambers and Masamitsu ⁸ 1980 Simulation	General 2-d code (RINGBEARER), realistic σ generation, initial conditions, etc.	Unstable for $-I_p/I_b \geq (.5 - .8)$
F. Chambers, J. Masamitsu, W. Barletta, K. Bruckner ⁹ 1980 Simulation	General and idealized return current models. J_p peaked on axis.	Beam hollowing observed. Equilibrium fails for $-I_p/I_b \geq .6$

beam profile results in phase mix damping of the oscillations which reduces or eliminates sausage growth. For example, simulation has shown the Bennett profile $[J_b(r) \propto (1 + r^2/a^2)^{-2}]$ to be stable. When return current is greater than about 50% of the beam current, the mode is predicted to be highly unstable, even though ample phase mix damping is present.

The experimental situation (as far as I am aware) is that no unstable sausage mode has ever been observed for a self pinched beam propagating in gas. Damped oscillations have been observed near the point of injection into gas for both Astron and FX-25 generated beams at LLNL. The observed stability of Astron is explained by the absence of return current during the long (300 ns) pulse. A bunched Astron pulse and an FX-25 pulse¹⁰ propagating in a few torr gas were subject to a large degree of current neutralization and would therefore seem to have violated the threshold condition for instability at zero frequency. However, it is likely that much of the return current in these experiments flowed at larger radius than the beam and the 50% threshold condition was inapplicable. A broad return current is expected for pressure in the range 1-10 torr, where electric breakdown considerably spreads conductivity. At moderate pressures (10-200 torr) sausage instability was also not observed in the FX-25 experiments, although the return current should have been concentrated close to the beam axis. A possible explanation is that the hose instability, which was present at these pressures, removed the pulse from the conducting channel before the sausage mode could grow to finite amplitude. Alternatively, sausage growth may have been present, but was not recognized due to the large hose amplitude.¹¹

The present concern over the sausage mode is generated by recent simulation results [F. Chambers and J. Masamitsu, and K. Bruckner], which indicate that under realistic conditions of propagation at moderate pressure (~200 torr) a beam will be expelled from its channel. Previously, the incorporation of realistic effects into propagation models worked in the direction of stability (e.g., rounded profiles were more stable than flat profiles). The recent simulations generate electrical conductivity through direct ionization and electric breakdown, which even at 200 torr pressure result in a large degree of current neutralization. In addition, the beam launch and head erosion dynamics act as a source of finite amplitude perturbations, which can be quickly amplified to an unacceptable magnitude if conditions for stability are not met.

In this report, an analytical calculation of the sausage mode growth rate is presented which (crudely) recovers all of the previous stability results [Table 1] in their appropriate domains of validity. The profile dependence appears through three "form factors," which may be evaluated for any particular choice of beam and plasma currents. The theory of the sausage mode is therefore unified into a consistent picture which is free from contradiction (except possibly with experiment); this adds strength to the predictions. Also, the derived dispersion relation is of simple enough form to be of aid in the interpretation of further simulation and experimental studies.

II. MODEL EQUATIONS

For the beam dynamics we adopt the 1-1 coupled envelope and emittance equations¹²

$$\frac{\partial^2 R}{\partial z^2} = \frac{E^2}{R^3} - \frac{k^2 r^2}{R}, \quad (1)$$

$$\frac{\partial E^2}{\partial z} = \frac{-2\alpha k^2 r^2 R^2}{\frac{E}{R} + \frac{k^2 r^2 R}{E}} \frac{\partial^2 R}{\partial z^2}. \quad (2)$$

Here R is the beam's rms radius and $E^2 = R^2 \overline{\theta^2}$ is the squared emittance with θ denoting the random angle of thermal motion. The independent variables are z and the "retarded time" $\tau = t - z/c$. Damping by phase mixing is modeled through Eq. (2) with a rate proportional to the profile dependent constant α [see Table 2];

$$\alpha = (0 - .5). \quad (3)$$

The betatron frequency $k(r, z, \tau)$ enters the dynamics only in the weighted average

$$\overline{k^2 r^2} = \frac{2 I_b}{I_A} \int_0^{\pi} d\tau \left(\frac{\partial}{\partial r} \frac{I_{br}}{I_b} \right) \left(\frac{I_r}{I_b} \right), \quad (4)$$

where I_r denotes net current contained inside radius r ; i.e., it is the sum of beam and plasma components

$$I_r = I_{br} + I_{pr} \quad (5)$$

with

$$I_{br} = \int_0^r dr' 2\pi r' J_b(r', z, \tau) , \quad (6)$$

and I_{pr} is similarly defined. I_A is the (constant) Alfvén current

$$I_A = \frac{8\gamma mc^3}{q} \approx (17 \text{ kA}) (\beta\gamma) \left(\frac{m}{m_e} \right) \left(\frac{t}{\tau} \right) , \quad (7)$$

and enters the theory as a dimensional scale factor. Note that all currents are defined with their true sign and the charge of an electron is $q_e = -e$.

It is clear that $I_b = I_{br} (r = \infty)$ is constant. We also assume that the net plasma current (I_p) is constant, but that the plasma current density varies in its radial distribution. That is, if we decompose the plasma current into steady and perturbed components

$$J_p = J_{p0}(r) + \Delta J_p(r, z, \tau) , \quad (8)$$

then the fixed return current is

$$I_p = \int_0^\infty dr 2\pi r J_{p0}(r) , \quad (9)$$

and the perturbed current satisfies

$$\int_0^\infty dr 2\pi r \Delta J_p(r, z, \tau) = 0 . \quad (10)$$

The justification for this assumption is that net current is nearly frozen-in over most of the beam pulse length due to the large value of the fundamental magnetic decay time

$$\tau_{\text{fund}} \approx \frac{\pi \sigma R^2}{c^2} L \quad (11)$$

where R_c is the channel radius, $\bar{\sigma}$ is an average electrical conductivity, and L is an inductance with magnitude in the range

$$L \approx \log \frac{b^2}{R_c^2} \approx 4-8 \quad (12)$$

for most experiments. The adjustments of profile (ΔJ_p) decay on the shorter time scale (by an order of magnitude)

$$\tau_m \approx \frac{\pi \bar{\sigma} R_0^2}{c^2} \quad (13)$$

where R_0 is the unperturbed beam radius. An accurate expression for τ_m is derived in Appendix A.

