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IS THERE A STATISTICAL MECHANICS OF TURBULENCE?

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The statistical-mechanical treatment of turbulence is made questionable by strong nonlinearity and strong disequilibrium that result in the creation of ordered structures imbedded in disorder. Model systems are described which may provide some hope that a compact, yet faithful, statistical description of turbulence nevertheless is possible. Some essential dynamic features of the models are captured by low-order statistical approximations despite strongly non-Gaussian behavior.

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1. INTRODUCTION

There are no universally agreed boundaries to statistical mechanics. A central goal certainly is the extraction and elucidation of features that (a) express significant physics; (b) can be described in a relatively compact fashion, without reference to the full details of motion in the configuration space; and (c) can be deduced in some satisfying way from the underlying dynamical equations. Navier-Stokes (N-S) turbulence presents a particularly interesting and difficult statistical dynamics. There is an interplay of order and disorder, associated with strong departure from absolute statistical equilibrium. The statistical properties may not even be space-extensive, because motions arise that are the order of system size. There is no clean scale separation: hydrodynamic modes of a wide and continuous range of spatial and temporal scales interact with each other. Well-defined and characteristic vortex structures arise in the midst of disorder. The basic question concerning statistical description is whether such complex behavior permits a closed representation that is simple enough to be tractable and insightful but powerful enough to be faithful to the essential dynamics.

The present paper first recalls briefly some basic mathematical and physical features of turbulence and discusses qualitatively some of the matters cited above. The relatively easy equilibrium statistical mechanics of the Euler equation stands in strong contrast to the highly difficult, but not totally intractable, dissipative disequilibrium of N-S turbulence. The results of applying two statistical approximation schemes to some simple model systems are discussed against this background. The chosen systems exhibit highly non-Gaussian behavior, yet are reasonably well described by approximations that close at the level of low-order

moments. Although the models are very elementary, these results perhaps may generate a certain optimism about continued progress in the statistical mechanical treatment of turbulence.

2. EQUATIONS OF MOTION AND INVARIANTS

The N-S equation may be written [1]

$$(\partial/\partial t - \nu \nabla^2) \mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) \times \boldsymbol{\omega}(\mathbf{x}, t) - \nabla h(\mathbf{x}, t), \quad (1)$$

where ν is the kinematic viscosity, $\mathbf{u}$ is the velocity field, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity field and $h = \frac{1}{2} |\mathbf{u}|^2 + p/\rho$ is the normalized pressure head (p is pressure and ρ is the uniform mass density). Incompressibility is expressed by

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0. \quad (2)$$

The constants of motion include the following integrals over the entire fluid volume: normalized momentum $\int \mathbf{u} d\mathbf{x}$ (conserved for all ν with cyclic boundaries), normalized kinetic energy $K(t) = \frac{1}{2} \int |\mathbf{u}|^2 d\mathbf{x}$ (conserved for $\nu = 0$ with cyclic or rigid boundaries) and helicity $H(t) = \int (\boldsymbol{\omega} \cdot \mathbf{u}) d\mathbf{x}$ (conserved when kinetic energy is). In addition there are the circulation constants $\int_C \mathbf{u} \cdot d\mathbf{l}$, where $d\mathbf{l}$ is the line element of any closed curve C that moves with the fluid (conserved for $\nu = 0$ with cyclic or rigid boundaries). In two-dimensional (2-D) flow, the vorticity has only one component (perpendicular to the flow plane) and the circulation constants imply that the vorticity in every surface element moving with the flow is constant for $\nu = 0$. In turn, this implies the inviscid constancy of $\int |\boldsymbol{\omega}|^n d\mathbf{x}$ for all n such that the integral exists; in particular the enstrophy $\Omega(t) = \int |\boldsymbol{\omega}|^2 d\mathbf{x}$ is conserved.

Suppose that $u(\mathbf{x}, t)$ can be expanded in some complete solenoidal, orthonormal set of vector functions $\phi^i(\mathbf{x})$:

$$u(\mathbf{x}, t) = \sum_i y_i \phi^i(\mathbf{x}), \quad (3)$$

where

$$\nabla \cdot \phi^i(\mathbf{x}) = 0, \quad \int \phi^i(\mathbf{x}) \cdot \phi^j(\mathbf{x}) d\mathbf{x} = \delta_{ij}.$$

Here i is not a vector index; instead, it orders the orthonormal functions in some convenient fashion. The kinetic energy becomes

$$K(t) = \frac{1}{2} \sum_i [y_i(t)]^2, \quad (4)$$

Equation (2) may be used to eliminate pressure and vorticity and (1) is thereby transformed to

$$dy_i/dt + \sum_j (v_{ij} + L_{ij}) y_j + \sum_{j,k} A_{ijk} y_j y_k = 0. \quad (5)$$

Here v_{ij} (arising from v), L_{ij} (arising from elimination of pressure via (2)), and $A_{ijk} = A_{ikj}$ (arising from the nonlinear term and pressure elimination) all are constant coefficients. If the boundary conditions are cyclic on a box, the $\phi^i(\mathbf{x})$ can be taken as transverse sine and cosine waves. Then $v_{ij} = vk^2 \delta_{ij}$, where k is wavenumber, and L_{ij} vanishes.

