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The Stability of Colliding Ion Beams

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ABSTRACT

We determine conditions for stability of two identical colliding ion beams in the presence of neutralizing electrons, but no background ions. Such a situation is envisioned for the Counterstreaming Ion Torus. The ion beams are taken to be Maxwellian in their frames of reference. The approximation of electrostatic and electromagnetic modes is made. The stability of the electrostatic modes depends on the relation between the ion electron temperature ratio and the relative beam velocities. The stability of the electromagnetic mode depends on the relation between the ion plasma β and the relative beam velocities.

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I: INTRODUCTION

A situation often encountered in plasma physics is that of two identical, homogeneous ion beams of different mean energies colliding with each other in the presence of a thermal background of neutralizing electrons. Such a situation is frequently unstable, being subject to the buildup of different types of plasma waves. In this paper we investigate the stability conditions for this situation.

The equilibrium we consider has a uniform magnetic field $\vec{B}$, an ion velocity distribution of

$$f_i(\vec{v}) = \frac{N_i}{(2\pi T_i/M)^{3/2}} \exp - \frac{M(\vec{v} - U\vec{z})^2}{2T_i} + \exp - \frac{M(\vec{v} + U\vec{z})^2}{2T_i} \quad (1)$$

and an electron velocity distribution of

$$f_e(\vec{v}) = \frac{N_e}{(2\pi T_e/m)^{3/2}} \exp - \frac{mv^2}{2T_e} \quad (2)$$

(see Fig. 1). U and $-U$ are the mean drift velocities of the two ion beams, T_i is the internal temperature and n_i the density of each beam. z is a cartesian coordinate taken parallel to $\vec{B}$. T_e is the temperature and $n_e = 2n_i$ the density of the neutralizing electron background. There is no ion background. M and m are the ion and electron masses respectively.

To carry out this analysis we employ the hot plasma dispersion relation¹ for waves with space-time behavior $\text{Re}[\text{const} \times \exp(-i\omega t + ik_1x + ik_3z)]$ (see Fig. 2). We examine this relation for both the cases of magnetized ($\vec{B} \neq 0$) and unmagnetized ($\vec{B} = 0$) ion beams. In order to simplify the analysis we make the approximation that the waves are either purely electrostatic or

electromagnetic. [For ($\vec{B} = 0$) these waves are uncoupled exactly.] We find for both magnetized and unmagnetized beams that the electrostatic stability depends on the relation between the two dimensionless quantities T_i/T_e and U/v_i , where $v_i \equiv (2T_i/M)^{1/2}$. For each value of U/v_i there is a critical ion electron temperature ratio $(T_i/T_e)_c$ such that the ion beams are unstable if T_i/T_e is less than this value. The electrostatic stability diagram for both $\vec{B} = 0$ and $\vec{B} \neq 0$ is displayed in Fig. 5.

If $\vec{B} = 0$, the plasma is stable against electromagnetic modes. If $\vec{B} \neq 0$, the stability depends on the relation between the quantity $\beta \equiv 8\pi n_i T_i / B^2 \equiv v_i^2 / v_A^2$ and U/v_i . Again for each value of U/v_i there is a critical value of β , β_c , such that if β exceeds β_c we have unstable electromagnetic modes. The electromagnetic stability diagram is shown in Fig. 8.

Our analysis is linear in the amplitude of the waves. However, it is generally accepted that if the ion beams are unstable to either mode, nonlinear wave-particle interactions from the resulting waves will lead to a rapid diffusion in velocity space. The relative velocities of the beams will be reduced and their thermal spreads will increase until a new stable equilibrium is reached.

Thus, the determination of the stability criteria has considerable practical importance since it puts limits on the possible interacting ion beams that may stably exist. Examples of devices involving interacting ion beams are the Counterstreaming Ion Torus, CIT,² and the thermal barrier in the Tandem Mirror.³ It is clear that our model also applies when two separate plasmas collide because the electron velocities in each separate plasma are large compared to the relative ion velocities. Thus, our results are also of interest for the study of collisionless shocks.⁴

Analytic investigations of the stability of two ion beams have been published by Stringer,⁵ Weibel,⁶ Perkins,⁷ and others.⁴ These investigations have been restricted to certain limiting cases. Our investigation is essentially numerical. We do make the approximation that the electrostatic and electromagnetic modes are decoupled and that the electrons are massless. Otherwise, our calculation is quite general.

We treat the electrostatic modes in Section II and the electromagnetic modes in Section III. In Section IV we discuss our results, compare them with the earlier calculations, and apply them to the CIT.

II: STABILITY OF ELECTROSTATIC MODES

We consider first magnetized beams, $\vec{B} \neq 0$. The dispersion relation for electrostatic modes is derived in Appendix A. It can be written

$$0 = T_i/T_e + 1$$

$$+ \frac{1}{2} \sum_{\mu=-\infty}^{+\infty} e^{-\mu} I_{|\mu|}(\mu) \left[\left(\frac{\omega + k_3 U}{k_3 v_i} \right) Z \left(\frac{\omega - \omega_i + k_3 U}{k_3 v_i} \right) \right. \\ \left. + \left(\frac{\omega - k_3 U}{k_3 v_i} \right) Z \left(\frac{\omega - \omega_i - k_3 U}{k_3 v_i} \right) \right]$$

$$\equiv T_i/T_e + F(\omega/k_3 v_i) \quad , \quad (3)$$

where $\mu \equiv k_1^2 T_i / M \omega_i^2$ and $\omega_i \equiv eB/Mc$ is the ion cyclotron frequency. Z is the plasma dispersion function defined by

$$Z(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - z} dx, \quad \text{Im}(z) > 0, \quad (4)$$

when $\text{Im}(z) > 0$ and by analytic continuation elsewhere. $I_{1/2}(\mu)$ is the modified Bessel function. The first term of Eq.(3) represents the electron contribution to the dispersion relation; we have taken the limit of zero electron mass and dropped the $k^2 \lambda_{De}^2$ term, where λ_{De} is the electron Debye length.

