MASTER

CONCERNING GRIBOV VACUUM COPIES IN NON-ABELIAN GAUGE THEORY*

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Abstract

Construction of gauge field configurations $A_\mu^a(x)$ in an $SU(2)$ Yang-Mills theory satisfying everywhere $F_{\mu\nu}^a(x) = 0$ is discussed. Using the method of sections, a field related to the zero-size limit of an instanton is presented. The corresponding limit for a multi-instanton solution requires a generalization of the Landau gauge condition. Finally, an alternative method and explicit solution is given for the case of $\partial_\mu A_\mu^a = 0$.

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I. INTRODUCTION

A new ingredient in the quantization of non-abelian gauge theories, and a potentially important step towards their physical interpretation, has been pointed out recently by Gribov.¹ The principal idea is that a choice of gauge, such as the Coulomb gauge, leaves a non-trivial residual freedom in the choice of gauge field, this extra freedom being absent in the case of electrodynamics. For example, it is possible to find gauge transforms of the pure vacuum such that the Yang-Mills field

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c \quad (1)$$

vanishes everywhere and such that the local gauge condition is preserved.

In what follows, we refer to such A_μ^a as vacuum copies.

It has been speculated that this copying phenomenon may be of fundamental importance in the physical implications of Yang-Mills theory, perhaps pertaining to the elusive property of confinement. We leave this question open, but the possibility provides a strong motivation for this work; indeed, the literature on this subject is already expanding rapidly.²

In Ref.1, the discussion centered on the Coulomb gauge which served only as an example. Here we shall explore vacuum copies in the Landau gauge and a generalization thereof. The principal difference is that Coulomb gauge copies resemble the Wu-Yang form³ while those of the Landau gauge are expected to have possible structures related to the instanton of Belavin et al.⁴

To construct these copies we initially break translational invariance, although this would be restored at the end in any realistic calculation, using e.g. the method of collective coordinates. In Sec.II we consider the zero-size limit of one instanton, centered around one preferred space-time point and use the method of sections to construct a copy. In Sec.III, we study the zero-size limit of the multi-instanton, and find that the Landau

condition must be generalized, for this case.

In Sec.IV we present an alternative approach to vacuum copies in the Landau gauge, together with an explicit solution. Finally, in Sec.V are the concluding remarks, including a brief comment on the Faddeev-Popov⁵ formalism.

II. ZERO-SIZE LIMIT OF ONE INSTANTON

It appears that to maintain any given gauge condition and to find a vacuum beyond the pure case $A_\mu^a = 0$ it is necessary to break translational invariance. Thus, we consider first a vacuum copy with one preferred space-time center. Full translational invariance can then be restored by averaging all possible locations of this center.

For the Landau gauge we start from the BPST instanton⁴ in Euclidean space given by

$$A_\mu^a = \frac{2}{g} \frac{\eta_{\mu\nu}^a x_\nu}{(x^2 + \lambda^2)} \quad (2)$$

$$F_{\mu\nu}^a = -\frac{4}{g} \eta_{\mu\nu}^a \left(\frac{\lambda}{x^2 + \lambda^2} \right)^2. \quad (3)$$

As is well known⁴ this instanton carries a topological charge given by

$$Q = \frac{g^2}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \quad (4)$$

$$= 1. \quad (5)$$

Here $\tilde{F}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}^a$. Using Gauss' theorem, this may be re-written

$$Q = \frac{g^2}{32\pi^2} \int_{x^2 \rightarrow \infty} d^3\sigma_\alpha T_\alpha \quad (6)$$

where

$$T_\mu = \epsilon_{\mu\nu\alpha\beta} \left[2A_\nu^a \partial_\alpha A_\beta^a + \frac{2}{3} g \epsilon^{abc} A_\nu^a A_\alpha^b A_\beta^c \right]. \quad (7)$$

The parameter λ in Eq.(2) is arbitrary and reflects the scale-invariance of the theory; it characterizes the size of the instanton. For general λ one has

$$T_\mu = \frac{16x_\mu}{g^2(x^2 + \lambda^2)^3} (x^2 + 3\lambda^2) \quad (8)$$

$$\partial_\mu T_\mu = \frac{192}{g^2} \left(\frac{\lambda}{x^2 + \lambda^2} \right)^4. \quad (9)$$

In the zero-size limit $\lambda \rightarrow 0$, we have $F_{\mu\nu}^a = 0$ for all x_μ except at the origin, where a singularity develops. This singularity is where the topological charge accumulates since one now has

$$F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = \frac{32\pi^2}{g^2} \delta^4(x). \quad (10)$$

Now, for a vacuum copy, defined by $F_{\mu\nu}^a = 0$ everywhere, it is clear from Eq.(4) that the net topological charge Q must vanish. Thus, the zero-size limit of the instanton is not a vacuum copy, as it stands.

