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AND ELECTROMAGNETIC INTERACTIONS

BY

SUSUMU OKUBO

THE UNIVERSITY OF ROCHESTER
DEPARTMENT OF PHYSICS AND ASTRONOMY
ROCHESTER, NEW YORK

MASTER

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Classification of Gauge Groups for Unified Weak
and Electromagnetic Interactions*⁺

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Susumu Okubo

Department of Physics and Astronomy

University of Rochester, Rochester, New York 14627

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Abstract

We investigate general structure of the gauge group G underlying the unified weak and electromagnetic interactions, assuming that G is compact and that all quarks have only electric charges $2/3$ and $-1/3$. Then, we prove that G cannot contain any exceptional Lie groups G_2 , F_4 , E_6 , E_7 , and E_8 as its factor. Moreover, the underlying representations for quark multiplets to be used must be one of fundamental representations for each component simple Lie group contained as a factor of G . For example, only spinor representation is allowed for $SO(2l+1)$ groups. If G is semi-simple in addition, then G must be a product of $SU(3l)$ groups. This latter fact is likely in conflict with the present experimental data.

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The unified electromagnetic and weak interaction theory of Weinberg and Salam appears to be now on solid ground. However, parity non-violation in atomic physics may require enlarging the group structure beyond the original $SU(2) \otimes U(1)$ group. Indeed, many larger groups have been proposed and discussed by various authors. However, these attempts are not systematic. Here in this note, we shall make a systematic classification of all possible gauge groups G and their multiplet representations to be used. We consider here only the hadronic sector with the following assumptions. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G . Then our assumptions are:

- (1) G is a compact Lie group.
- (2) The electric charge operator Q is a member of a Cartan sub-algebra of $\mathfrak{g}$.
- (3) Q can assume only two eigenvalues $2/3$ and $-1/3$ in any irreducible representation for quark multiplets.

These ansätze are physically reasonable and normally satisfied in any conventional model of unified weak and electromagnetic interactions. First of all, if G is not compact, then its non-trivial unitary representations are automatically of infinite dimension so as to necessitate introduction of infinite number of quarks. This may be undesirable at least for the present. With respect to the second ansatz, it is satisfied within the framework of standard gauge theories¹⁾. Finally, although we have assumed the (flavor) quark electric charges to be either $2/3$ or $-1/3$, this fact is not crucial for most of our discussions except for the case of G being semi-simple.

Now, before going into details, we shall first state a summary of the main results. Since G is assumed to be a compact Lie algebra, it

is essentially equivalent to²⁾ a product of a semi-simple Lie group and $U(1)$ groups, as far as its Lie algebra is concerned. Hence, we may write

$$G = G_1 \otimes G_2 \otimes \dots \otimes G_p \otimes U(1) \otimes U(1) \otimes \dots \otimes U(1), \quad (1)$$

where G_j ($j=1,2,\dots,p$) are simple Lie groups. If G is semi-simple, then $U(1)$ groups should not be present in Eq. (1). Corresponding to Eq. (1), we may write the electric charge operator Q as a direct sum

$$Q = Q_1 + Q_2 + \dots + Q_p + R \quad (2)$$

where Q_j ($j=1,2,\dots,p$) are components of Q in the Lie algebra $\mathfrak{g}_j$ of group G_j and where R is a direct sum of all infinitesimal generators of $U(1)$ groups present in Eq. (1). Here, we consider only irreducible representations of G_j and $U(1)$ groups. Then, we can prove the following:

(1) None of the simple groups G_j ($j=1,2,\dots,p$) can be³⁾ one of exceptional Lie algebras G_2 , F_4 , E_6 , E_7 , and E_8 .

(2) If G is semi-simple, then G must be a product of $SU(3l)$ groups.

If we relax the condition (3) into a weaker ansatz that Q can assume only two distinct real eigenvalues u and v such that $u + v \neq 0$, then G is still a product of $SU(n)$ ($n \geq 3$) groups although n need not now be an integer multiple of three in contrast to the previous case.

(3) Any irreducible representation $\{\lambda\}$ to be used for quark multiplets is trivial one for all simple groups G_j ($j=1,2,\dots,p$) except for one of them, say G_1 to be definite. In that case only irreducible representation to be used for G_1 must be one of fundamental ones.