It is evident that the radial dependence of beam and plasma current profiles must be treated in order to evaluate the mean betatron frequency. Only the simple modifications of structure which can be approximated by self similar expansion are considered here. That is, we look at the lowest order perturbed mode and ignore the higher order (radial) modes, which cannot be treated with the envelope formalism anyway. A justification is that the conditions of stability are expected to be most severe for the lowest mode, higher modes being more strongly damped by phase mixing. For a perturbation of the rms radius

$$R = R_0 + \Delta R(z, \tau) \quad (14)$$

the self-similar beam current perturbation is

$$\Delta J_b = - \frac{\Delta R}{R_0} \frac{1}{r} \frac{\partial}{\partial r} r^2 J_{b0} \quad (15)$$

It is readily verified that this form conserves net beam current, and the perturbation of R satisfies

$$2R_0 \Delta R = \int_0^{\infty} dr \left(\frac{2\pi r \Delta J_b}{I_b} \right) r^2 = \Delta(R^2) \quad (16)$$

as required. The time development of the plasma current perturbation is modeled by the simple "circuit equation"

$$\frac{\partial \Delta J_p}{\partial \tau} + \frac{\Delta J_p}{\tau_m} = - \frac{\partial \Delta J_b}{\partial \tau} \quad (17)$$

Thus a change in beam profile induces a change in the plasma current profile, which then dies away in time τ_m . The form of Eq. (17) guarantees the conservation of net plasma current.

Electrical conductivity is also characterized by a radial profile $\sigma(r)$, however it is effectively removed from the calculation by the use of the constant decay time (τ_m) and specified profile $J_{p0}(r)$. These profiles are needed to evaluate the form factors which appear in the dispersion relation so the analysis may be regarded as having 3/2 dimensions plus retarded time ($z, 1/2 \tau; \tau$).

III. EQUILIBRIUM CONDITION

In order to satisfy Eq. (1) the unperturbed beam variables must satisfy the condition

$$\overline{(k^2 r^2)}_0 = \frac{E_0^2}{R_0^2} > 0 \quad (18)$$

In the present work E_0 and R_0 are considered to be independent of z and τ , although in reality they must at least vary slowly in the pulse body due to scattering, energy loss and changes in I_p . An important point is that the mean squared betatron frequency cannot be guaranteed to be positive in all physically realizable cases. If it is negative, then equilibrium fails and the beam is blown apart. A useful formula for $\overline{(k^2 r^2)}_0$ is readily derived from Eq. (4); integrating by parts and substituting $J_p = J - J_b$, one obtains (in general)

$$\begin{aligned} \overline{k^2 r^2} &= \frac{2I_b}{I_A} \left[\frac{I}{I_b} - \int_0^\infty dr \left(\frac{I_{br}}{I_b} \right) 2\pi r \left(\frac{J_b + J_p}{I_b} \right) \right] \\ &= \frac{I_b}{I_A} \left[1 + \frac{2I_p}{I_b} - 2 \int_0^\infty dr \left(\frac{I_{br}}{I_b} \right) \left(\frac{2\pi r J_p}{I_b} \right) \right]. \end{aligned} \quad (19)$$

We have for the unperturbed beam

$$(\overline{k^2 r^2})_0 = \frac{I_b}{I_A} \left[1 + (2 - C) \frac{I_p}{I_b} \right] \frac{I_b}{I_A} \Gamma, \quad (20)$$

where we define the form factor

$$C = 2 \int dr \left(\frac{I_{br0}}{I_b} \right) \left(\frac{2\pi r J_{p0}}{I_p} \right), \quad (21)$$

and the "effective current neutralization" is $f_m = 1 - \Gamma$. For equilibrium to be possible we must have $\Gamma > 0$, i.e.

$$-\frac{I_p}{I_b} < \frac{1}{2 - C}. \quad (22)$$

If J_{p0} has the same profile as J_{b0} , then $C = 1$ and $-I_p/I_b < 1$ is required (no surprise). A more interesting case is that of a spike return current on axis:

$$J_p = I_p \frac{\delta(r)}{2\pi r}. \quad (23)$$

Then $C = 0$ and $-I_p/I_b < 1/2$ is required. Physically, the failure of equilibrium corresponds to half of the beam particles being subjected to a repulsive force.

It is assumed here that the equilibrium condition [Eq. (22)] is satisfied. A mean betatron frequency is then defined as

$$\bar{k} = \sqrt{\frac{(\overline{k^2 r^2})_0}{R_0^2}} = \sqrt{\frac{I_b}{I_A} \frac{\Gamma}{R_0^2}}. \quad (24)$$

IV. PERTURBED EQUATIONS AND DISPERSION RELATION

Equations for the perturbed quantities δR , $\delta(E^2)$, and $\delta(k^2 r^2)$ are obtained by linearizing Eqs. (1), (2), and (19) around the equilibrium values; we have

$$\frac{\partial^2 \delta R}{\partial z^2} = - \frac{3E_0^2 R}{R_0^4} + \frac{\Delta(E^2)}{R_0^3} + \frac{(k^2 r^2)_0 \Delta R}{R_0^2} - \frac{\Delta(k^2 r^2)}{R_0} \quad (25)$$

$$\frac{\partial \Delta(E^2)}{\partial z} = - \frac{2\alpha (k^2 r^2)_0 R_0^2 (\partial^2 \delta R / \partial z^2)}{\frac{E_0}{R_0} + \frac{(k^2 r^2)_0 R_0}{E_0}} \quad (26)$$

$$\Delta(k^2 r^2) = - \frac{2I_b}{I_A} \int_0^r dr \left[\frac{\Delta I_{br}}{I_b} \left(\frac{2\pi r J_{p0}}{I_b} \right) + \frac{I_{br0}}{I_b} \left(\frac{2\pi r \Delta J_p}{I_b} \right) \right] \quad (27)$$

Equations for ΔJ_b and ΔJ_p have already been given [Eqs. (15) and (17)], and by integrating Eq. (15) once in r we obtain

$$\Delta I_{br} = \int_0^r dr' 2\pi r' \Delta J_b' = - \frac{\Delta R}{R_0} 2\pi r^2 J_{b0} \quad (28)$$

In order to simplify this system, the unperturbed emittance is eliminated using the equilibrium condition [Eq. (18)]. Further, the (0) notation for unperturbed quantities may now be dropped.