Inspection of Eq.(3) shows that $F(\omega/k_3 v_i)$ depends only on the parameters $\omega/k_3 v_i$, $\theta \equiv k_1/k_3$, μ and U/v_i . μ and θ characterize the wave vector $\vec{k}$, see Fig. 2, and $\omega/k_3 v_i$ is to be determined by solving Eq.(3). The parameter T_i/T_e does not occur in F . Thus, since we are primarily concerned with determination of stability criteria, a Nyquist technique is suggested. Fix U/v_i . Then for each choice of μ and θ imagine $F(\omega/k_3 v_i)$ to be traced out in the complex plane as $\omega/k_3 v_i$ describes a semicircle in the upper half $\omega/k_3 v_i$ plane. An example is displayed in Fig. 3 for $\mu = 0$ and $U/v_i = 1.2$ (if $\mu = 0$, F is independent of θ). It is clear that we would get a corresponding plot of the right-hand side of Eq. (3) by shifting the curve of Fig. 3 to the right by T_i/T_e units. Now according to the Nyquist criteria, there is a root for $\omega/k_3 v_i$ in the upper half plane, corresponding to a growing mode, if this displaced curve surrounds the origin. This is the case for all values of T_i/T_e up to $T_i/T_e = 0.217$. Thus, for this particular case, if $T_i/T_e > 0.217$ the mode is stable. To determine the actual limit on T_i/T_e we must repeat this procedure for every value of μ , and for each μ , for every value of θ .

Our actual strategy for determining stability is as follows. First, fix U/v_i . Then, for each μ , we maximize the unstable range of T_i/T_e over all possible values of θ by plotting a sequence of curves of $F(\omega/k_3 v_i)$ for each θ . We actually need only keep track of the places where the curves cross the negative real axis and maximize the distances of these crossings from the origin. It is the case that $F(+\infty) = 0$, so that the curves always start from the origin. The resulting values of T_i/T_e are then maximized over μ to get $(T_i/T_e)_C$ as a function of U/v_i . These curves for T_i/T_e are given in Fig. 4 for each value of μ . The upper envelope of these curves gives the stability curve $(T_i/T_e)_C$, plotted separately in Fig. 5. It is seen from this figure that for a given value of U/v_i all values of T_i/T_e below $(T_i/T_e)_C$ are unstable.

We determine $e^{-\mu} I_{|l|}(\mu)$ from a table for $e^{-\mu} I_0(\mu)$ and $e^{-\mu} I_1(\mu)$, and use a recursion relation to calculate them for $|l|$ larger than 1. (We used double precision for these calculations to overcome the instability inherent in this recursion relation.) It was found adequate to take only those values of l for which $|l| \leq \max(3\mu, 3)$. The function $Z(z)$ was evaluated from a table for $|z| \leq 7$ and from its asymptotic result for $|z| > 7$. It was found by trial that only a moderate range of θ 's were needed to maximize T_i/T_e for any μ and U/v_i .

Only values of μ up to 5 were considered. For larger values of μ one may employ the asymptotic limit of Eq. (3). It is shown in Appendix B that in this limit Eq. (3) reduces to

$$0 = \frac{T_i}{T_e} + 1 + \frac{1}{2} \left(\frac{\omega + kU'}{k_3 v_i} \right) Z \left(\frac{\omega + kU'}{k_3 v_i} \right) + \frac{1}{2} \left(\frac{\omega - kU'}{k_3 v_i} \right) Z \left(\frac{\omega - kU'}{k_3 v_i} \right), \quad (5)$$

where $k = (k_1^2 + k_3^2)^{1/2}$, and $U' \equiv Uk_3/k$ is the projected streaming velocity on

the wave vector $\vec{k} = k_1 \hat{x} + k_3 \hat{z}$. The results of the $\mu = \infty$ limit are in excellent agreement with those for $\mu = 5$. Therefore, $\mu = 5$ is a satisfactory upper limit.

The Nyquist technique only establishes conditions for existence of an unstable root, and does not give its location. Thus, with this method we have no direct information about the growth rate, $\gamma = \text{Im}(\omega)$, of such a root. However, an estimate of the growth rate near a point on the $(T_i/T_e)_c$ curve can be obtained by using a Taylor expansion about this critical point. Linearizing Eq. (3) about this point and the corresponding value of $\omega/k_3 v_i$ gives

$$\delta\left(\frac{T_i}{T_e}\right) + \frac{\partial F}{\partial(\omega/k_3 v_i)} \delta\left(\frac{\omega}{k_3 v_i}\right) = 0, \quad (6)$$

where $\delta(T_i/T_e) = T_i/T_e - (T_i/T_e)_c$.

Then,

$$\frac{\delta\gamma}{k_3 v_i} \equiv \text{Im}\left(\frac{\delta\omega}{k_3 v_i}\right) = \text{Im}\left(-1 \frac{\partial F}{\partial(\omega/k_3 v_i)} \delta\left(\frac{T_i}{T_e}\right)\right), \quad (7)$$

$\partial F/\partial(\omega/k_3 v_i)$ is available from our computation and we may thus evaluate $d(\gamma/k_3 v_i)/d(T_i/T_e)$ along the $(T_i/T_e)_c$ curve. This derivative is plotted in Fig. 6 and can be used to estimate nearby growth rates.