More precisely, we adopt a lexiconal ordering of simple roots as in Patera and Sankoff⁴⁾. Then, the highest weight Λ for the irreducible representation must be:

- (i) A_n ($n \geq 1$), $\Lambda = \Lambda_j$ ($1 \leq j \leq n$) any fundamental representation.
- (ii) B_n ($n \geq 2$), $\Lambda = \Lambda_n$ = spinor representation
- (iii) C_n ($n \geq 2$), $\Lambda = \Lambda_1$ = basic representation
- (iv) D_n ($n \geq 3$), $\Lambda = \Lambda_1$ or Λ_{n-1} or Λ_n = either basic or spinor representation.

Now, let us sketch a proof for these statements. We first note the following facts. Only one of Q_j ($j=1,2,\dots,p$), say Q_1 , can assume at most two distinct eigenvalues, while the rest Q_j ($p \geq j \geq 2$) must assume zero eigenvalues in any given multiplet. To show it, we rewrite Eq. (2) as a direct sum

$$Q = Q_1 + \tilde{Q}$$

where $\tilde{Q}$ is a sum of the rest. Suppose now that both Q_1 and $\tilde{Q}$ can assume at least two distinct eigenvalues. Then, their direct sum Q must assume at least three distinct eigenvalues as we may easily check. But this contradicts our ansatz (3) that Q can assume only two distinct eigenvalues. Therefore, only one of Q_1 and $\tilde{Q}$ can assume two distinct eigenvalues. Suppose that Q_1 can take a single eigenvalue. Then, its value must be zero because the simplicity of G_1 demands

$$\text{Tr } Q_1 = 0 \quad (3)$$

In that case, the simplicity of G_1 requires the representation to be trivial for G_1 . We replace Q by $\tilde{Q}$ and repeat the same reasoning to

show that only one of Q_j can take two distinct eigenvalues while the rest of Q_j is identically zero. Suppose that Q_1 can assume two distinct eigenvalues x and y . Then, since all other Q_j with $j \neq 1$ are identically zero in the representation under consideration, we can rewrite Eq. (2) as

$$Q = Q_1 + R.$$

Since the Abelian generator R can take only a given common eigenvalue z in the irreducible representation, the eigenvalues of Q must be $x + z$ and $y + z$ which must agree with $2/3$ and $-1/3$. This requires

$$\begin{aligned} x - y &= \pm 1, \\ x + y + 2z &= \frac{1}{3}. \end{aligned} \quad (4)$$

Next, let us assume that the Lie algebra $\mathfrak{g}_1$ is not of the type A_n ($n \geq 2$) which corresponds to $SU(n+1)$ ($n \geq 2$) group. Then, as we have proved elsewhere⁵⁾, we must have

$$\text{Tr } Q_1 = \text{Tr } (Q_1)^3 = 0. \quad (5)$$

Let n_1 and n_2 be the number of states in the representation corresponding to eigenvalues x and y of Q_1 , respectively. Then, Eq. (5) leads to

$$\begin{aligned} n_1 x + n_2 y &= 0, \\ n_1 x^3 + n_2 y^3 &= 0 \end{aligned}$$

which demands $x + y = 0$ and $n_1 = n_2$. Therefore, Eq. (4) requires

$$x = -y = \frac{1}{2}, \quad z = \frac{1}{6} \quad (6)$$

where we assumed $x > y$. Note that $z = 1/6$ implies a presence of the $U(1)$ group in Eq. (1). Hence, if G is semi-simple, this is not possible and we conclude that the group G_1 must be $SU(n)$ ($n \geq 3$) group. The same conclusion should be applicable to all other G_j as long as there is a multiplet in which Q_j is not trivial. The same result equally applies when we relax our condition (3) into a weaker one that Q assumes two distinct eigenvalues u and v such that $u + v \neq 0$.

Next, let us designate a simple root system by

$$\pi = \{ \alpha_1, \alpha_2, \dots, \alpha_n \} \quad (7)$$

and adopt a lexiconal ordering of simple roots as in reference (4).