A dispersion relation for the sausage mode is obtained by assuming the free wave form

$$\delta R = \exp(-i\Omega kz + p\tau/\tau_L) \quad (29)$$

with $p \equiv -i\omega$. The imaginary parts of Ω and ω are regarded as positive at this stage. Equations (25) and (26) become

$$-\Omega^2 \frac{\delta R}{R} = -3 \frac{\delta R}{R} + \frac{\Delta(E^2)}{k^2 R^4} + \frac{\delta R}{R} - \frac{\Delta(k^2 r^2)}{k^2 R^2} \quad (30)$$

$$-i\Omega \frac{\Delta(E^2)}{k^2 R^4} = \alpha\Omega^2 \frac{\Delta R}{R} , \quad (31)$$

and from Eqs. (15) and (17)

$$\Delta J_p = \frac{-p}{1+p} \Delta J_b = \frac{p}{1+p} \frac{\Delta R}{R} \frac{1}{r} \frac{\partial}{\partial r} r^2 J_b . \quad (32)$$

Substituting from Eqs. (28) and (32) into Eq. (27) and integrating by parts in the second term yields

$$\Delta(k^2 r^2) = \frac{\Delta R}{R} \frac{I_b}{I_A} \int_0^\infty dr \, 8\pi r^3 \left(\frac{J_b J_p}{I_b^2} + \frac{p}{1+p} \frac{J_b^2}{I_b^2} \right) . \quad (33)$$

It is convenient to write this expression in the form

$$\Delta(k^2 r^2) = \frac{\Delta R}{R} \frac{I_b}{I_A} A \left(B \frac{I_p}{I_b} + \frac{p}{1+p} \right) , \quad (34)$$

where A and B are the constant form factors

$$A = \int_0^\infty dr \, 8\pi r^3 \left(\frac{J_b}{I_b} \right)^2 , \quad (35)$$

$$B = \frac{1}{A} \int_0^\infty dr \, 8\pi r^3 \left(\frac{J_b}{I_b} \frac{J_p}{I_p} \right) . \quad (36)$$

Equations (31) and (34) are used to eliminate $\Delta(E^2)$ and $\Delta(k^2 r^2)$ from Eq. (30). For a non-trivial solution, the coefficient of $\Delta R/R$ must vanish, this yields the dispersion relation

$$0 = -D(p, \Omega) = \Omega^2 + i\alpha\Omega - 2 - \frac{A}{\Gamma} \left(B \frac{I_p}{I_b} + \frac{p}{1+p} \right) . \quad (37)$$

More generally, $D(p, \Omega)$ appears as the beam response function for an imposed perturbation or initial conditions [see Section V].

At this point it is useful to recall the definitions of the various factors which appear in the dispersion relation and to write it in a form which is convenient for the stability analysis. Three shape dependent constants (A, B, and C) have been defined. The constant A, which depends only on J_b , is a measure of roundness and is a maximum for the flat profile ($A_{\text{flat}} = 2$). Values of A calculated for several beam profiles are given in Table 2. The constants B and C depend on the return current profile and are defined such that they equal unity whenever $J_p/I_p = J_b/I_b$ (independent of magnitudes). The factor $\Gamma = [1 + (2 - C) I_p/I_b] = (1 - f_m)$ enters through the equilibrium condition. All of these factors may be lumped into two quantities appearing in D:

$$S = (2\Gamma + A + AB I_p/I_b)/\Gamma, \quad (38)$$

$$T = A/\Gamma. \quad (39)$$

We have the two equivalent forms for the dispersion function:

$$D(p, \Omega) = S - \frac{T}{p + 1} - (\Omega^2 + i\alpha\Omega) \quad (40a)$$

$$= \frac{T [p - p(\Omega)]}{(1 + p) [1 + p(\Omega)]}, \quad (40b)$$

where

$$p(\Omega) = \frac{(\Omega^2 + i\alpha\Omega) - (S - T)}{S - (\Omega^2 + i\alpha\Omega)}. \quad (41)$$

The function $p(\Omega)$ is the growth rate in τ/τ_m for a pulse driven with real frequency (Ω). It is evident that there will be growth when $\alpha = 0$ for $(S - T) < \Omega^2 < S$. However, sufficiently large α can eliminate this. A qualitative change in mode growth is indicated when $S - T$ becomes negative (due to large enough I_p/I_b). Growth with $\Omega = 0$ is then possible and cannot be eliminated by finite α .

V. STABILITY ANALYSIS

To apply the derived dispersion relation without ambiguity it is useful to evaluate the perturbed amplitude resulting from initial conditions imposed at $z = 0$. The perturbation is turned on at $\tau = 0$, and in general we may

specify the two functions ΔR and $\partial \Delta R / \partial z$ at $z = 0$ for $\tau \geq 0$. It is also possible to specify a perturbed component of current at $\tau = 0$ for $z \geq 0$; however, since there is no causal basis for such a current, it is not considered here. This is guaranteed by the form of Eq. (32).

For convenience we use dimensionless variables:

$$\Delta R/R \rightarrow \Delta R, \quad kz \rightarrow z, \quad \tau/\tau_m \rightarrow \tau \quad (42)$$

Then the disturbance of beam radius has the form

$$\Delta R = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \int_{-i\infty}^{+i\infty} \frac{dp e^{-i\Omega z + p\tau}}{2\pi i} \frac{F(p, \Omega)}{D(p, \Omega)}, \quad (43)$$

with the initial conditions contained in the function $F(p, \Omega)$. Integration paths in Eq. (43) pass above (Ω) or to the right (p) of any singularities. Equations (25) and (26) yield (after some algebra)

$$F = \left(1 + \frac{i\alpha}{\Omega}\right) \int_0^{\infty} d\tau e^{-p\tau} \left[\frac{\partial \Delta R}{\partial z}(z=0, \tau) - i\Omega \Delta R(z=0, \tau) \right] \quad (44)$$