Now consider the unmagnetized beams, $\vec{B} = 0$. If $\vec{B} = 0$, the dispersion relation is Eq. (5). (This is also the correct dispersion relation if the electrons are magnetized.) The direction of $\vec{k}$ enters Eq. (5) only through U' : as $\vec{k}$ varies in direction U' ranges from 0 to U . Thus, we obtain the unmagnetized stability condition by maximizing the $k_1 = 0$ (i.e., $\mu = 0$) curve of Fig. 4 over all $U' < U$. The resulting value of $(T_i/T_e)_c$ is the long-dashed line in Fig. 5.

III: STABILITY OF ELECTROMAGNETIC MODES

We now consider the stability of two colliding magnetized ion beams to electromagnetic modes. We approximate these modes by the assumption that the parallel electron flow is so effective that E_z is zero. Thus, we need only keep the upper left hand two by two determinant in the dispersion relation and set it to zero.

As shown in Appendix C, the dispersion relation can be written as

$$G(\omega/k_3 v_i) \equiv (K_{11}' - \frac{2}{\beta} \frac{\mu}{2}) [K_{22}' - \frac{2}{\beta} \mu (1 + \frac{1}{2})] - (K_{12}')^2 = 0 ; \quad (8)$$

where

$$\beta = 8\pi n_i T_i \frac{2}{B} = 4\pi n_e T_i \frac{2}{B} ,$$

$$K_{11}' = 4\pi \frac{\omega}{\omega_i} \frac{2}{2} K_{11} = \sum_{\pm} \frac{\omega \pm k_3 U}{k_3 v_i} \sum_{l=-\infty}^{\infty} \frac{l^2}{2} e^{-\mu} I_{|l|}(\mu) Z \left(\frac{\omega - l\omega_i \pm k_3 U}{k_3 v_i} \right) , \quad (9)$$

$$K_{22}' = 4\pi \frac{\omega}{\omega_i} \frac{2}{2} K_{22} = \sum_{\pm} \frac{\omega \pm k_3 U}{k_3 v_i} \sum_{l=-\infty}^{\infty} e^{-\mu} \left[\left(\frac{l^2}{\mu} + 2\mu \right) I_{|l|}(\mu) - 2\mu I_{|l|}'(\mu) \right] \\ \times Z \left(\frac{\omega - l\omega_i \pm k_3 U}{k_3 v_i} \right) , \quad (10)$$

and

$$K'_{12} = 4\pi i \frac{\omega^2}{\omega_i^2} K_{12} = \int_{\pm} \frac{\omega \pm k_3 u}{k_3 v_i} \sum_{\ell=-\infty}^{\infty} \ell e^{-\mu} [I_{|\ell|}(\mu) - I'_{|\ell|}(\mu)] \\ \times Z \left(\frac{\omega - \ell \omega_i \pm k_3 U}{k_3 v_i} \right) + \frac{2\omega}{\omega_i} . \quad (11)$$

The last term in Eq. 11 represents the electron contribution to the current. The remaining terms in Eqs. (9) to (11) are all due to the ions. Since the electron contribution to K_{11} and K_{22} are smaller than the ion contributions by the square root of the mass ratio, we have dropped them.

Inspection of Eqs. (8) to (11) shows that the dispersion relation can be expressed entirely in terms of the quantities $\omega/k_3 v_i$, β , U/v_i , μ , and θ . Note that the ion-electron temperature ratio does not enter into the dispersion relation for electromagnetic modes, but the plasma parameter β does appear.

As before, to determine the stability of a plasma with equilibrium parameters U/v_i and β we must solve Eq. (8) for $\omega/k_3 v_i$ for all values of the wave parameters μ and θ and find those values for which $\text{Im}(\omega/k_3 v_i) > 0$. As in the electrostatic case a Nyquist technique is indicated and we note that the parameter β does not occur in the K' .

However, it is a little more difficult to imitate the method of Section II in reducing the number of parameters from four to three because $1/\beta$ occurs quadratically, rather than linearly, as T_i/T_e does in the electrostatic case. Let us factor this quadratic and write Eq. (8) as

$$G \left(\frac{\omega}{k_3 v_i} \right) = \frac{4\mu^2}{\theta^4} \left[F_+ \left(\frac{\omega}{k_3 v_i} \right) - \frac{1}{\beta} \right] \left[F_- \left(\frac{\omega}{k_3 v_i} \right) - \frac{1}{\beta} \right] = 0 , \quad (12)$$

where

$$F_{\pm} \left(\frac{\omega}{k_3 v_i} \right) = \frac{\theta^2}{2\mu} \frac{(1 + \theta^2) K'_{22} + K'_{11} \pm \left\{ [(1 + \theta^2) K'_{22} - K'_{11}]^2 + 4K_{12}^2 \right\}^{1/2}}{2(1 + \theta^2)} \quad (13)$$

Let $\omega/k_3 v_i$ trace out a semicircle C in the upper half plane, Fig. 7a. Then by the argument principle, we know that there will be a root of $G(\omega/k_3 v_i)$ in the upper half plane if and only if G traces out a contour which surrounds the origin.

Instead of following $G(\omega/k_3 v_i)$ we trace out the functions $F_+(\omega/k_3 v_i)$ and $F_-(\omega/k_3 v_i)$. Because these factors involve a square root they are not analytic everywhere, but instead represent two branches of a multivalued analytic function. Thus, it is not immediately obvious that the Nyquist technique should work on the separate factors.