Correspondingly, the fundamental weight system Ω is given by

$$\Omega = \{ \Lambda_1, \Lambda_2, \dots, \Lambda_n \} \quad (8)$$

which satisfies

$$2(\Lambda_j, \alpha_k) = \delta_{jk} (\alpha_j, \alpha_j) \quad (9)$$

for $j, k=1, 2, \dots, n$. Also, let

$$S_j \equiv S_{\alpha_j} \quad (10)$$

be a special Weyl reflection operator with respect to a plane perpendicular to $\alpha_j \in \pi$. Then, we have of course

$$S_j \Lambda_k = \Lambda_k - \delta_{jk} \alpha_k. \quad (11)$$

The highest weight Λ of any irreducible representation of G_1 is written as

$$\Lambda = m_1 \Lambda_1 + m_2 \Lambda_2 + \dots + m_n \Lambda_n \quad (12)$$

where m_j ($j=1,2,\dots,n$) are non-negative integers specifying the representation.

Let us use the standard Cartan-Weyl basis of $\mathfrak{g}_1$, specified by H_j , E_β , and $E_{-\beta}$. Let

$$\xi = (\xi^1, \xi^2, \dots, \xi^n) \quad (13)$$

be a constant numerical vector in the root space, and write

$$Q_1 = (\xi, H) \equiv \sum_{j=1}^n \xi^j H_j, \quad (14)$$

since Q_1 is a member of a Cartan sub-algebra of $\mathfrak{g}_1$.

Let M be the generic weight belonging to the representation with the highest weight Λ . Then eigenvalues of Q_1 are of the form

$$(\xi, M) \quad (15)$$

which must assume only two values x and y for all weights M . This fact restricts the type of the representation to be used and the form of the vector ξ . Hereafter, let us discuss mainly the case of $\mathfrak{g}_1$ being not of the algebra A_n ($n \geq 2$), although the latter case can be dealt with similarly with some modifications. In that case, (ξ, M) can assume only two values $+1/2$ and $-1/2$. By changing the sign of ξ if necessary, we can normalize

$$(\xi, \Lambda) = \frac{1}{2}. \quad (16)$$

We can prove the following lemma.

Lemma

Let β be any positive root. Then,

- (i) (ξ, β) can assume values only either 1 or 0 or -1.

(ii) If we have

$$\frac{2(\lambda, \beta)}{(\beta, \beta)} \geq 2,$$

then we must have $(\xi, \beta) = 0$.

(iii) If we have

$$\frac{2(\lambda, \beta)}{(\beta, \beta)} = 1,$$

then (ξ, β) assumes values 0 or 1 but not -1.

The proof goes as follows. Let M be a weight and consider its β -series of weights:

$$M_k = M + k\beta, \quad k \geq l \geq -j.$$

Then, it is known⁶⁾ that

$$\left| \frac{2(M, \beta)}{(\beta, \beta)} \right| = k + j = \text{integer}.$$

Especially, if $k + j \geq 2$, then we have more than three weights belonging to the series. In that case, eigenvalues (ξ, M_k) of Q_1 would assume more than three distinct eigenvalues, unless we have $(\xi, \beta) = 0$.

Therefore, if we have

$$\left| \frac{2(M, \beta)}{(\beta, \beta)} \right| \geq 2,$$

then we must have $(\xi, \beta) = 0$. Replacing M by λ , this gives the second part of the lemma. Next, consider the case $k + j = 1$. Then,

$$S_{\beta} M = M - \frac{2(M, \beta)}{(\beta, \beta)} \beta$$

is also a weight so that both (ξ, M) and

$$(\xi, S_\beta M) = (\xi, M) - \frac{2(M, \beta)}{(\beta, \beta)} (\xi, \beta)$$

must assume values either $1/2$ or $-1/2$. This implies that

$$\frac{2(M, \beta)}{(\beta, \beta)} (\xi, \beta)$$

can take values either 1 , or -1 , or 0 . Now, for a given non-zero root β , we can always find⁷⁾ a weight M such that $(\beta, M) \neq 0$, since we are dealing with non-trivial irreducible representation. Then, for such a M , the value of $2(\beta, M)/(\beta, \beta)$ is non-zero integer. Combining this fact with several statements found above, we conclude that the range of (ξ, β) is limited to three values 1 , 0 , and -1 . This proves the (i) of the lemma. The last part (iii) of the lemma can be similarly proved by considering two weights Λ and $S_\beta \Lambda$ together with the normalization condition Eq. (16).