We examine the response to an impulsive disturbance at time $\tau = \tau_0 > 0$:

$$\Delta R(z=0, \tau) = \delta(\tau - \tau_0) \quad (45a)$$

$$\frac{\partial \Delta R}{\partial z}(z=0, \tau) = 0 \quad (45b)$$

where $\delta(\tau - \tau_0)$ is the Dirac delta function. Equation (44) yields

$$F = -i(\Omega + i\alpha) e^{-p\tau_0} \quad (46)$$

Inserting this expression and the expression for D [Eq. (40b)] into Eq. (43) gives

$$\Delta R = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \int_{-i\infty}^{+i\infty} \frac{dp e^{-i\Omega z + p\tau}}{2\pi i} \left[(-i)(\Omega + i\alpha) e^{-p\tau_0} \right] \frac{(1+p)[1+p(\Omega)]}{T[p-p(\Omega)]} \quad (47)$$

The integration over p is performed first; we have

$$\begin{aligned} \int_{-i\infty}^{+i\infty} \frac{dp}{2\pi i} e^{p(\tau - \tau_0)} \frac{1+p}{p - p(\Omega)} &= \int_{-i\infty}^{+i\infty} \frac{dp}{2\pi i} e^{p(\tau - \tau_0)} \left[1 + \frac{1+p(\Omega)}{p - p(\Omega)} \right] \\ &= \delta(\tau - \tau_0) + e^{i\Omega(\tau - \tau_0)} [1 + p(\Omega)] H(\tau - \tau_0) , \end{aligned} \quad (48)$$

where $H(\tau - \tau_0)$ is the unit step at τ_0 . It is apparent that the beam response can be represented as the sum of a propagating initial condition and an induced wave:

$$\Delta R = \Delta R_i(z) \delta(\tau - \tau_0) + \Delta R_+(z, \tau) H(\tau - \tau_0) , \quad (49)$$

with

$$\Delta R_i = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi i} e^{-i\Omega z} (\Omega + i\alpha) \frac{1+p(\Omega)}{T} , \quad (50)$$

$$\Delta R_+ = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi i} e^{-i\Omega z + p(\Omega)(\tau - \tau_0)} (\Omega + i\alpha) \frac{[1+p(\Omega)]^2}{T} . \quad (51)$$

Without loss of generality we set $\tau_0 = 0$ for the remainder of this study.

The evaluation of ΔR is greatly simplified by a change of variable.

Note that we may write

$$1 + p(\Omega) = \frac{T}{s - (\Omega^2 + i\alpha\Omega)} = \frac{T}{\left(s - \frac{\alpha^2}{4}\right) - \left(\Omega + \frac{i\alpha}{2}\right)^2} . \quad (52)$$

Defining

$$U^2 = s - \frac{\alpha^2}{4} \quad (\text{assumed positive}) , \quad (53)$$

$$\Omega' = \left(\Omega + \frac{i\alpha}{2}\right)/U , \quad (54)$$

Equation (52) becomes

$$1 + p(\Omega) = \frac{T/U^2}{1 - \Omega'^2} \quad , \quad (55)$$

and expressions (50) and (51) become

$$\Delta R_i = \int_{-\infty}^{+\infty} \frac{d\Omega'}{2\pi i} \exp\left(\frac{-\alpha z}{2} - i\Omega' U z\right) \left(\Omega' + \frac{i\alpha}{2U}\right) \frac{1}{1 - \Omega'^2} \quad , \quad (56)$$

$$\Delta R_+ = \frac{T}{U^2} \int_{-\infty}^{+\infty} \frac{d\Omega'}{2\pi i} e^g \left(\Omega' + \frac{i\alpha}{2U}\right) \frac{1}{(1 - \Omega'^2)^2} \quad , \quad (57)$$

with

$$g(z, \tau, \Omega') = -\frac{\alpha z}{2} - \tau - i\Omega' U z + \frac{T\tau/U^2}{1 - \Omega'^2} \quad . \quad (58)$$

The propagating initial condition is readily evaluated by the method of residues to be

$$\Delta R_i(z) = e^{-\alpha z/2} \left[\cos Uz + \frac{\alpha}{2U} \sin Uz \right] \quad , \quad (59)$$

which is just a decaying oscillation of frequency U .

The induced wave $[\Delta R_+(z, \tau)]$ contains the unstable response (if any) and is much more difficult to evaluate in detail. Its amplitude immediately behind the propagating initial condition is evaluated by the method of residues to be

$$\Delta R_+(z, \tau = 0+) = \frac{T}{4U^2} e^{-\alpha z/2} [2Uz \sin(Uz) - z\alpha \cos(Uz) + \frac{\alpha}{U} \sin(Uz)] \quad . \quad (60)$$

This disturbance initially increases quadratically with z ($\sim Tz^2/2$), but at large z exhibits decaying oscillations of frequency U .

For general $\tau > 0$ an exact evaluation of ΔR_+ has not been found, so we resort to approximation by the method of steepest descents. A sketch of the

topography of the complex Ω' plane associated with the factor e^{δ} is given in Fig. 1. Note that there are intrinsic singularities at $\Omega' = \pm 1$ and the deformed contour must pass above these. To determine the saddle points, we first write $g(z, \tau, \Omega')$ in the form

$$g = U z \left[- \left(\frac{\alpha}{2U} + \frac{2 U^2 \delta^2}{T} \right) + G(\delta, \Omega') \right] , \quad (61)$$

where

$$\delta = \sqrt{\frac{T \tau}{2 U^3 z}} , \quad (62)$$

and

$$G(\delta, \Omega') = -i\Omega' + \frac{2 \delta^2}{1 - \Omega'^2} . \quad (63)$$

The saddle points are the solutions of

$$0 = \frac{\partial g}{\partial \Omega'} = -i + \frac{4 \delta^2 \Omega'}{(1 - \Omega'^2)^2} . \quad (64)$$

Any positive value of δ identifies a ray in the (z, τ) plane, and there are four associated saddle points [see Fig. 1]. Two of these saddles always lie in the lower Ω' half-plane and make little contribution to ΔR_+ ; we will only be concerned with the pair which lie in the upper half plane. It is found that when δ is less than a critical value ($\delta_c = 2/3^{3/4}$), that the upper saddle pair is located between the singularities and the point $\Omega'_c = i/\sqrt{3}$. The deformed contour passes through both of the upper saddle points in this case. However, when $\delta = \delta_c$, the upper saddle pair coalesces on the imaginary axis (at Ω'_c). For $\delta > \delta_c$, these saddles both lie on the positive imaginary axis, and the deformed contour (path of steepest descent) passes through the upper one.