To see that it can, recall that if the complex angle of the analytic function G increases by a nonzero multiple of 2π , G must have a root in the upper half plane. But the increase in the angle of G is the sum of the increases of the angles of the two factors $(F_+ - 1/\beta)$ and $(F_- - 1/\beta)$. Thus, if we plot F_+ and F_- , as $\omega/k_3 v_i$ describes the semicircle C , we can easily determine by shifting these curves, which values of $1/\beta$ produce a non-zero increase in the sum of the angles. For example, if F_+ returns on itself after $\omega/k_3 v_i$ describes C , then any value of $1/\beta$ which makes $(F_+ - 1/\beta)$ surround the origin produces an unstable root $\omega/k_3 v_i$. Similarly for the factor $(F_- - 1/\beta)$. On the other hand, if F_+ does not return to itself but returns to F_- we must let $\omega/k_3 v_i$ describe C twice and then determine for which values of $1/\beta$ the total curve surrounds the origin. In this way we have again reduced our parameters by one, just as for the electrostatic case.

To make the method precise, let us first note that when $\omega/k_3 v_i$ is very large K'_{12} is approximately $2\omega/\omega_i$, while K'_{11} and K'_{22} are finite. Thus, the magnitude of the square root term is $4|\omega/\omega_i|$; and when $|\omega/k_3 v_i| = \infty$, F_+ and F_- are of magnitude ∞ . We arbitrarily define $F_+(-\infty) = +\infty$ and $F_-(-\infty) = -\infty$. Then, on the rest of the curve the meaning of each branch is defined by continuity as $\omega/k_3 v_i$ traces out the real part of C. In one case, F_+ goes to $-\infty$ when $\omega/k_3 v_i$ reaches $+\infty$ and traces out a semicircle in the lower half plane [see Fig. 7(b)], as $\omega/k_3 v_i$ traces out the upper part of C, finally returning to $+\infty$. At the same time, F_- will pass from $-\infty$ to $+\infty$ and trace out a semicircle in the upper half plane returning to itself [Fig. 7(b)]. We then examine the two contours separately for stability. In the other case, F_+ goes from $+\infty$ at $\omega/k_3 v_i = -\infty$ to $+\infty$ at $\omega/k_3 v_i = +\infty$ and to $-\infty$ after the semicircle [Fig. 7(c)]. During a second pass of $\omega/k_3 v_i$ along C the F_- curve is traced, returning to the starting point of F_+ . We then examine the combined closed curve for stability. [It is a fact that $G(-\omega/k_3 v_i) = G^*(\omega/k_3 v_i)$ so that in the first case $F_-(-\omega/k_3 v_i) = F_+^*(\omega/k_3 v_i)$ and in the second case $F_+(-\omega/k_3 v_i) = F_+^*(\omega/k_3 v_i)$, $F_-(-k_3 v_i) = F_-^*(\omega/k_3 v_i)$. It is also easy to show that $\text{Im}(F_+) < 0$ and $\text{Im}(F_-) > 0$ when $\omega/k_3 v_i = -\infty$. These relations help to interpret the curves.]

Again, as in the electrostatic case, we need only keep track of the places where F_+ and F_- cross the real axis. However, we must be careful to take sufficiently small steps in $\omega/k_3 v_i$ that we stay on the correct branch of the square root. Otherwise, it is possible to jump inadvertently from the F_+ curve to the F_- curve, producing a spurious root. [See the dotted line in Fig. 7(b).]

To determine the criteria for stability against the electromagnetic modes, we first fix the value of U/v_i . Then for a fixed μ we maximize the range of $1/\beta$ for which $\omega/k_3 v_i$ is unstable over θ . It turns out that this

range in $1/\beta$ is always an interval from 0 to some upper limit given by the largest positive root of F_+ or F_- . We subsequently maximize this root over μ to find $(1/\beta)_c$ as a function of U/v_i . This critical curve of $1/\beta$ is plotted in Fig. 8. In Fig. 9 we plot the critical values of β^* defined by

$$\beta^* = \beta \left(2 + \frac{4}{3} \frac{U^2}{v_i^2} \right). \quad (14)$$

β^* is two thirds the total contained energy of the ions, including the directed energy of the beams, divided by the magnetic energy.

IV: DISCUSSION OF RESULTS

IVa: Electrostatic Modes

The criteria for stability of the two ion beam equilibrium against electrostatic modes is given in Fig. 5. The critical mode for stability is always the $\omega = 0$ mode. Near marginal stability this mode is driven unstable by either a Cherenkov interaction with the beams ($\vec{k} \cdot \vec{v} = 0$) or a cyclotron interaction ($k_3 v_{||} = \omega_i$).

We may understand the stability results for the Cherenkov interaction as follows:

For $U/v_i < 0.71$ the ion distribution indicated in Fig. 1 is actually single humped so by the Penrose criterion⁸ it must be stable. It remains stable up to $U/v_i = 0.9$. This is because no mode can propagate between the two humps for smaller U/v_i .

For U/v_i greater than 0.9, the parallel propagating modes become unstable for a finite range of T_i/T_e . We expect the mode to be unstable when T_i/T_e is just the correct value for an $\omega = 0$ mode. That is, the electron temperature should be such that the perturbed electron density, proportional to $1/k_3^2 \lambda_{De}^2$,

just balances the perturbed ion density proportional to $\text{Re } Z'(U/v_i)/k_3^2 \lambda_{Di}^2$. Thus, the equation of the curve for marginal instability of the parallel propagating mode is $(T_i/T_e)_c = Z'(U/v_i)$. This is the result of Stringer⁵ and is indicated in Fig. 5 by the short-dashed line.