The important property of inviscid conservation of energy by individual pair and triad interactions follows immediately from overall energy conservation and the fact that (4) is a sum of squares: if only two or three modes are excited at an instant, then the rate of change of $K(t)$ at that instant involves contributions only from those modes. This implies the detailed conservation identities

$$L_{ij} + L_{ji} = 0, \quad A_{ijk} + A_{jki} + A_{kij} = 0, \quad (6)$$

Moreover, it is easy to see in the case of the cyclic box, that A_{ijm} vanishes if any two indices are the same. This immediately yields the inviscid Liouville theorem:

$$\sum_i \partial \dot{y}_i / \partial y_i = 0. \quad (7)$$

It follows from (6), and corresponding identities associated with helicity and enstrophy, that $K(t)$, $H(t)$ and $\Omega(t)$ remain inviscid constants of motion under a Galerkin truncation that discards all modes $\phi(\mathbf{x})$ lying above a cutoff wavenumber k_c . In contrast, the circulation constants either do not survive the truncation, or become unrecognizable.

3. INVISCID EQUILIBRIUM AND THE DIRECT-INTERACTION APPROXIMATION

The quadratic energy constant (4) and the Liouville property (7) imply that there is a formal equilibrium statistical mechanics for the inviscid N-S equation (Euler equation). This system suffers an ultraviolet catastrophe, in common with all classical field theories. If all modes above a cutoff wavenumber k_c are eliminated, both energy conservation and Liouville property survive, and the equilibrium statistical mechanics becomes realizable. In contrast to most Hamiltonian systems of interest, the energy constant K is a simple sum of squares; all the dynamical complexity resides in the nonlinear equation of motion. Energy equipartition is then a transparent consequence of a canonical equilibrium ensemble of the form $\exp(-\beta K)$. The equilibrium partition is more complicated if the equilibrium ensemble is of the more general form $\exp(-\beta K - \alpha H)$ in 3-D or $\exp(-\beta K - \alpha \Omega)$ in 2-D [2,3].

The energy-equipartition equilibrium $\exp(-\beta K)$ becomes more interesting when the associated fluctuation-dissipation relations are

considered. These relations show that the truncated Euler system can imitate a N-S fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wavenumber modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared to the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity. When the wavenumber ratio is not asymptotically large, the dynamical damping is nonlocal in space and time; it exhibits long-time tails like those obtained from the kinetic theory of a gas [4].

Numerical experiments on the evolution of the truncated Euler system show that low-order statistics actually do approach values corresponding to the absolute equilibria described above, provided that k_c is large enough to leave the order of one hundred or more modes in the system. [5,6]

The canonical equilibrium statistical mechanics based on $\exp(-\beta K)$ appears to be adequately portrayed by a relatively simple statistical approximation, the direct-interaction approximation (DIA). The closed DIA integro-differential equations involve only the covariance matrix of the mode amplitudes and the mean response matrix of the amplitudes to infinitesimal perturbations. DIA represents the lowest-order term in a line-renormalized perturbation expansion for the truncated Euler system, under the assumption of Gaussian initial statistics. It is importantly distinguished from other truncations of the expansion by the fact that it exactly reproduces the statistics of certain stochastic model systems. [7-9] Thereby some essential consistency properties are guaranteed: existence of solutions; conservation of K , H and Ω ; survival of the equilibrium ensembles; and realizability of the covariance matrix.

The stochastic models underlying DIA can be constructed in several different ways. For present purposes, consider a collection of N identical but dynamically independent systems, each of which obeys (5) (with $\nu_{ij} = \delta_{ij}\nu_i$ and L_{ij} set to zero to give the N-S equation in a cyclic box):

$$(d/dt + \nu_i)y_i^\alpha + \sum_{j=1}^N A_{ij\alpha} y_j^\alpha y_j^\alpha = 0. \quad (\alpha = 1, 2, \dots, N) \quad (8)$$

Here α labels the systems. The DIA model (random-coupling model) [7,8] is obtained in the limit $N \rightarrow \infty$ by replacing (8) with

$$(d/dt + \nu_i)y_i^\alpha + N^{-1} \sum_{\beta, \gamma} \phi_{\alpha\beta\gamma} \sum_{j=1}^N A_{ij\alpha} y_j^\beta y_j^\gamma, \quad (9)$$

where β, γ run from 1 to N and, for each trio of indices, $\phi_{\alpha\beta\gamma}$ takes the value 1 or -1 at random, subject only to symmetry under all index permutations. This model clearly retains the conservation and Liouville properties of (8). The random ϕ values make all terms of the line-renormalized expansions except the lowest vanish in the limit, so that closed equations result for low-order statistics.