Using this lemma, we can prove that G_1 cannot be any one of exceptional Lie algebras G_2 , F_4 , E_6 , E_7 , and E_8 as follows. By (i) of the lemma, (ξ, α_j) can take values only of 1 , 0 , and -1 . Then, studying explicit forms of all fundamental weights Λ_j in terms of simple roots, we can easily see without any calculation that (ξ, Λ) can never take half-integer value $1/2$ except possibly for cases of E_7 . For the E_7 , we utilize a procedure based upon the lemma to exclude this case also. The details of the procedure will be explained shortly for example of the algebra B_n .

Another proof for impossibility of exceptional Lie algebras is as follows. We note that all exceptional Lie algebras G_2 , F_4 , E_6 , E_7 , and E_8 as well as A_1 and A_2 do not have⁸⁾ any genuine fourth-order

Casimir invariants. On the basis of this fact, we can prove⁹⁾ the validity of

$$T_V(Q_i)^4 = K(\lambda) [T_V(Q_i)^2]^2 \quad (17)$$

in any irreducible representation $\{\lambda\}$ of these algebras, where $K(\lambda)$ is given by

$$K(\lambda) = \frac{1}{2[2 + d(\lambda_0)]} \frac{d(\lambda_0)}{d(\lambda)} \left\{ 6 - \frac{I_2(\lambda_0)}{I_2(\lambda)} \right\}. \quad (18)$$

Here, $\{\lambda_0\}$ designates the adjoint representation, where $d(\lambda)$ and $I_2(\lambda)$ refer to dimension and eigenvalue of the second order Casimir invariants of the representation $\{\lambda\}$, respectively. Now, for algebras, A_1 , G_2 , F_4 , E_6 , E_7 , and E_8 (but not for A_2), we have also the validity of Eq. (5) and hence of Eq. (6). As the result, Eqs. (17) and (18) give

$$\frac{I_2(\lambda_0)}{I_2(\lambda)} = 4 \left[1 - \frac{1}{d(\lambda_0)} \right].$$

We can check that this relation cannot be satisfied for any irreducible representation of all exceptional Lie algebras. With respect to the algebra A_1 , we can easily verify that it is satisfied only for two dimensional spinor representation of A_1 , i.e. only two dimensional representation is permissible for $G_1 = SU(2)$ group.

Next, our lemma is also sufficient to show that the irreducible representation $\{\lambda\}$ must coincide with one of fundamental representation, i.e., $\lambda = \lambda_j$. However, its demonstration depends upon case by case investigation. Here as an illustration, let us consider the algebra B_n . In that case, we note first

$$\beta = \Lambda_2 = \alpha_1 + 2[\alpha_2 + \alpha_3 + \dots + \alpha_n] \quad (19)$$

is a dominant root of B_n , and second that

$$\frac{2(\Lambda, \beta)}{(\beta, \beta)} = m_1 + 2[m_2 + m_3 + \dots + m_{n-1}] + m_n. \quad (20)$$

Also from an inspection of explicit forms of all fundamental weights Λ_j for B_n , we see that (ξ, Λ) can assume half integer value $1/2$ only when m_n is an odd integer so that $m_n \geq 1$. Therefore, as long as one of $m_1, m_2, \dots, m_{n-1}$ is non-zero, we have

$$\frac{2(\Lambda, \beta)}{(\beta, \beta)} \geq 2$$

so that by our lemma, we find

$$(\xi, \beta) = 0. \quad (21)$$

Let us suppose for example $m_1 \geq 1$. Then, all roots β_j ($2 \leq j \leq n-1$) of form

$$\beta_j = s_j s_{j-1} \dots s_3 s_2 \beta = \beta - (\alpha_2 + \alpha_3 + \dots + \alpha_j) \quad (22)$$

satisfy still

$$\frac{2(\Lambda, \beta_j)}{(\beta_j, \beta_j)} \geq 2.$$

As the consequence, we must have

$$(\xi, \beta_j) = 0 \quad (23)$$

also for $j = 2, 3, \dots, n-1$. Together with Eq. (21), these imply

$$(\xi, \alpha_2) = (\xi, \alpha_3) = \dots = (\xi, \alpha_{n-1}) = 0, \quad (24)$$

$$(\xi, \alpha_1) + 2(\xi, \alpha_n) = 0. \quad (25)$$

However, since (ξ, α_j) and (ξ, α_n) can assume only values 0, +1, or -1 by our lemma, the validity of (25) requires also