For T_i/T_e less than $(T_i/T_e)_c$ the mode is purely growing. This is because any propagating wave with $\text{Re } \omega > \text{Im } \omega$ would propagate closer to one beam than the other and this beam would dominate both the real and imaginary parts of the dispersion relation. Hence, such a mode must be damped. The actual instability may not be viewed as a pure resonance instability. In fact, for positive imaginary ω the resonant ion terms contribute to the real parts of the density perturbation. It turns out that (T_i/T_e) must be less than $(T_e/T_i)_c$ to balance this change. In this way we may understand why smaller T_i/T_e 's are unstable.

As U/v_i increases, the unstable range of T_i/T_e increases until $U/v_i = 1.5$ above which it decreases. (This is because when $U/v_i > 1.5$, the $\omega = 0$ is propagating in the tail of both ion distributions, so the perturbed ion density decreases for fixed electric field perturbation. The electron temperature must correspondingly increase to reduce the electron density perturbation.) For $0.9 < U/v_i < 1.5$ the parallel propagating mode is the most unstable mode. For U/v_i larger than 1.5 the worst mode is an obliquely propagating mode with wavelength so short that the ions are effectively unmagnetized. The dispersion relation of a wave propagating at an angle ϕ is the same as that of a parallel propagating wave with ion drifts $\pm U \cos \phi$. Thus, for $U/v_i > 1.5$ one can always choose ϕ to make $U \cos \phi/v_i = 1.5$ and obtain a critical value of T_i/T_e equal to its value at $U/v_i = 1.5$. As a consequence, the critical T_i/T_e curve is a horizontal straight line above $U/v_i = 1.5$.

For an obliquely propagating mode a cyclotron resonance interaction is possible. However, until U/v_i reaches about 3 it is not very important compared to the Cherenkov interaction. This is because in order to have an unstable interaction the cyclotron resonance must be with ions on the lower velocity side of the beam distribution function (see dotted line in Fig. 1). If μ is order unity this requires [from $k_z(U - v_i) = \omega_i$] that the angle of propagation satisfy $\tan \phi = [(U - v_i)/v_i \sqrt{2\mu} = (U/v_i - 1)]$. The angle of the Cherenkov wave satisfies $\cos \phi = 1.5v_i/U$. Thus, the critical angle for cyclotron resonance exceeds that for Cherenkov resonance at about $U/v_i = 3$. Above this value the waves for moderate μ propagate for smaller T_e than do oblique waves and the corresponding critical T_i/T_e ratio is larger.

Weibel⁶ investigated the critical temperature for cyclotron resonance alone and his results are given by the dot-dashed curve in Fig. 5. It is seen that our results interpolate nicely between those of Stringer and those of Weibel.

To summarize the electrostatic stability results, we have stability for an ion electron temperature ratio less than a certain value $(T_i/T_e)_C$ depending on U/v_i . If $U/v_i < 0.9$, the modes are stable for all T_i/T_e . For U/v_i between 0.9 and 1.5 parallel propagating modes are worst and $(T_i/T_e)_C$ rises. For U/v_i between 1.5 and 3 obliquely propagating modes are the worst and the critical curves is level. For $U/v_i > 3$ cyclotron resonance modes dominate and the critical curve rises steadily.

As an application of our results, we consider the stability of the CIT.² Numerical simulations of this device indicate that the range of values of T_i/T_e and U/v_i are those delimited by the rectangle in the upper left hand corner of Fig. 5 denoted CIT. It is thus seen that, as far as electrostatic modes are concerned, the CIT is stable. There is the possibility that it be

unstable because the perpendicular and parallel temperatures of the ion beam are not equal, while our stability results only apply when they are equal. This seems unlikely, but even if it were so, these instabilities should lead to a situation with equal perpendicular and parallel temperatures in the same general region of the T_i/T_e , U/v_i diagram.

IVb: Electromagnetic Modes

Our results concerning the stability of electromagnetic modes may be interpreted very simply. In previous stability investigations of the case when one beam has negligible density compared to the other, it was found that the most unstable mode was the Alfvén mode, and that instability set in when the secondary beam velocity exceeded the propagation speed of the Alfvén mode carried by the primary beam. We find that the analogous result holds for our case of two equal density beams. The worst mode is the Alfvén mode "carried" by one beam, and it becomes unstable when the relative velocity of the two beams first exceeds the propagation speed of the Alfvén wave by a small amount. Under these circumstances the wave is situated in velocity space at the dotted line in Fig. 1. The ions of the second beam interact resonantly with the wave through the variation in the magnetic field strength and thus amplify the wave. It is a remarkable fact that the second beam has little effect on the wave propagation. One finds that for the frequency of the wave in the frame of the other beam ($\omega + k_3 U$), which is about one half the ion cyclotron frequency, and the known angle of propagation the phase velocity of the wave is about equal to that predicted by cold plasma theory. Now $v_A/v_i = \sqrt{1/\beta}$ so the wave will propagate more slowly than the other beam when $1/\beta$ is smaller than a critical value depending on the angle of propagation and its doppler shifted frequency $\omega + k_3 U \approx 2k_3 U$. Our stability

analysis selects out optimum values for instability. Because the ratio $v_A/v_i = \sqrt{1/\beta}$ and the wave carried by the first beam has relative propagation velocity of order v_A we can say roughly that $1/\beta$ is of order $(2U/v_i)^2 \alpha$, where α represents the reduction from v_A deduced from cold plasma wave theory.

The instability curve corresponds to small values of μ as indicated in Fig. 8.

The cyclotron resonances interaction seems to play a small role in the stability analysis and all the large μ modes are not critical for stability.

As can be seen from Fig. 10, the growth rates are a reasonable fraction of $k_3 v_i$ when β differs significantly from β_c . Thus, these are definitely strong modes.