The DIA closure also implies an alternative stochastic model, in the form of a generalized Langevin equation [9] that yields the same covariance functions as (9), but inequivalent higher statistics. In cases where the modal covariance matrix is diagonal, which includes the absolute equilibrium distributions, the Langevin equation is

$$(d/dt + \nu_i)y_i(t) + \int_0^t \eta_i(t,s)y_i(s)ds = b_i(t), \quad (10)$$

where $\eta_i(t,s)$ is a time-lagged dynamical damping kernel and $b_i(t)$ is an effective random forcing, which may be taken to have Gaussian statistics. The quantities $\eta_i(t,s)$ and $b_i(t)$ satisfy

$$\eta_i(t,s) = -4 \sum_{j=1}^N A_{ij\alpha} A_{j\alpha i} G_j(t,s) Y_\alpha(t,s), \quad (11)$$

$$\langle b_i(t)b_j(t') \rangle \equiv B_{ij}(t,t') = 2 \sum_{j \neq i} (A_{ij})^2 Y_j(t,t') Y_i(t,t'), \quad (12)$$

where $Y_i(t,t') = \langle y_i(t)y_i(t') \rangle$. $G_i(t,s)$ is the response function associated with (10):

$$(d/dt + \nu_i)G_i(t,t') + \int_0^t \eta_i(t,s)G_i(s,t')ds = 0, \quad G_i(t',t') = 1. \quad (13)$$

The DIA equations are closed by multiplying (10) with $y_i(t')$, averaging to obtain

$$(\partial/\partial t + \nu_i)Y_i(t,t') + \int_0^t \eta_i(t,s)Y_i(s,t')ds = \langle b_i(t)y_i(t') \rangle, \quad (14)$$

and using (13) to find

$$\langle b_i(t)y_i(t') \rangle = \int_0^t G_i(t,s)B_{ij}(t,s)ds. \quad (15)$$

In (10) the explicit conservative nonlinear interactions of (5) have been replaced by a dynamical damping term and a random forcing term. The two together express conservation in the mean.

If all ν_i vanish, (13)-(15) readily yield an equilibrium solution in which $Y_i(t,t)$ is independent of i and t and $Y_i(t,t')$ satisfies the fluctuation-dissipation relation

$$Y_i(t,t') = Y_i(t,t)G_i(t,t') \quad (t \geq t'). \quad (16)$$

It also can be verified [4] that when y_i is a mode of wavenumber $k \ll k_c$, the integral in (10) reduces to the form $\eta_{dyn} k^2 y_i(t)$ with η_{dyn} independent of i . Thus the Euler fluid in equilibrium exhibits a dynamical damping like molecular viscosity. When k/k_c is $O(1)$, the time lag in the integral corresponds to the well-known long-time tails of the form $(t-s)^{-3/2}$ in three dimensions and $1/(t-s)$ in two dimensions. In all essential respects, DIA appears to give faithful qualitative and semi-

quantitative predictions for $Y_i(t,t')$, $G_i(t,t')$, and $\eta_i(t,s)$ in absolute equilibrium.

4. VISCOUS DISEQUILIBRIUM: STRUCTURE WITHIN RANDOMNESS

The differences encountered in passing from the absolute equilibrium of the truncated Euler equation to actual N-S turbulence are extreme. Energy equipartition, which places most of the energy at wavenumbers the order of k_c , is replaced by a disequilibrium spectrum in which typically most of the energy is at $k \ll k_d$, where k_d is a typical dissipation wavenumber. There is a cascade of energy from these lower wavenumbers to wavenumbers $\sim k_d$ where the energy is dissipated by viscosity. The turbulence may decay or it may be maintained in a steady state. In the steady state, energy input may come from explicit external stirring terms added to the N-S equation or from the interaction between a pressure-driven mean flow and rigid boundaries. In the latter case, the energy budget is complicated: small-scale (high-wavenumber) processes near the boundaries pump energy into the large-scales (low-wavenumbers) of the turbulence and this energy in turn is passed to high wavenumbers in the interior of the flow where it is dissipated [10]. There appears to be a non-uniform dependence of flow dynamics on viscosity: The dissipation of energy from large scales appears to approach a constant rate, independent of viscosity, as viscosity goes to zero [1].