To clarify completely the topological properties it is instructive to make the singular gauge transformation

$$U(x) = (x_\mu - i\sigma \cdot \underline{x}) / \sqrt{x^2} \quad (11)$$

to arrive at the so-called "singular" gauge⁶

$$\hat{A}_\mu^a = U^\dagger A_\mu^a U - \frac{i}{g} U^\dagger \partial_\mu U \quad (12)$$

$$\hat{A}_\mu^a = \frac{\lambda^2}{x^2} A_\mu^a \quad (13)$$

with A_μ^a given by Eq.(2). This is now singular at the origin for all $\lambda \neq 0$

and correspondingly falls away too quickly at large distance to sustain a topological charge. From Eq.(13) one checks that

$$\hat{T}_\mu = - \frac{16\lambda^4 x_\mu}{g^2 (x^2 + \lambda^2)^{3/4}} \quad (14)$$

$$\hat{F}_{\mu\nu}^a = \partial_\mu \hat{T}_\nu - \partial_\nu \hat{T}_\mu \quad (15)$$

$$= \frac{192}{g^2} \left(\frac{\lambda}{x^2 + \lambda^2} \right) - \frac{32\pi^2}{g^2} \delta^4(x). \quad (16)$$

The crucial point is that now the topological charge is $\hat{Q} = 0$ for all λ . That is, Q defined by Eq.(4) is not gauge invariant under singular transformations.

A vacuum copy should be such that all truly gauge invariant quantities coincide with their values for the pure vacuum $A_\mu^a = 0$.

Such a copy may be defined from Eqs.(2) and (13) by using the method of sections.⁷ We therefore define $A_\mu^a(x)$ by

$$A_\mu^a(x) = \begin{cases} \lim_{\lambda \rightarrow 0} (\hat{A}_\mu^a) = \frac{2}{g} \frac{\eta_{\mu\nu}^a x_\nu}{x^2} & x^2 > \epsilon^2 \\ \lim_{\lambda \rightarrow 0} (\hat{A}_\mu^a) = 0. & x^2 < \epsilon^2 \end{cases} \quad (17)$$

Here ϵ^2 may be made arbitrarily small but non-zero. At the boundary between the two regions, the two functions are related by the gauge transformation Eq.(11) which is regular where it is used.

III. ZERO-SIZE LIMIT OF MULTI-INSTANTON

Although an explicit formula for the most general N instanton configuration is not at hand, 't Hooft⁶ has provided one with $5N$ parameters (rather than the allowed $8N-3$) that is sufficiently general for our purpose. It is given by

$$A_{\mu}^a = \frac{1}{g} \eta_{\mu\nu}^a \partial_{\nu} \ln \left(1 + \sum_{i=1}^N \frac{\lambda_i^2}{x_{i\mu}^2} \right) \quad (18)$$

in which $X_{i\mu} = (x - x_i)_{\mu}$ and where $x_{i\mu}$, λ_i are the location, size respectively of the i th instanton.

This formula is written in the singular gauge $A_{\mu}^a \sim x^{-3}$, $T_{\mu} \sim x^{-7}$ and hence has net topological charge $Q = 0$. In the zero-size limit, it becomes the pure vacuum $A_{\mu}^a = 0$.

Thus, to investigate such vacuum copies it is essential to transform to a non-singular gauge. For the case $N = 2$ the appropriate singular gauge transformation is given by

$$U(x) = \left(\hat{z}_{\mu}(x) + i \underline{\sigma} \cdot \hat{z}(x) \right) \quad (19)$$

$$z_{\mu}(x) = \alpha \frac{x_{1\mu}}{x_1^2} + \beta \frac{x_{2\mu}}{x_2^2} \quad (20)$$

with $\hat{z}_{\mu} = z_{\mu} / \sqrt{z^2}$.

Actually, one needs $\alpha = -\beta$ but we have considered the more general case where α, β are any real constants. For the zero-size limit one then has

$$A_{\mu}^a = \frac{2}{g} \eta_{\alpha\beta}^a (\partial_{\mu} \hat{z}_{\alpha}) \hat{z}_{\beta} \quad (21)$$

$$\partial_{\mu} A_{\mu}^a = \frac{2}{g} \eta_{\alpha\beta}^a (\square \hat{z}_{\alpha}) \hat{z}_{\beta}. \quad (22)$$

After some algebra this gives

$$\partial_{\mu} A_{\mu}^a = - \frac{8\alpha\beta}{g x_1^2 x_2^2} \frac{1}{z^2} (x_2^2 - x_1^2) \eta_{k\lambda}^a x_{1k} x_{2\lambda} \quad (23)$$

and the right-hand side of (23) is non-vanishing for all $\alpha, \beta \neq 0$. If we specialize to the zero-size limit of the two instanton solution ($\alpha = -\beta$) one finds⁸

$$A_{\mu}^a = \frac{2}{g(x_1 - x_2)^2} \left[\eta_{\mu\nu}^a \left(\frac{x_{1\nu}}{x_1^2} - \frac{x_{2\nu}}{x_2^2} \right) (x_2^2 - x_1^2) - 2 \left(\frac{x_{1\mu}}{x_1^2} - \frac{x_{2\mu}}{x_2^2} \right) \eta_{k\lambda}^a x_{1k} x_{2\lambda} \right]. \quad (24)$$

It is easily checked that the divergence now satisfies Eq.(23).