The approximation of the electromagnetic dispersion relation is definitely not rigorous as discussed in the last half of Appendix C. The accuracy of the points on the marginal β curve are indicated by the lengths of the bars in Fig. 8. It is seen that the corrections are small except for $U/v_i > 6$. In general, the small μ results on stability are valid in the electromagnetic approximation but the larger μ results may not be. Since the smaller μ modes seem most unstable, we may place some confidence in the stability curves Figs. 8 and 9. (It is the case that the entire unstable region is unstable to small μ modes.) However, to be certain of the stability of the $\beta < \beta_c$ region it will be necessary to examine the full dispersion relation for larger μ modes. This has not yet been done.

The only previous calculation of electromagnetic modes stability is that of Perkins.⁷ He makes a different approximation, namely large θ , and obtains a different dispersion relation. This is the dispersion relation obtained by treating the 2 - 2 element as large and setting the minor of the first and

third columns and rows to zero. His result is analytic and his stability condition is indicated in Fig. 8 by dashes. Because the larger θ modes were more stable in our calculation, it is not surprising that his unstable region lies inside ours. It is interesting that when our electromagnetic dispersion relation breaks down at $U/v_i = 6$, it seems to approach his results.

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APPENDIX A

DERIVATION OF THE ELECTROSTATIC DISPERSION RELATION

The dispersion relation for a hot plasma is

$$\begin{vmatrix} 4\pi K_{11} - \frac{c^2 k_3^2}{\omega^2} & 4\pi K_{12} & 4\pi K_{13} - \frac{c^2 k_1 k_3}{\omega^2} \\ 4\pi K_{21} & 4\pi K_{22} - \frac{c^2 k^2}{\omega^2} & 4\pi K_{23} \\ 4\pi K_{31} - \frac{c^2 k_1 k_3}{\omega^2} & 4\pi K_{32} & 4\pi K_{33} - \frac{c^2 k_1^2}{\omega^2} \end{vmatrix} = 0, \quad (A-1)$$

where the components K_{ij} of the dielectric tensor are given in Akhiezer et al.¹ The electrostatic approximation corresponds to $c^2 k^2 / \omega^2 \gg |K_{ij}|$; that is, electrostatic waves are those propagating at speeds slow compared to the speed of light in a plasma. Since, for ω near or smaller than ω_i , we generally have $4\pi K_{11} \approx 4\pi K_{22} \approx c^2 / v_A^2$ and the rest of the elements are smaller, the electrostatic approximation reduces essentially to $\omega/k \ll v_A$. Expanding the determinant as a polynomial in $c^2 k^2 / \omega^2$ we find the coefficient of $(c^2 k^2 / \omega^2)^3$ cancels and we have

$$(4\pi K_{11} k_1^2 + 8\pi K_{13} k_1 k_2 + 4\pi K_{33} k_3^2 + k^2) \frac{c^4}{\omega^4} + O(c^2) = 0. \quad (A-2)$$

Keeping only the largest term we find the dispersion relation in the electrostatic limit reduces to

$$k_1^2 (4\pi K_{11}^i + 4\pi K_{11}^e + 1) + 2k_1 k_3 (4\pi K_{13}^i + 4\pi K_{13}^e) + k_3^2 (4\pi K_{33}^i + 4\pi K_{33}^e + 1) = 0 \quad (A-3)$$

We restrict ourselves to $\omega \ll \omega_{ce}$ and zero electron gyroradius so only the K_{33}^e component of the electron dielectric tensor need be kept. In this limit

$$4\pi K_{33}^e = \frac{1}{k_3^2 \lambda_D^2} \quad (A-4)$$

where $\lambda_{de} \equiv (T_e / 4\pi n_e e^2)^{1/2}$ is the electron Debye length. The components K_{11}^i , K_{13}^i , K_{33}^i of the ion dielectric tensor are in the notation of Section II,

$$4\pi K_{11}^i = \frac{\omega_{pi}^2}{\omega^2} \sum_{\pm} \frac{\omega \pm k_3 U}{k_3 v_i} \sum_{l=-\infty}^{\infty} \frac{l^2 e^{-\mu_I} |l|(\mu)}{\mu} Z\left(\frac{\omega \pm k_3 U - l\omega}{k_3 v_i}\right) \quad (A-5)$$

$$4\pi K_{33}^i = \frac{2\omega_{pi}^2}{\omega^2} \sum_{\pm} \frac{\omega \pm k_3 U}{k_3 v_i} \sum_{l=-\infty}^{\infty} \left(\frac{\omega - l\omega_i}{k_3 v_i}\right)^2 e^{-\mu_I} |l|(\mu) Z\left(\frac{\omega \pm k_3 U - l\omega_i}{k_3 v_i}\right) + 4\omega_{pi}^2 \left(\frac{\omega}{k_3 v_i}\right)^2 \quad (A-6)$$

$$4\pi K_{13}^i = 4\pi K_{31}^i = \frac{\omega_{pi}^2}{\omega^2} \sum_{\pm} \frac{\omega - k_3 U}{k_3 v_i} \sum_l \frac{\omega - l\omega_i}{k_3 v_i} \frac{e^{-\mu_I} |l|(\mu)}{\sqrt{\mu/2}} Z\left(\frac{\omega + k_3 U - l\omega_i}{k_3 v_i}\right) \quad (A-7)$$

where $\omega_{pi}^2 \equiv 4\pi n_i e^2 / m$.