Apart from changes in gross statistics, such as the energy distribution, the dissipative disequilibrium that is turbulence exhibits

characteristic, but plastic, structures that exist together with more random background excitation. The structures can appear as nonlinear waves and consequent strong vortices that grow from a boundary region or as sheets and ropes of intense vorticity that appear in the interior of even homogeneous, isotropic turbulence [10-15]. The role played by such structures in the energy budget and in other basic statistics is not yet clear; certainly the structures may be more important in some turbulence flows than in others.

An essential question is whether any tractable, low-order dynamical approximation exists that can handle the effects of flow structures. It should be stated emphatically that no low-order statistical description can provide the information needed to draw pictures of these structures like those obtainable from flow visualizations. Low-order moments, or small- n n -variate distributions of velocity or vorticity, are insufficient to specify uniquely the spatial form of structures. From a dynamical point of view, three possibilities come to mind:

- (a) The nonlinearity of the N-S equation, together with the presence of structures, makes it impossible to construct successful approximations involving only low-order statistics. That is to say, a statistical mechanics of turbulence does not exist.
- (b) The structures do not really affect low-order statistics. It is possible to ignore structures and construct successful low-order approximations as in absolute equilibrium.
- (c) Structures do affect low-order statistics, but it is not necessary to capture their detailed form. Low-order approximations can be constructed (perhaps at a level that includes third- and fourth-

order moments) that crudely, but sufficiently, account for structures and their effects.

Arguments can be made for each of these possibilities. In the case of turbulent channel flows, vortex formation near the boundaries and subsequent penetration into the flow appear to be essential parts of the mechanism of momentum transport from wall to the flow interior [10-13]. On the other hand, theories that take no account of such structures have considerable success in describing the momentum transport. For example, one may note the bounding and optimizing theories of Busse, Malkus and their colleagues [16,17], and the recent success of Yakhot, Orszag and their colleagues [18] in calculating shear flows with simple eddy-viscosity models.

Structure is also apparent in many laboratory studies and computer simulations of decaying isotropic turbulence. For example, a recent simulation by Yamamoto and Hosokawa [15] provides a clear illustration: the univariate distribution of $|\omega|$ is found to be very close to exponential, in contrast to the almost universal finding that the univariate distribution of u is very close to Gaussian. This $|\omega|$ distribution is associated with a marked spatial spottiness of vorticity. Regions of intense vorticity are confined to thin rope segments that occur sparsely throughout the cyclic box. Contributions from these ropes dominate $\langle |\omega|^4 \rangle$ so that $\langle |\omega|^4 \rangle / \langle |\omega|^2 \rangle^2$ has a value ~ 6 , which is far above the value $5/3$ for a Gaussian distribution. However, the authors note that most of the mean energy dissipation (energy dissipation is related to $|\omega|^2$) comes from the weakly-excited flow regions outside the ropes. Thus it seems possible that the mean energy budget can be treated without taking account of the rope-like vorticity structures.

How do approximations like DIA perform in the disequilibrium environment of actual N-S turbulence? The applications to date have been principally to isotropic turbulence. At low Reynolds numbers (but high enough for nonlinear effects to be dominant), DIA appears to predict quite adequately the evolution modal covariance $Y_i(t, t')$, both for $t = t'$ (energy spectrum) and for $t \neq t'$ [19]. At higher Reynolds numbers, the nonlinear transfer of energy among the higher wavenumbers is inaccurately predicted due to the following difficulty: $Y_i(t, t')$ for a high-wavenumber mode decays, as a function of $t - t'$, not only because of stretching and other distortions associated with energy cascade but also because of convective dephasing. The latter simply represents the bodily carrying about of small-scale structures by the dominant large-scale velocity. Because $Y_i(t, t')$ plays so a central role in DIA, the approximation is not able to distinguish between these two effects. This problem can be handled by replacing the Eulerian covariance $Y_i(t, t')$, which measures time correlations of velocities at fixed points in space, with a Lagrangian covariance that measures correlations along fluid-element trajectories. DIA modified in this way appears to predict successfully the energy cascade at high Reynolds numbers as well low Reynolds numbers [20].

Recently DIA has had a limited success in predicting statistics higher than second-order. Decaying isotropic turbulence exhibits a depression of the mean-square total nonlinear term in the N-S equation below the values corresponding to a Gaussianly distributed velocity field that, at each time, has the same wavenumber power spectrum as the actual decaying field. The effect is that the mean-square time derivative of the velocity field is also depressed below values for the corresponding Gaussian field. Depressions of the order of 50% have been observed in

simulations [21,22] Although the closed DIA equations involve only second-order statistics, fourth-order and higher statistics can be calculated by going back to the stochastic model (9) [see (18)]. In this way, DIA is found to predict a depression of nonlinearity that is remarkably close to that observed in simulations (Figure 1) [22].