To aid in the saddle analysis, we denote a saddle point by

$$\Omega'_s = \psi + i\lambda . \quad (65)$$

Then

FIG. 1a. Topography of the Ω' plane for $\delta > \delta_c$.

FIG. 1b. Topography of the Ω' plane for $\delta < \delta_c$.

$$\operatorname{Re} G(\Omega'_s) = \lambda + \frac{2 \delta^2 (1 - \psi^2 + \lambda^2)}{(1 - \psi^2 + \lambda^2)^2 + 4\psi^2 \lambda^2}, \quad (66)$$

$$\operatorname{Im} G(\Omega'_s) = -\psi + \frac{4 \delta^2 \psi \lambda}{(1 - \psi^2 + \lambda^2)^2 + 4\psi^2 \lambda^2}. \quad (67)$$

The saddle condition becomes

$$0 = \operatorname{Re} \frac{\partial G}{\partial \Omega'_s}(\Omega'_s) = \frac{4\delta^2 \{ \psi[(1 - \psi^2 + \lambda^2)^2 - 4\psi^2 \lambda^2] - 4\lambda^2 \psi(1 - \psi^2 + \lambda^2) \}}{[(1 - \psi^2 + \lambda^2)^2 - 4\psi^2 \lambda^2]^2 + 16\psi^2 \lambda^2 (1 - \psi^2 + \lambda^2)^2}, \quad (68)$$

$$0 = \operatorname{Im} \frac{\partial G}{\partial \Omega'_s}(\Omega'_s) = -\psi + \frac{4 \delta^2 \{ 4\psi^2 \lambda(1 - \psi^2 + \lambda^2) + \lambda[(1 - \psi^2 + \lambda^2)^2 - 4\psi^2 \lambda^2] \}}{[(1 - \psi^2 + \lambda^2)^2 - 4\psi^2 \lambda^2]^2 + 16\psi^2 \lambda^2 (1 - \psi^2 + \lambda^2)^2}. \quad (69)$$

Unfortunately, G_s and Ω'_s cannot be written as simple algebraic functions of δ . However, these equations can be solved parametrically in terms of λ . One finds after considerable algebra:

Case of $\delta > \delta_c = 2/3^{3/4}$:

$$\psi = 0, \quad (70)$$

$$\delta = \frac{1 + \lambda^2}{2\sqrt{\lambda}}, \quad (71)$$

$$\operatorname{Im} G(\Omega'_s) = 0, \quad (72)$$

$$\operatorname{Re} G(\Omega'_s) = \frac{3\lambda}{2} + \frac{1}{2\lambda}. \quad (73)$$

The critical (coalescence) point is $\lambda_c = 1/\sqrt{3}$, with $\operatorname{Re} G(\Omega'_s) = \sqrt{3}$.

Case of $\delta < \delta_c$:

$$\psi = \pm (1 + \lambda^2 - 2\lambda \sqrt{1 + \lambda^2})^{1/2}, \quad (74)$$

$$\delta = \sqrt{2} \lambda(1 + \lambda^2)^{1/4}, \quad (75)$$

$$\operatorname{Im} G(\Omega'_s) = -\psi(\lambda) \left[\frac{1 - 2\lambda/\sqrt{1+\lambda^2}}{1 - \lambda/\sqrt{1+\lambda^2}} \right], \quad (76)$$

$$\operatorname{Re} G(\Omega'_s) = \lambda \left[\frac{2 - \lambda/\sqrt{1+\lambda^2}}{1 - \lambda/\sqrt{1+\lambda^2}} \right]. \quad (77)$$

Both $\operatorname{Re} G(\Omega'_s)$ and δ^2 are increasing functions of λ through the range $0 < \lambda < \infty$, so a one-to-one relationship is established.

For large and small δ we find the limits

$$\operatorname{Re} G_s \rightarrow \left(\frac{27}{2} \delta^2 \right)^{1/3} \quad (\delta^2 \rightarrow \infty), \quad (78a)$$

$$\operatorname{Re} G_s \rightarrow (2 \delta^2)^{1/2} \quad (\delta^2 \rightarrow 0). \quad (78b)$$

An approximate formula for $\operatorname{Re} G$ (good to $\pm 8\%$) is

$$\operatorname{Re} G_s \approx \frac{1}{1 + \delta^2} \left[(2 \delta^2)^{1/2} + \delta^2 \left(\frac{27}{2} \delta^2 \right)^{1/3} \right]. \quad (79)$$

The important questions are whether there is unstable growth, what is the maximum amplitude, and where does it occur? To answer these questions, we find the maximum of $\operatorname{Re} g_s$ for fixed τ , i.e., we follow an arbitrary but fixed beam segment and determine the maximum amplitude of the perturbation it suffers. The condition for a maximum is [using Eqs. (61)-(64)]

$$0 = \frac{\partial}{\partial z} \operatorname{Re} g(z, \tau, \Omega'_s) = \frac{\partial \operatorname{Re} g}{\partial z} + \frac{\partial \operatorname{Re} g}{\partial \Omega'_s} \frac{\partial \Omega'_s}{\partial z} = -\frac{\alpha}{2} + U\lambda. \quad (80)$$

Maximum amplitude for fixed τ (denote by $\tilde{\lambda}$) therefore occurs with the unique value

$$\tilde{\lambda} = \alpha/2U, \quad (81)$$

i.e. the saddle point lies on the real Ω axis (independent of $\lambda \leq \lambda_c = 1/\sqrt{3}$). The condition that the maximum occurs with $\tilde{\delta} \leq \delta_c$ is therefore

$$\frac{\alpha}{2U} \leq \lambda_c = 1/\sqrt{3}, \quad (82)$$

which yields (using $U^2 = S - \alpha^2/4$)

$$\alpha^2 \leq S . \quad (83)$$

For the case where maximum amplitude occurs with $\delta > \delta_c$ we find, with the aid of Eqs. (61), (62), (71), (73), and (81)

$$\text{Re } g_{\max} = \tau(T/S - 1) , \quad (84)$$

so there is growth when

$$T > S . \quad (85)$$

This is just the condition for the appearance of a zero frequency mode. The ray of maximum growth is

$$\tau/z = S^2/\alpha T . \quad (86)$$

For maximum amplitude occurring with $\delta < \delta_c$ we find from Eqs. (61), (62), (75), (77) and (81),

$$\text{Re } g_{\max} = \tau \left[\frac{\frac{\alpha^2 + T}{2\alpha\sqrt{S}} - 1}{1 - \frac{\alpha}{2\sqrt{S}}} \right] . \quad (87)$$

The condition for growth is

$$\frac{\alpha^2 + T}{2\alpha\sqrt{S}} > 1 , \quad (88)$$

and the ray of maximum growth is

$$\tau/z = \alpha^2 \sqrt{S}/T . \quad (89)$$

Eqs. (84) - (86) connect continuously with Eqs. (87) - (89) at the critical point (where $\alpha = \sqrt{S}$).