From Eqs. (A-4) - (A-7) we have

$$\begin{aligned}
 & 4\pi (k_1^2 K_{11}^i + 2k_1 k_2 K_{13}^i + k_3^2 K_{33}^i) \\
 &= \frac{\omega_{pi}^2}{\omega^2} \sum_{\pm, l} \frac{\omega \pm k_3 U}{k_3 v_i} Z \left(\frac{\omega + k_3 U - l\omega_i}{k_3 v_i} \right) e^{-\mu} I_{|l|}(\mu) \\
 & \times \left[\frac{l^2 k_1^2}{\mu} + \frac{2\sqrt{2}}{\sqrt{\mu}} k_1 k_3 l \frac{\omega - l\omega_i}{k_3 v_i} + \frac{2k_3^2 (\omega - l\omega_i)^2}{k_3^2 v_i^2} \right] + \frac{4\omega_{pi}^2}{\omega^2} \left(\frac{\omega}{k_3 v_i} \right)^2 k_3^2. \quad (A-8)
 \end{aligned}$$

The bracket reduces to $2\omega^2/v_i^2$. Making use of this result and Eq. (A-4) we can reduce Eq. (A-3) to

$$\begin{aligned}
 & \frac{1}{2} \frac{T_e}{T_i} \frac{1}{\lambda_{De}^2} \sum e^{-\mu} I_{|l|}(\mu) \frac{\omega \pm k_3 U}{k_3 v_i} Z \left(\frac{\omega - l\omega_i \pm k_3 U}{k_3 v_i} \right) \\
 & + \frac{T_e}{T_i \lambda_{De}^2} + \frac{1}{\lambda_{De}^2} k^2 = 0. \quad (A-9)
 \end{aligned}$$

Finally, if we multiply this equation by $\lambda_{De}^2 T_i/T_e$ and assume $k^2 \lambda_{De}^2 \ll 1$ we get Eq. (3).

Now let us consider a plasma with unmagnetized ions. The electrons may be magnetized or not, their contribution to the dispersion relation (A-9) is simply $1/\lambda_{De}^2$ in either case. The contribution of the ions may be obtained by projecting the equilibrium ion distribution on the direction of $\vec{k}$. The resulting distribution is that of Eq. (1) with U replaced by $U' \equiv U \cos \phi$. Then the dispersion relation is obtained from (A-9) by taking $k_1 = 0$ in (A-9) and setting $k_3 = k$ (taking the z axis along $\vec{k}$).

$$\frac{1}{2} \frac{T_e}{T_i} \frac{1}{\lambda_{De}} \int \frac{\omega \pm kU}{kv_i} z \left(\frac{\omega \pm kU}{v_i} \right) + \frac{T_e}{T_i \lambda_{De}^2} + \frac{1}{2} + k^2 = 0 \quad (A-10)$$

This reduces to Eq. (5) on multiplying by $(T_i/T_e)\lambda_{De}^2$ and dropping $k^2\lambda_{De}^2$.

The physical significance of the electrostatic mode can be seen if we consider the three equations in E_x , E_y , and E_z corresponding to the determinant of Eq. (A-1) and keep only the c^4 coefficients. Then, $E_y = 0$ and $E_x:E_z = k_1:k_3$. That is, $\vec{E}$ is parallel to $\vec{k}$ and has no curl, it is purely electrostatic.

In Section II the critical frequency at marginal stability is shown to be $\omega = 0$. Thus, at this point the electrostatic mode is an exact solution of the full dispersion relation. The stability or instability of other regions of Fig. 5 is derived from consideration of waves with approximate phase velocity $\omega/k = v_i$. Thus, the sufficient condition for accuracy of the electrostatic mode solutions is $\beta \ll 1$.

APPENDIX B

DERIVATION OF THE LARGE μ LIMIT OF THE ELECTROSTATIC DISPERSION RELATION

If μ is large, we must take many terms in the sum over the Bessel function in Eq. (3). In fact, we may replace the sum over l by an integral.

First, we make use of the result that for large μ

$$e^{-\mu} I_l(\mu) \approx \frac{e^{-l^2/2\mu}}{\sqrt{2\pi\mu}}, \quad (B-1)$$

a result that can be obtained by asymptotic evaluation of the expression

$$e^{-x} I_l(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-x(1 - \cos\phi) - il\phi] d\phi, \quad (B-2)$$

for x large. This result is valid unless $l^2 \gg 2\mu$ in which case we may neglect $I_l e^{-\mu}$ anyway. Substitution of (B-2) into the summation Eq. (3) and replacing the sum by an integral we get for each term

$$\frac{1}{2} \int e^{-\mu} I_l(\mu) \times Z\left(x - \frac{l\theta}{\sqrt{2\mu}}\right) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dl e^{-l^2/2\mu}}{\sqrt{2\pi\mu}} Z\left(x - \frac{l\theta}{\sqrt{2\mu}}\right), \quad (B-3)$$

where $x \equiv \omega \pm k_3 U/k_3 v_i$, and $\theta \equiv k_1/k_3$. Now, if $\text{Im } z > 0$

$$Z(z) = 2i \int_0^{\infty} dt \exp[-t^2 + (1/2)zt]. \quad (B-4)$$

Substitute Eq. (B-4) into Eq. (B-3) and let $l = \sqrt{2\mu} y$. We obtain for this term

$$\frac{2ix}{\sqrt{\pi}} \int_0^{\infty} dt e^{-t^2} \int_{-\infty}^{\infty} dy \exp[-y^2 + 2i(x-y\theta)t]$$

$$= 2ix \int_0^{\infty} dt \exp[-t^2(1 + \theta^2 + 2ixt)]$$

$$= \frac{x}{\sqrt{1 + \theta^2}} {}_2F_1\left(\frac{x}{\sqrt{1 + \theta^2}}\right).$$

Making use of this result we easily reduce Eq. (3) to Eq. (5).