However, DIA totally fails to predict the spottiness of vorticity and dissipation that is observed in the laboratory and in simulations. In fact it gives precisely the Gaussian value 5/3 for the ratio $\langle |\omega|^4 \rangle / \langle |\omega|^2 \rangle^2$ in contrast to the value ~ 6 found by Yamamoto and Hosokawa [15].

5. TWO HIGHLY NON-GAUSSIAN MODEL SYSTEMS

The model systems to be considered here exhibit two kinds of marked departure from Gaussian statistics: first, a very strong depression of the nonlinear terms; second, highly intermittent distributions of mode amplitudes. They are presented in order to illustrate the behavior of DIA and a second statistical approach, a method of mode decimation, in the face of highly non-Gaussian statistics. The first system is one of the simplest implementations of (5) and (6). It consists of three modes that obey

$$dy_1/dt = y_1 y_2, \quad dy_2/dt = y_2 y_1, \quad dy_3/dt + y_3 = -2y_1 y_2. \quad (17)$$

The behavior of this system was explored by taking an ensemble of 1000 realizations with initial independent, Gaussian distributions of mode amplitudes. The initial mode variances were taken as

$$Y_1(0,0) = Y_2(0,0) = 4, \quad Y_3(0,0) = 0.$$

Figure 2 shows the evolution of the mode intensities $Y_i(t,t)$ and of the

nonlinear ratios R_i defined by

$$R_1(t) = \langle [y_2(t)y_3(t)]^2 \rangle / [Y_2(t,t)Y_3(t,t)], \text{ etc.}$$

In general,

$$R_i(t) = \sum_{j,k,m} A_{ijk} A_{imn} \langle y_j(t)y_k(t)y_m(t)y_n(t) \rangle / B_i(t,t).$$

[In an independent Gaussian distribution, $R_i(t) = 1$.] Because of the symmetry in (17) and the initial conditions, the statistics of modes 1 and 2 are identical.

Note that the drain of energy from undamped modes 1 and 2 into damped mode 3 stops while modes 1 and 2 still contain finite energy; the nonlinear term seen by mode 3 goes to zero! What happens is that $y_1(t)$ goes to zero in every realization in which $y_2(t)$ remains finite, and vice versa. This occurs because the system has a second constant of motion, in addition to (4):

$$D(t) = [y_1(t)]^4 - [y_2(t)]^2.$$

$D(t)$ is constant even when mode 3 is damped. In consequence, the energy transferred from modes 1 and 2 to mode 3 must come equally from each mode. This drives the amplitude of the more weakly excited of modes 1 and 2 in each realization to zero. In turn, $R_3(t)$ goes to zero and the energy transfer stops.

The DIA equations (11)-(15) for this system are easily written down. Perturbative analysis of (9) in the limit $N \rightarrow \infty$ yields the following expression for $R_i(t)$ [22]:

$$R_i(t)B_i(t,t) = B_i(t,t) + \dots$$

$$4 \sum_{j,k,r,s} b_{ijk} b_{lmn} \int_0^t ds \int_0^s dr G_j(t,s) G_m(t,r) Y_k(t,s) Y_n(t,r) Y_l(s,r) + \\ 8 \sum_{j,k,l,m,n} b_{ijk} a_{lmn} \int_0^t ds \int_0^s dr G_j(t,s) G_l(s,r) Y_k(t,s) Y_m(t,r) Y_n(t,r), \quad (18)$$

where

$$a_{ijk} = (A_{ijk})^2, \quad b_{ijk} = 2A_{ijk}A_{jki}.$$

$B_1(t,t)$ is the Gaussian value for the mean-square total nonlinear term.

Figure 3 shows the evolution of $Y_1(t,t)$ and $R_1(t)$ obtained by numerical integration of these integro-differential equations. The inhibition of energy transfer and the decline of $R_1(t)$ both are more-or-less reproduced by DIA. Major oscillations in $R_1(t)$ appear in DIA that are not present in the exact dynamics. But it is remarkable that DIA, based on a closure at the level of second-order moments, can exhibit a strong depression of $R_1(t)$. The constancy of $D(t)$ survives in DIA, but only in the averaged form $d\langle D(t) \rangle / dt = 0$. The latter is empty for the present problem because $\langle D(t) \rangle = 0$ by symmetry. Thus the success of DIA in predicting $R_1(t)$ does not have a trivial explanation.