$$(\xi, \alpha_1) = (\xi, \alpha_n) = 0$$

so that we have $(\xi, \alpha_j) = 0$ for all $j = 1, 2, \dots, n$. Since n simple roots α_j span the whole root space, this leads to result $\xi=0$ and consequently $Q_1=0$, in contradiction with $Q_1 \neq 0$. This shows that any case with $m_1 \neq 0$ is impossible. Similarly, we can prove that only possible case is $\Lambda = \Lambda_n$, i.e. that the representation must be the spinor representation. Moreover, we can show that ξ must be given by

$$\xi = \frac{2}{(\alpha_\ell, \alpha_\ell)} \Lambda_\ell - \frac{2}{(\alpha_{\ell-1}, \alpha_{\ell-1})} \Lambda_{\ell-1} \quad (26)$$

where ℓ could be any integer satisfying $n \geq \ell \geq 1$. For the case $\ell=1$, we omit the second term in Eq. (26). Similarly, we can prove the fact that $\Lambda = \Lambda_1$ is the only possible case for the algebra C_n , while the algebra D_n ($n \geq 3$) admits $\Lambda = \Lambda_1$, or Λ_{n-1} , or Λ_n .

For the algebra A_n ($n \geq 2$), we have to modify our argument slightly, since Q_1 can assume now two eigenvalues x and y such that $x-y = \pm 1$. Nevertheless, we can still derive that only $\Lambda = \Lambda_j$ ($1 \leq j \leq n$) is allowed. Moreover, the condition $(\xi, \Lambda) = x$ or y imposes a constraint that x (or y) must have a form

$$x = \frac{N}{n+1}$$

where N is an integer. Therefore, if G is semi-simple, and consequently if x is either $2/3$ or $-1/3$, then $n+1$ must be an integer multiple of three in agreement with the result proven elsewhere¹⁰⁾ by a different method. Then, $G_1 = SU(3l)$.

We would like to make a comment that the case of G being semi-simple is perhaps experimentally ruled out by the following reason. As we noted already, G must be in that case a product of $SU(3l)$ groups. The model, based upon $G=SU(3)$ has been studied by several authors¹¹⁾, and these models predict a pure axial vector for neutral current in contradiction with the present experimental data¹²⁾. We could modify the model so as to evade the pure axial vector current structure. However, some other experimental difficulties persist¹³⁾. Hence, the group $G = SU(3)$ is perhaps unlikely to be compatible with the experiment. The model on the basis¹⁴⁾ of $G=SU(3) \otimes SU(3)$ will not show such experimental difficulties. However, recently Chanowitz et al¹⁵⁾ noted that any gauge theory involving any $SU(n)$ ($n \geq 3$) group would require the so-called unnatural suppression of flavor-changing neutral currents. This could be a problem.

Finally, let us classify all compact gauge groups according to their rank N . Excluding trivial cases of pure abelian groups, we see that the lowest possible case $N=1$ corresponding to the $SU(2)$ group is forbidden since it is a simple group not of a form $SU(3l)$. The next case $N=2$ admits simple groups of $SU(3)$, $S_p(4) \approx SO(5)$, $SO(4) \approx SU(2) \otimes SU(2)$, G_2 , as well as non-semi-simple group $SU(2) \otimes U(1)$. However, only $SU(3)$ among these semi-simple groups satisfies our

criteria found in this note, so that only permissible groups with $N=2$ are $SU(3)$ and $SU(2) \otimes U(1)$. However, we have seen the experimental difficulty for the $SU(3)$. This leaves only the Weinberg-Salam gauge group $SU(2) \otimes U(1)$ as the only viable lowest rank gauge group. The next case $N=3$ which are compatible with our criteria are all non-semi-simple of the form $SU(2) \otimes U(1) \otimes U(1)$, $SU(2) \otimes SU(2) \otimes U(1)$, $Sp(4) \otimes U(1) \simeq SO(5) \otimes U(1)$, and $SU(3) \otimes U(1)$. We note that models based upon all these groups have been already proposed¹⁶⁾ by many authors. Hence, only new groups which have been likely unattempted before arise only for $N \geq 4$, such as $Sp(6) \otimes U(1)$ and $SO(7) \otimes U(1)$.

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