APPENDIX C

DERIVATION OF THE ELECTROMAGNETIC DISPERSION RELATION

The electromagnetic dispersion relation is obtained by setting the 2×2 minor in the upper left hand corner of the full dispersion relation Eq. (A-1) to zero.

$$\begin{vmatrix} 4\pi K_{11} - \frac{c^2 k_3^2}{\omega^2} & 4\pi K_{12} \\ 4\pi K_{21} & 4\pi K_{22} - \frac{c^2 k^2}{\omega^2} \end{vmatrix} = 0 \quad (C-1)$$

The justification for expecting this procedure to be valid is that K_{33}^e is expected to be large compared to the other components of the determinant.

K_{11} is given by Eq. (A-5). Also

$$4\pi K_{22} = 4\pi K_{22}^i = \frac{\omega^2}{\omega^2} \sum_{\pm} \frac{\omega - k_3 U}{k_3 v_i} \sum_{\ell=-\infty}^{\infty} e^{-\mu} \left[\left(\frac{\ell^2}{2\mu} + 2\mu \right) I_{|\ell|}(\mu) - 2\mu I'_{|\ell|}(\mu) \right] \\ \times Z \left(\frac{\omega \pm k_3 U - \ell \omega_i}{k_3 v_i} \right) \quad (C-2)$$

and $4\pi K_{12} = -4\pi K_{21} = 4\pi (K_{12}^i + K_{12}^e)$ where

$$4\pi K_{12}^i = -i \frac{\omega_{pi}^2}{\omega^2} \sum_i \frac{\omega + k_3 U}{k_3 v_i} e^{-\mu} \sum_{l=-\infty}^{\infty} l (I_{|l|} - I'_{|l|}) z \left(\frac{\omega \pm k_3 U - l \omega_i}{k_3 v_i} \right)$$

$$4\pi K_{12}^e = -i 2 \omega_{pi}^2 / \omega_i \omega \quad (C-3)$$

Expanding the determinant, multiplying it by $(\omega^2/\omega_{pi}^2)^2$ and introducing our notation, we have

$$G(\omega) = \left(K'_{11} - \frac{2}{\beta} \frac{\mu k_3^2}{k_1^2} \right) \left[K'_{22} - \frac{2}{\beta} \mu \left(1 + \frac{k_3^2}{k_1^2} \right) \right] - K_{12}'^2 = 0 \quad (C-4)$$

where $K'_{11} = 4\pi\omega^2/\omega_{pi}^2 K_{11}$, $K'_{22} = 4\pi\omega^2/\omega_{pi}^2 K_{22}$ and $K'_{12} = 4\pi(\omega^2/\omega_{pi}^2) K_{12}$. This is just Eq. (8). Substituting for K'_{11} from Eq. (A-5) for K'_{22} from Eq. (C-2) for K_{12}^e and K_{12}^i from Eq. (C-3), we obtain Eqs. (9) - (11) for K_{11}' , K_{22}' and K_{12}' .

In order to estimate the validity of our solutions for the electromagnetic modes we include a correction to the Eq. (C-1) when K_{33} is not infinite. Still regarding K_{31} and K_{32} as small this correction obtained from Eq. (A-1) is

$$\left(K_{33} - \frac{c^2 k_3^2}{\omega^2} \right) G(\omega) - \left(\frac{\omega^2}{2} \right)^2 \left(4\pi K_{22} - \frac{ck^2}{\omega^2} \right) \frac{c^4 k_1^2 k_3^2}{4\omega} = 0 \quad (C-5)$$

Writing $\omega = \omega_0 + \delta\omega$ where $G(\omega_0) = 0$, and setting $4\pi K_{33} - c^2 k_3^2/\omega^2 = 4\pi K_{33}^e$

$\sim 1/k_3^2 \lambda_{De}^2$ we find

$$\frac{\delta \omega_G(\omega_o)}{k_3^2 \lambda_{De}^2} = (4\pi^2 k_{22}^2 - \frac{c^2 k_2^2}{\omega_o^2}) \frac{c^4 k_1^2 k_3^2}{\omega_{pi}^4} \quad (C-6)$$

The right hand side of Eq. (C-5) is evaluated at ω_o . From this estimate of $\delta \omega$ the shift in the initial $1/\beta$ curve of Fig. 8 may also be estimated. The size of resulting correction to the marginal stability curve is indicated in Fig. 8 by the vertical bars.

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Fig. 1. The ion and electron velocity distributions. The dashed line indicates the ions that are in cyclotron resonance. (PPPL-802306)

$$\mu \equiv \frac{k_1^2 T_1}{\omega_i^2 M}$$

$$\theta \equiv \tan \phi = \frac{k_1}{k_3}$$

Fig. 2. The wave parameters. (PPPL-802307)

Fig. 3. The Nyquist diagram for electrostatic waves. (PPPL-802309)

Fig. 4. Critical electrostatic T_i/T_e curves for fixed μ .
(PPPL-802302)

Fig. 5. The stability diagram for electrostatic modes. Previous results are indicated. (See text.) (PPPL-802308)

Fig. 6. The derivative of the electrostatic growth rate on the marginal curve.
(PPFL-802305)

Fig. 7. Nyquist diagrams for electrostatic modes. (See text.)
(PPPL-802301)

Fig. 8. The stability diagram for electromagnetic modes.
(PPPL-802304)

Fig. 9. The stability diagrams for electromagnetic modes with β^* . (PPPL-802310)

Fig. 10. The derivative of the electromagnetic growth rate on the marginal curve. (PPPL-802303)