The second model system is constructed as a combination of elementary conservative triad interactions of the form (17). It imitates, in a very primitive way, the tree-like structure of interactions present in the N-S equation. There is a sequence of stages; the n -th mode in the s -th stage is labeled $y_{ns}(t)$ where $n = 1, 2, \dots, N_s$ and N_s is the number of modes in the stage. Each of the possible pairs of modes in stage s drives just one mode in stage $s+1$, which implies

$$N_{s+1} = N_s(N_s - 1)/2.$$

The equations of motion for the system of S stages are then

$$(d/dt + \nu_1)y_{11} = y_{12}y_{21} + y_{13}y_{31} + (N_1 - 3) \text{ similar terms,}$$

$$\begin{aligned}
 (d/dt + \nu_1)y_{12} &= y_{13}y_{21} + y_{11}y_{23} + (N_1 - 3) \text{ similar terms,} \\
 (d/dt + \nu_1)y_{13} &= y_{11}y_{22} + y_{12}y_{21} + (N_1 - 3) \text{ similar terms,} \\
 &\dots\dots\dots (19) \\
 (d/dt + \nu_2)y_{21} &= -2y_{12}y_{13} + [y_{22}y_{13} + y_{23}y_{32} + (N_2 - 3) \text{ similar terms}], \\
 &\dots\dots\dots \\
 (d/dt + \nu_S)y_{S1} &= -2y_{S-1,2}y_{S-1,3}, \\
 &\dots\dots\dots
 \end{aligned}$$

Note that the damping factor ν_a is the same for each mode in stage a . The structure of the equations for a mode in stage a is that there is one term quadratic in amplitudes of stage $a-1$ (except for $a=1$) and $N_a - 1$ terms consisting of a product of an amplitude from stage a and an amplitude from stage $a+1$ (except for $a=S$). Because of the symmetry of the equations, they are unchanged by any permutation of mode labels within each stage. The number of modes, and the number of terms in the equations rise very rapidly with S and N_1 .

Numerical results are presented here for three stages ($S=3$) and for the two values $N_1=3$ or $N_1=4$. There are then a total of 9 or 25 modes and the populations of the three stages are 3-3-3, or 4-6-15. The integrations were performed by the 4th-order Runge-Kutta-Gill method for an independent Gaussian ensemble of 1000 realizations. The conditions were

$$Y_{1a}(0,0) = 4, \quad Y_{2a}(0,0) = Y_{3a}(0,0) = 0, \quad \nu_1 = \nu_2 = 0, \quad \nu_3 = 1,$$

where $Y_{aa}(t,t') = \langle y_{aa}(t)y_{aa}(t') \rangle$. Figure 4 shows two selected statistics for

each case: the evolution of $Y_{1n}(t,t)$ (averaged over n to reduce finite-ensemble fluctuation effects) and (similarly averaged over n) the quantity

$$J_{2n}(t) = \frac{1}{3} \langle [y_{2n}(t)]^4 \rangle / [Y_{2n}(t,t)]^2.$$

The latter has the value 1 if $y_{2n}(t)$ has a Gaussian distribution. Both cases show an arrest of energy transfer out of stage 1, similar to that seen in the first model. The plateau in $Y_{1n}(t,t)$ is lower in the 4-6-15 case. The curves for $J_{2n}(t)$ show that the distribution of $y_{2n}(t)$ becomes intermittent and that the effect is markedly stronger for the 4-6-15 case.

If DIA is worked out for the present model, it succeeds as before, in more-or-less reproducing the arrest of energy transfer. But it totally fails to predict the intermittency of $y_{2n}(t)$ and, instead, gives the exactly Gaussian values $J_{2n}(t) = 1$. It is therefore of interest to apply a more powerful, but related, statistical approximation that is capable of predicting non-Gaussian values of $J_{2n}(t)$. This is the method of constrained decimation (CDS) [23-25]. The basic idea of CDS is to treat explicitly only a fraction of the mode amplitudes (in the present application the fraction decreases as stage population increases) and to represent the dynamical effects of the remaining modes by constrained stochastic forcing terms in the equations of motion for the explicit modes. The constraints express statistical symmetries that result from initial conditions and the symmetries built into the original equations of motion. Converging approximations are expected as more such constraints are systematically imposed. In the limit of very strong decimation, CDS can be analyzed perturbatively, and then, with a particular choice of

constraints, it yields DIA.

In the illustration to be given here, the 4-6-15 case is decimated to three explicit modes in each stage (3-3-3). The equations of motion for the nine explicit modes ($y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33}$) are

$$\begin{aligned}
 (d/dt + v_1)y_{11} &= p_{11} + q_{11}, & p_{11} &= y_{12}y_{22} + y_{13}y_{22}, \\
 (d/dt + v_1)y_{12} &= p_{12} + q_{12}, & p_{12} &= y_{13}y_{21} + y_{11}y_{23}, \\
 (d/dt + v_1)y_{13} &= p_{13} + q_{13}, & p_{13} &= y_{11}y_{22} + y_{12}y_{21}, \\
 (d/dt + v_2)y_{21} &= -2y_{12}y_{13} + p_{21} + q_{21}, & p_{21} &= y_{22}y_{33} + y_{13}y_{32}, \\
 & \dots \dots \dots & & \\
 (d/dt + v_3)y_{31} &= -2y_{12}y_{23}, \\
 & \dots \dots \dots
 \end{aligned} \tag{20}$$

Here the p_{aa} are the terms that represent interaction of y_{aa} with other explicit modes in stage a [they are written out explicitly in (19)], and the q_{aa} represent all the interaction terms with modes outside the explicit set [they consist of the $N_a - 3$ "similar terms" that appear in (19)]. With the present choice $N_1 = 4$, each q_{1a} consists of one term (e.g., $q_{11} = y_{14}y_{14}$) and each q_{2a} consists of three terms. Note that q_{3a} vanishes because it is assumed that there is no stage 4.