Thus, it is possible to construct vacuum copies from such zero-size limits, using the method of sections, if one generalizes the Landau condition to

$$\partial_{\mu} A_{\mu}^a = \sum_{\substack{i,j=1 \\ i < j}}^N c_{ij}(x) \eta_{k\lambda}^a x_{ik} x_{j\lambda} \quad (25)$$

where N is the number of instantons. Only in the case $N = 1$ does the right-hand side of (25) vanish, thus restoring the Landau gauge as was discussed in the previous sub-section.

We now turn to an alternative approach in the Landau gauge.

IV. ALTERNATIVE APPROACH IN LANDAU GAUGE

For the general gauge transformation of the form, Eq.(19) above, the field A_{μ}^a is given by Eq.(21) and hence its divergence by Eq.(22). In order to satisfy the Landau condition, therefore, we need

$$\eta_{\mu\nu}^a (\hat{\square} \hat{z}_{\mu}) \hat{z}_{\nu} = 0 \quad (26)$$

together with

$$\hat{z}_{\mu} \hat{z}_{\mu} = 1 \quad (27)$$

in order that the gauge transformation U be unitary and unimodular ($\hat{z}_{\mu} = \text{real}$).

To fulfill the nonlinear constraint, Eq.(27), we parametrize

$$\hat{z}_{\mu} = \begin{pmatrix} \cos\phi \sin\theta \sin\alpha \\ \sin\phi \sin\theta \sin\alpha \\ \cos\theta \sin\alpha \\ \cos\alpha \end{pmatrix} \quad (28)$$

where the three functions $\alpha(x)$, $\theta(x)$, $\phi(x)$ are constrained only by Eq.(26).

The necessary and sufficient condition to satisfy Eq.(26) is the generalized Klein-Gordon-type equation

$$\left(\square + \beta(x) \right) \hat{z}_\mu = 0 \quad \mu = 1, 2, 3, 4 \quad (29)$$

with $\beta(x)$ an arbitrary function, independent of the choice of μ .

To find an explicit solution, we may take as an example $\theta = \frac{\pi}{2}$ whereupon Eq.(29) gives just two equations

$$\alpha = (\partial_\mu \phi)^2 \sin \alpha \cos \alpha \quad (30)$$

$$\phi = -2 \cos \alpha \partial_\mu \phi \partial_\mu \alpha. \quad (31)$$

We may satisfy Eq.(31) by the ansatz

$$\phi = Ax_4 + B \quad (32)$$

$$\alpha = \alpha(x_1, x_2, x_3). \quad (33)$$

Defining $s = \hat{c}_i x_i$ where $\hat{c}_i$ is an arbitrary unit vector, Eq.(30) may be re-written

$$\nabla^2 (2\alpha) = \frac{d^2}{ds^2} (2\alpha) = A^2 \sin 2\alpha. \quad (34)$$

This is an equation familiar from Sine-Gordon theory, or equivalently from the undamped pendulum, and there is a solution

$$\tan \frac{\alpha}{2} = e^{\pm As}. \quad (35)$$

Substituting Eqs.(32) and (35) into Eqs.(21) and (28) we find

$$\underline{A}_4^a = \frac{2A \sin \alpha}{g} \left(-\cos \alpha \sin (x_4 + B), \cos \alpha \cos (Ax_4 + B), \sin \alpha \right) \quad (36)$$

$$\underline{A}^a = \frac{2A \hat{c} \sin \alpha}{g} \left(\cos (Ax_4 + B), \sin (Ax_4 + B), 0 \right) \quad (37)$$

where the components $a = (1,2,3)$ are written explicitly.

For a suitable re-definition of the coordinate axes $\mu = 1,2,3,4$, this field A_μ^a satisfies also the Coulomb gauge $\nabla \cdot \underline{A}^a = 0$; it is then a vacuum copy which avoids the spherical-symmetry assumptions made in Ref.1.

V. CONCLUDING REMARKS

Given the existence of vacuum copies, one must next decide how to incorporate them into calculations of physical quantities.

The most natural formalism is that of Faddeev and Popov,⁵ suitably modified. The original version may be written symbolically for a vacuum-vacuum amplitude

$$\int \mathcal{D}A_\mu \delta(\partial_\mu A_\mu) \Delta(A) e^{iS}. \quad (38)$$

One method of modification is to write⁹

$$\int \mathcal{D}A_\mu \delta(\partial_\mu A_\mu) \left(\sum_n \Delta(A^{(n)})^{-1} \right)^{-1} e^{iS} \quad (39)$$

where $\Delta(A^{(n)})$ is the jacobian evaluated for the n^{th} copy. An equivalent prescription is easily seen to be

$$\int \mathcal{D}A_\mu \frac{\delta(\partial_\mu A_\mu) \Delta(A)}{N(A)} e^{iS} \quad (40)$$

where $N(A)$ is a functional giving the number of copies for a prescribed $A_\mu^a(x)$. This alternative form may be more useful for practical calculations.

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