To avoid confusion, it should be noted that the criterion of growth is that $\text{Re } g$ be positive for fixed τ . Typically, $\text{Re } g$ will be negative for small z and increase to a maximum which is either positive or negative and be located in the oscillatory or non-oscillatory zone according to the conditions given here.

An important point is that the sausage mode is convective—an impulsive disturbance eventually dies out if the pulse has finite length (in τ). To see this we note that for large z ,

$$\delta^2 = \frac{T\tau}{2U^3 z} \rightarrow 0, \quad (90)$$

and by Eqs. (61) and (78b)

$$\text{Re } g_s \rightarrow Uz \left[-\left(\frac{\alpha}{2U} + \frac{2U^2 \delta^2}{\tau} \right) + \sqrt{2\delta^2} \right] \quad (91a)$$

$$\rightarrow -\frac{\alpha z}{2}. \quad (91b)$$

For the special case of $\alpha = 0$ (no damping—flat profile)

$$\text{Re } g \rightarrow Uz \sqrt{2\delta^2} = \sqrt{\frac{T\tau z}{s^{1/2}}}; \quad (92)$$

there is continued growth but at a decreasing rate.

Finally, an explicit formula for ΔR_+ is obtained from Eq. (57) by carrying through the integration in Ω close to the saddle point(s). Details are omitted here. For the case $\delta > \delta_c$ one finds

$$\Delta R_+ = \frac{T}{U^2} \frac{e^{g_s}}{\sqrt{2\pi}} \frac{(\lambda + \alpha/2U)}{\sqrt{Uz}} \sqrt{\frac{\lambda}{(1 + \lambda^2)^3 (3\lambda^2 - 1)}}, \quad (93)$$

with g_s given by Eqs. (61) and (73). This formula breaks down as $\lambda \rightarrow \lambda_c = 1/\sqrt{3}$. In the limit $\delta \gg \delta_c$ Eq. (93) becomes

$$\Delta R_+ = \frac{T}{U^2} \frac{e^{g_s}}{\sqrt{2\pi}} \left[\sqrt{3} (4\delta^2)^{5/6} (Uz)^{1/2} \right]^{-1}, \quad (94)$$

with

$$g_s = U_z \left[\frac{-\alpha}{2U} - \frac{2U^2}{T} \delta^2 + \frac{3}{2} (4 \delta^2)^{1/3} - \frac{1}{2} (4 \delta^2)^{-1/3} \right] \quad (95)$$

The saddle analysis is valid in this limit when

$$U_z (\delta^2)^{2/3} \propto \left(\frac{T}{U^2} \right)^{2/3} (U_z)^{1/3} \gg 1 \quad (96)$$

Due to mathematical complexity, ΔR_+ has not been worked out for the case $\delta < \delta_c$.

VI. SUMMARY OF RELATIONS

Notation:

$$\Delta R = \exp(p\tau/\tau_m - i\bar{\omega}kz)$$

$p, \bar{\omega}$

$$\tau = t - z/\beta c$$

$$\tau_m = \pi \bar{\omega} R^2/c^2$$

$$\bar{k} = (I_b / R^2 I_A)^{1/2}$$

$$(I_b, I_p, I_A)$$

R

$\bar{\omega}$

$$f_m = (1 - \Gamma)$$

free wave

dimensionless frequencies $\leftrightarrow$ Laplace transform variables

retarded time

magnetic decay time of perturbation

mean betatron frequency

constant currents (beam, plasma, Alfvén)

rms radius

mean electrical conductivity

effective current neutralization factor

Form Factors:

$$A = \int dr 8\pi^2 r^3 J_b^2 / I_b^2$$

$$B = 1/A \int dr 8\pi^2 r^3 J_b J_p / I_b I_p$$

$$C = 2 \int dr (I_{br} / I_b) 2\pi r J_p / I_p$$

$$\Gamma = 1 + (2 - C) I_p / I_b > 0$$

$$S = 2 + (A + AB I_p / I_b) / \Gamma$$

$$T = A/\Gamma$$

$$\alpha = (0 - .5)$$

roundedness factor

perturbation form factor

equilibrium form factor

equilibrium condition

frequency form factor

frequency form factor

damping constant

Dispersion Relation:

$$D(p, \Omega) = S - T/(p+1) - (\Omega^2 + i\alpha\Omega) = 0 \quad \text{dispersion relation}$$

$$= T[p - p(\Omega)]/(1+p)[1 + p(\Omega)] \quad \text{alternate form}$$

$$p(\Omega) = [(\Omega^2 + i\alpha\Omega) - (S - T)]/[S - (\Omega^2 + i\alpha\Omega)]$$

$$U = \sqrt{S - \alpha^2/4}$$

growth rate in τ

useful definition

$$AR \propto \exp - (\alpha/2 \pm iU)\bar{k}z$$

damped free oscillation (single beam segment)

$$g(\tau, z, \Omega) = p(\Omega) \tau/\tau_m - i\Omega\bar{k}z$$

growth factor

Variable Changes:

$$\tau/\tau_m \rightarrow \tau, \quad \bar{k}z \rightarrow z$$

$$\Omega' = (\Omega + i\alpha/2)/U$$

$$\delta = \sqrt{T/2} U^3 z$$

defines ray in (z, τ) plane

$$g = -\alpha z/2 - \tau - i\Omega' U z + (T\tau/U^2)/(1 - \Omega'^2)$$

$$= U z [- (\alpha/2U + 2U^2 \delta^2/T) + G]$$

$$G = -i\Omega' + 2\delta^2/(1 - \Omega'^2)$$

Saddles in upper Ω' 1/2-plane:

$$\Omega'_s = \psi + i\lambda$$

saddle point

Character of Ω'_s is determined by value of δ compared with "critical value" $\delta_c = 2/3^{3/4}$. λ is an increasing function of δ .