In the present examples, the symmetry constraints imposed upon the q_{aa} will consist of some or all of following set ($n, m = 1, 2, 3$):

$$\langle q_{1a} y_{1a} \rangle = \frac{1}{2} \sum_m \langle y_{1m} p_{1m} \rangle, \tag{21}$$

$$\langle q_{2a} y_{1a} \rangle = \frac{1}{2} \sum_m \langle y_{1m} p_{2m} \rangle, \tag{22}$$

$$\langle q_{1n} y_{1n}^3 \rangle = \frac{1}{6} \sum_n \langle y_{1n}^3 p_{1n} \rangle \quad (23)$$

$$\langle q_{2n} y_{2n}^3 \rangle = \frac{1}{2} \sum_n \langle y_{2n}^3 p_{2n} \rangle, \quad (24)$$

$$\langle q_{1n}^2 \rangle = \frac{1}{6} \sum_n \langle p_{1n}^2 \rangle_{\text{diag}}, \quad (25)$$

$$\langle q_{2n}^2 \rangle = \frac{1}{2} \sum_n [\langle p_{2n}^2 \rangle_{\text{diag}} + 4 \langle p_{2n}^2 \rangle_{\text{cross}}], \quad (26)$$

All amplitudes in these relations have the same time argument. The quantity $\langle p_{2n}^2 \rangle_{\text{diag}}$ represents the sum of the two diagonal terms (squares) in p_{2n}^2 while $\langle p_{2n}^2 \rangle_{\text{cross}}$ represents the sum of the two cross terms. If the definitions of the q_{2n} are substituted, all of the constraints (21)-(25) are easily seen simply to state the equality of moments that differ only by index changes; these constraint equations can easily be verified, in the limit of infinite ensemble size, directly from the assumed initial statistical symmetry among modes in a stage and the symmetry properties of the equations of motion. Fluctuation error terms properly should be added to the constraint equations when the ensemble is finite (as it must be in direct numerical integrations), but this refinement will be ignored here.

The symmetry constraints are used by doing the following sequence of operations at each integration time step: (a) calculate the right hand sides of (21)-(26) from the ensemble of explicit y_{1n} amplitudes values; (b) find the q_{2n} amplitudes, in each realization, that minimize $\langle q_{2n}^2 \rangle$ while satisfying whatever of the linear constraints (21)-(24) are imposed; (c) add, to these least-square q_{2n} values, random increments normalized so that the sums satisfy the quadratic constraints (25) and (26), which always are imposed; (d) integrate equations (20), for each realization, one step in time. Gram-Schmidt orthogonalization and a Newton-Raphson

method are used to find the constrained least-square values for q_{22} [23].

Figure 5 shows the results of performing the above CDS algorithm with an ensemble of 1000 realizations and the same initial conditions as for the full-system integrations of Figure 4. Two cases are shown. In the first (Figure 5a), the linear constraints (21) and (22) are imposed, along with the quadratic constraints (25) and (26). In the second (Figure 5b), all the constraints (21)-(26) are imposed. It is easy to verify that the (21) and (22) constrain the q_{22} so that energy conservation for the full system (explicit and implicit modes) survives in the mean. They are indispensable constraints. The plots show that both cases reproduce the inhibition of energy transfer from stage 1 with excellent fidelity and give an energy plateau close to the exact values for the 4-5-15 system rather than for the 3-3-3 system. But only the second case, with constraints (23) and (24) imposed, reproduces well the full-system behavior of $J_{22}(t)$. Constraints (23) and (24) directly affect the evolution of $\langle y_{22}^4 \rangle$. The behavior of $J_{22}(t)$ is a marked quantitative improvement over DIA (not shown). As noted before, DIA totally fails to capture the rise of $J_{22}(t)$ above the Gaussian value one.

In the cases shown, the decimation is moderate (from 25 modes to 9 modes) and provides no substantial saving in computing time over integration of the full system. The savings go up rapidly as the number of stages increases. If $N_1 = 4$ and there are 6 stages, the total number of modes is $\sim 1.5 \times 10^7$ which is reduced to 18 by decimation.

6. CONCLUDING REMARKS

The equilibrium statistical mechanics of the 3-D Euler equation, truncated in wavenumber, is essentially simple because the hydrodynamic kinetic energy is a sum of squares. This equilibrium appears to be well-described by DIA, which yields good approximations to the fluctuation-dissipation relations in addition to reproducing the energy equipartition. The introduction of viscous dissipation, to turn the Euler equation into the N-S equation, produces a profoundly different statistical dynamics, in which the modal intensities are far from the equilibrium equipartition and in which ordered flow structures coexist and interact with random excitation. In the case of isotropic turbulence, the structures are associated with strong spatial intermittency of viscous dissipation. DIA and related approximations appear to do an adequate job of predicting energy spectra. As illustrated in this paper, they can also handle certain types of substantial departures from Gaussian statistics. However DIA fails totally to capture the spatial intermittency of dissipation [22].

Perhaps the principal achievement of DIA-type theories is that they deduce from the equations of motion a generalized dynamical damping which embodies the idea of eddy viscosity but takes account of nonlocalness in space and time due to the absence of clean scale separations in turbulence [26]. It is not easy to go beyond statistical approximations at the DIA level, in which the closure retains explicitly only second-order moments of the velocity field. Renormalization group methods, as they have so far been applied to turbulence, represent a more drastic approximation in which much of the nonlocalness effects is ignored [27-30]. There has not yet been any indication that the

apparatus of contemporary dynamical systems theory can be of help in the problem of fully developed turbulence. There are two fundamental difficulties: first, the attractor for steady-state, high Reynolds-number turbulence probably has very large dimensionality; second, the characteristic time in which interesting things happen in turbulence is the order of one large-eddy turnover time and is exceedingly small compared to the time that must be required for the system to settle onto an attractor and explore very much of it [31]. Moreover, phenomena in decaying turbulence, which never reaches the attractor, are not essentially different from those in steady-state turbulence.

As illustrated in this paper, constrained decimation methods appear to hold some promise for describing intermittency effects. They also serve to imbed DIA in a sequence of converging approximations so that it need not be regarded simply as the first term in a divergent renormalized perturbation series. It is not clear to what extent the decimation methods can be developed either analytically or as a computational tool. If such an approach (or any other) is to capture phenomena like the nearly exponential distribution of vorticity, it must deal directly with x -space variables as well as with Fourier modes. The probability distribution of a single Fourier mode, or any small set of Fourier modes, is nearly Gaussian in homogeneous turbulence in a large box, whatever the departure from Gaussian statistics in x -space. This is because the Fourier modes are collective coordinates over the entire flow volume.

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FIGURE CAPTIONS

Figure 1. Depression of nonlinearity in decay of isotropic turbulence at initial $R_\lambda \sim 35$. The mean-square (over space) of the total nonlinear term in the Navier-Stokes equation, normalized by the mean-square value for a Gaussian field with the same instantaneous spectrum, is plotted against time. Squares represent a full computer simulation and the curve represents DIA values (see Sec. 5).

Figure 2. (a) Evolution of variance $Y_n(t,t)$ for the three-mode model. Curve 1 is for $n=1,2$ and curve 3 is for $n=3$. (b) Evolution of the normalized mean-square of the nonlinear term $R_n(t)$. Same labeling.

Figure 3. Same as Figure 3, except that the DIA approximations are now shown.

Figure 4. Evolution of the three-stage model. (a) modal variance for a first-stage mode $Y_{1,1}(t,t)$ for stage populations 3-3-3, (b) flatness measure $J_{1,1}(t)$ for a second-stage mode for populations 3-3-3; (c) same as (a) for populations 4-6-15; (d) same as (b) for populations 4-6-15.

Figure 5. Evolution of the three-stage model for stage populations 4-6-15 decimated to 3-3-3. (a) and (b) show $Y_{1,1}(t,t)$ and $J_{1,1}(t)$ under constraints (21), (22), (25) and (26). (c) and (d) show the same quantities under the full set of constraints (21)-(26).

AAPT ONLY

NL RATIO FOR DIA (CURVE) AND DNS

FIG. 1

DRAFT ONLY

EXACT (1000 REALIZATIONS) MODE 3

$$\begin{aligned}v_y &= 0 & v_z &= 1 \\ \langle |y_1(0)|^2 \rangle &= \langle |y_2(0)|^2 \rangle = 4 \\ \langle |z(0)|^2 \rangle &= 0\end{aligned}$$

FIG. 2(a)

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EXACT (1000 REALIZATIONS) MODE 3

FIG. 2(b)

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DIA MODE 3

FIG 3(a)

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AFX 880513 D1201 MODE 1 STG 2 PLT 6

FIG 3(b)

DRAFT ONLY

(a)

(b)

(c)

(d)