$$\infty > \delta > \delta_c:$$

$$\psi = 0$$

$$\lambda \rightarrow (2\delta)^{2/3}$$

$$\text{Re } G_s \rightarrow (27 \delta^2/2)^{1/3}$$

$$\left. \begin{array}{l} \psi = 0 \\ \lambda \rightarrow (2\delta)^{2/3} \\ \text{Re } G_s \rightarrow (27 \delta^2/2)^{1/3} \end{array} \right\} \delta \rightarrow \infty$$

$$\delta = \delta_c:$$

$$\psi = 0$$

$$\lambda = \lambda_c = 1/\sqrt{3}$$

$$\text{Re } G_s = \sqrt{3}$$

$$0 < \delta < \delta_c:$$

$$\psi = \pm (0 - 1)$$

$$\lambda \rightarrow \delta/\sqrt{2}$$

$$\text{Re } G_s \rightarrow (2 \delta^2)^{1/2}$$

$$\left. \begin{array}{l} \psi = \pm (0 - 1) \\ \lambda \rightarrow \delta/\sqrt{2} \\ \text{Re } G_s \rightarrow (2 \delta^2)^{1/2} \end{array} \right\} \delta \rightarrow 0$$

Maximum of $\text{Re } g_s$ for fixed τ (denoted by $\cdot$):

$$\lambda = \alpha/2U \quad \text{saddle value}$$

- equivalent to Ω_s on real axis--

$$\alpha^2 \lesssim S \leftrightarrow \delta \lesssim \delta_c$$

$$\delta > \delta_c \quad (\alpha^2 > S):$$

$$\text{Re } g_{\max} = \tau(T/S - 1)$$

maximum growth factor

$$T > S$$

growth condition

$$\tau/z = S^2/\alpha T$$

ray of maximum growth

$$\delta < \delta_c \quad (\alpha^2 < S):$$

$$\text{Re } g_{\max} = \tau[(\alpha^2 + T)/(2\alpha \sqrt{S}) - 1]/(1 - \alpha/2\sqrt{S})$$

maximum growth factor

$$(\alpha^2 + T)/2\alpha \sqrt{S} > 1$$

growth condition

$$\tau/z = \alpha^2 \sqrt{S}/T$$

ray of maximum growth

VII. DISCUSSION

An analytical theory of the sausage mode has been developed which takes into account the detailed radial profiles of J_b and J_p ; these, along with the ratio of net plasma to beam currents ($-I_p/I_b$) largely determine the stability character of a pulse. Profile dependence appears in the formulas for growth rate, etc. through several dimensionless constants which may be evaluated if profiles are known--e.g., from simulation. Constants A and α characterize beam profile alone, while B and C depend on the degree of overlap of beam and plasma current profiles. Dimensional factors are the mean betatron frequency ($\bar{k}$) for z scale and magnetic decay time (τ_m) for retarded time ($\tau = t - z/\beta c$). The derived dispersion function [Eqs. (40a, b)] is of a very simple form containing only three dimensionless parameters (α , S, T) derived from the original set of 5. Essentially, S is the natural resonance

frequency with frozen field and $S = T$ is the natural frequency with frozen plasma current density. Specific conclusions follow.

(a) The sausage mode exhibits either convective growth or damping (depending on the values of α , S , and T). The possibility of absolute instability exists only for the case $\alpha = 0$, which corresponds to an (unphysical) absence of phase mix damping.

(b) The mode has an oscillatory or non-oscillatory character depending on the observed zone of (z, τ) space. There is a critical ray ($\tau/z = \text{constant}$) which separates these zones; for τ/z above this ray the mode is non-oscillatory.

(c) A fixed segment of beam ($\tau = \text{constant}$) passes through both zones as it propagates to large z . If $\alpha^2 > S$ then the maximum of growth (if any) appears in the non-oscillatory zone and the maximum growth factor is $g_{\max} = (\tau/\tau_m)(T/S - 1)$. If $\alpha^2 < S$ then the maximum growth appears in the oscillatory zone with

$$g_{\max} = \frac{\tau}{\tau_m} \left[\frac{\frac{\alpha^2 + T}{2\alpha\sqrt{S}} - 1}{1 - \frac{\alpha}{2\sqrt{S}}} \right].$$

However, for sufficiently large z the mode damps, as required for convective instability.

(d) The threshold for non-oscillatory growth is signaled by the parameter relation $T = S$. This may be realized by raising the ratio of plasma to beam current to a value typically in the range .5 - .8, which is lower than the value required for equilibrium to fail. This threshold is equivalent to the appearance of a "zero frequency mode" of the system, i.e., the growth rate in τ is $p(\Omega = 0) = 0$. When $T > S$ positive kinetic energy (in the transverse plane) is generated by an expansion of the pulse if the plasma current profile does not change in the process. However, a perturbed plasma current is generated by expansion, and only dies away in time scale τ_m . The effect is to insure a convective nature for the mode instead of catastrophic (absolute) instability.

(e) The condition for oscillatory growth is also dependent on the degree of current neutralization. An example is the Gaussian Profile beam, which shows only damped oscillation in the range $0 < -I_p/I_b < .16$ (if beam and plasma currents have the same profiles). In the range $.16 < -I_p/I_b < .67$

there is growth which peaks in the oscillatory zone, and for $.67 < -I_p/I_b < 1$ there is also growth in the non-oscillatory zone. These threshold conditions are given for several other profiles in Table 3. If the ratio of plasma current to beam current radius is varied, then the thresholds change. Figure 2 plots this dependence for Gaussian beam and plasma current profiles--it is intuitively clear that increase plasma current radius aids stability. A recent report¹³ gives a more thorough analysis of these conditions and a comparison with simulation.

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FIG. 2. Threshold conditions for Gaussian profile beam and plasma currents. Critical values of $(-I_p/I_b)$ are plotted vs R_p/R_b .

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APPENDIX I CALCULATION OF τ_m

The free decay of the perturbed component of the plasma current is described by the diffusion equation

$$\nabla_{\perp}^2 \Delta A - \frac{4\pi\sigma(r)}{c^2} \frac{\partial \Delta A}{\partial \tau} = 0, \quad (A-1)$$

where A is the z component of the vector potential, and

$$\Delta J_p = -\frac{c}{4\pi} \nabla_{\perp}^2 \Delta A. \quad (A-2)$$

For the perturbed component we assume the single mode form

$$\Delta A = (\text{function of } r) e^{-\tau/\tau_m}. \quad (A-3)$$

We then obtain from Eq. (A-1) the variational expression

$$\tau_m = \frac{\int d^2 r \frac{4\pi\sigma}{c^2} (\Delta A)^2}{\int d^2 r |\nabla_{\perp} \Delta A|^2}. \quad (A-4)$$

That is, if an approximate form for ΔA is selected, say

$$\Delta A_{\text{approx}} = \Delta A_{\text{true}} + (\Delta A)^1, \quad (A-5)$$

then Eq. (A-4) gives τ_m correctly up to terms of order $[(\Delta A)^1]^2$.

We select the self similar form

$$(\Delta A)_{\text{approx}} = (I_b - I_{br}), \quad (A-6)$$

so the perturbation contains no net current. From Eq. (A-6) we find

$$\frac{\partial \Delta A_{\text{approx}}}{\partial \tau} = -2\pi r J_b, \quad (A-7)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Delta A_{\text{approx}}}{\partial r} = -2\pi \frac{1}{r} \frac{\partial}{\partial r} r^2 J_b. \quad (A-8)$$

Inserting expressions (A-6) and (A-7) into Eq. (4) yields

$$\tau_m = \frac{\int dr r \frac{4\pi\sigma(r)}{c^2} (I_b - I_{br})^2}{\int dr r (2\pi r J_b)^2} \quad (A-9)$$

We evaluate τ_m assuming that σ and J_b profiles are the same. First define the scale quantity

$$\tilde{\tau}_m \equiv \int dr \frac{2\pi\sigma(r)}{c^2} = \frac{1}{R_c^2} \quad (A-10)$$

where R_c is a channel resistance and is considered to be a known constant independent of the profiles. Then we have

$$\frac{\tau_m}{\tilde{\tau}_m} = \frac{\int dr r \frac{4\pi\sigma}{c^2} (I_b - I_{br})^2}{\left(\int dr \frac{2\pi\sigma}{c^2} \right) \int dr r (2\pi r J_b)^2} \quad (A-11)$$

$$= \frac{2 \int dr 2\pi r J_b (I_b - I_{br})^2}{\left(\int dr 2\pi r J_b \right) \int dr r (2\pi r J_b)^2} \quad (A-12)$$

$$= \frac{-\frac{2}{3} \int dr \frac{\partial}{\partial r} (I_b - I_{br})^3}{(I_b) \left(\frac{A I_b^2}{2} \right)} = \frac{4}{3A} \quad (A-13)$$

where A is the structure factor previously defined in the report:

$$A = \int dr \frac{8\pi^2 r^3 J_b^2}{I_b^2} \quad (A-14)$$

There is thus a simple inverse relation between τ_m and A , which is a measure of profile rounding. The more rounded the profile, the smaller is A and the larger is τ_m . For Gaussian J_b and σ we find $\tau_m = 4\pi\sigma(o)R^2/3 c^2$. The mean conductivity ($\bar{\sigma}$) previously mentioned may be defined as

$$\pi R^2 \bar{\sigma} = \frac{1}{R_c^2} \quad (A-15)$$

and we have

$$\tau_m = \frac{4}{3A} \bar{\tau}_m = \frac{4}{3A} \frac{1}{c^2} = \frac{4}{3A} \frac{\pi R^2 \sigma}{c^2} \quad (A-16)$$

For Gaussian $\sigma(r)$ and $J_b(r)$, $A = 1$ and $\bar{\sigma} = \sigma(0)$.

TABLE 2. Beam profile parameters.

Profile Name	$J_b(r)/(I_b/na^2)$	R^2/a^2	$\alpha \left(\begin{array}{l} \text{See} \\ \text{Ref. 12} \\ \alpha = \sqrt{\lambda} \end{array} \right)$	$A = \int_0^\infty dr 8\pi^2 r^3 \frac{J_b^2}{I_b^2}$
Flat	$H(a-r)$	.5000	.0000	2.0000
Parabolic	$2 \left(1 - \frac{r^2}{a^2} \right) H(a-r)$	.3333	.1491	1.3333
Gaussian	e^{-r^2/a^2}	1.0000	.3882	1.0000
Bennett	$(1 + r^2/a^2)^{-2}$	$R = \infty$; $n = 2$ case commonly used for simulation		.6667
Bennett truncated at na	$\left(\frac{1+n^2}{n^2} \right) \left(1 + \frac{r^2}{a^2} \right)^{-2} H(na - r)$	$\left(\frac{1+n^2}{n^2} \right) \log(1+n^2) - 1$	$\left[\frac{2}{3} \left(1 + \frac{3}{n^2} \frac{R^2}{a^2} - 1 \right) \right]^{1/2}$	$\alpha \left(\frac{1+n^2}{n^2} \right)^2 \left[\frac{1}{6} - \frac{1}{2(1+n^2)^2} + \frac{1}{3(1+n^2)^3} \right]$
$n = .5$	"	.1157	.0537	1.7333
$n = 1.0$	"	.3863	.1735	1.3333
$n = 1.5$	"	.7025	.3046	1.0769
$n = 2.0$	"	1.0118	.4248	.9333
$n = 3.0$	"	1.5584	.6207	.8000

TABLE 3. Critical ratio of plasma to beam current ($-I_p/I_b$) is tabulated vs profile for the onset of loss of equilibrium, non-oscillatory sausage growth and oscillatory growth. The profile of beam and plasma currents is assumed identical in all cases. We use $\alpha = .7$ for the Bennett profile to compute the oscillatory threshold.

Profile name	Non-oscillatory		
	Equilibrium fails	growth	Oscillatory growth
Flat	1.000	.5000	- ∞
Parabolic	1.000	.6000	-1.5533
Gaussian	1.000	.6667	+ .1625
Bennet ($n = \infty$)	1.000	.7500	+ .6288
$n = .5$	1.000	.5353	-3.4705
$n = 1.0$	1.000	.6000	-1.2096
$n = 1.5$	1.000	.6500	- .1036
$n = 2.0$	1.000	.6818	+ .2678
$n = 3.0$	1.000	.7143	+ .5272