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THE SECOND REGION OF STABILITY

BY

J. M. GREENE AND M. S. CHANCE

MASTER PLASMA PHYSICS LABORATORY

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The Second Region of Stability

J. M. Greene, and M. S. Chance

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544

ABSTRACT

A new type of axisymmetric magnetohydrodynamic equilibrium is presented. It is characterized by a region of pressure and safety factor variation with a short scale length imposed as a perturbation. The equilibrium consistent with these profile variations can be calculated by means of an asymptotic expansion. The flexibility obtained by generating such equilibria allows for a close examination of the mechanisms that are relevant to ballooning instabilities--ideal MHD modes with large toroidal mode number. The so-called first and second regions of stability against these modes are seen well within the limits of validity of the asymptotic expansion. It appears that the modes must be localized in regions with small values of the local shear of the magnetic field. The second region of stability occurs where the local shear is large throughout the range where the magnetic field line curvature is destabilizing.

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1. INTRODUCTION

The stability predictions of ideal MHD in axisymmetric tokamak configurations are generally obtained from the numerical solution of a partial differential eigenvalue equation. A useful approximation to this can be derived in the limit of localized modes with large toroidal mode number [1-4]. The resulting ordinary differential eigenvalue equation, known as the ballooning equation, can be easily solved numerically. It is useful to explore this equation thoroughly in order to identify the various physical mechanisms which can promote, or inhibit, instabilities. The insights thus gained are of considerable aid in developing an intuition into the solutions of the full equation.

Ballooning instabilities are driven by pressure gradients, and tend to be localized in regions with unfavorable magnetic field line curvature. Previous work has shown that these modes are stable in the limit of small pressure gradients, are destabilized at some moderate pressure, and then, remarkably, become stable again for quite large pressures [5-9]. This phenomenon is known as the existence of a second, or high pressure, region of stability. Unfortunately, these large pressures cause large distortions of the equilibrium. As a result, previous calculations of the second region of stability have utilized equilibria, whether estimated analytically or evaluated numerically, that were only marginally accurate. Improvements in this state of affairs would be welcome since stability in the presence of large pressure gradients is important in the tokamak program.

Closer consideration of the previous work led us to the conclusion that the critical stabilizing factors are associated with the pressure gradient, rather than the pressure itself. Furthermore, since the modes are localized radially [10,11], only the local pressure gradient can be important. These observations led us to consider the effect on the ballooning equation of inducing small-amplitude, short scale-length pressure variations in the equilibrium. This scale length can be chosen to be longer than that of the radial localization of the ballooning modes, so that the previous ballooning approximations are still valid. As described in the next section, this approximation is internally consistent, and can be evaluated with excellent accuracy for very large pressure gradients. The results are more than adequate to demonstrate the reality of the second region of stability.

It is useful to vary the safety factor profile in the same manner as the pressure profile since these two profiles work together to determine the tokamak properties.

Examining the effect of these short scalelength profile variations has yielded some new insights into the nature of ballooning instabilities, particularly the effect of shear in stabilizing these modes. They tend to be localized along the lines of force in regions of unfavorable magnetic field line curvature, and so it is the local shear in these regions that plays a critical role. In fact, ballooning instabilities are found only in regions where the local shear is small or vanishes, and it is this that leads to the second region of stability. The latter occurs where the plasma distortion is sufficiently large that the local shear is strongly reversed from its usual value.

The results of this paper also suggest a means of spanning the interval between the first and second region of stability. When the global shear is weak, the local shear can be more easily reversed. Furthermore, under these conditions the modes are less localized in regions of unfavorable curvature, and so are more sensitive to the stabilizing effects of the average curvature. Thus, it is reasonable to expect that ballooning modes can be completely stabilized at sufficiently low global shear.

The next section of this paper treats the effect of short scale-length, small-amplitude perturbations on the equilibrium, including their effect on the local shear. In Sec. III the changes, induced by these profile variations, are evaluated for the Mercier criterion for stability against localized modes. The modification of the ballooning equation is discussed in Sec. IV. The results of some numerical computations are given in Sec. V, with emphasis on the effect of local shear. Section VI contains the conclusions.

II. EQUILIBRIUM

The first step in examining the effect of a small, localized perturbation is to evaluate the induced displacement of the equilibrium configuration. This will be done here in PEST [12] (Princeton Equilibrium, Stability, and Transport) notation. The formulation of the axisymmetric equilibria used in this system is based on the paper of Greene, Johnson, and Weimer [13]. There, equilibria were characterized through the relation between two coordinate systems, cylindrical coordinates centered on the major axis, (X, ϕ, Z) and flux coordinates, (ψ, θ, ϕ) . Here ψ is a labeling of the magnetic flux surfaces, $0 \leq \psi \leq 1$, and θ is an angular measure, $0 \leq \theta \leq 2\pi$, around these

surfaces. The labeling of the flux surfaces and the shape of the lines of constant θ are determined by requiring the Jacobian of the transformation between these coordinates to be

$$J \equiv (\nabla\psi \times \nabla\theta \cdot \nabla\phi)^{-1} = \frac{uR^2}{2\pi R} \quad (1)$$

Here R is the major radius of the tokamak discharge, and

$$u \equiv \frac{R}{2\pi} \int_P \frac{d\tau}{x^2} \quad (2)$$

where the integral is taken over the plasma volume. A typical equilibrium, the one used for the numerical work of this paper, is shown in Fig. 1.

The magnetic field is denoted, in PEST notation, by

$$\underline{B} = f(\psi) \nabla\phi \times \nabla\psi + Rg(\psi) \nabla\phi \quad (3)$$

yielding

$$\underline{B} \cdot \nabla = \frac{2\pi Rf}{uR^2} \left(\frac{\partial}{\partial\theta} + \frac{ug}{2\pi f} \frac{\partial}{\partial\phi} \right) \quad (4)$$

Thus, the field lines are straight in (θ, ϕ) and the safety factor is given by

$$q(\psi) \equiv \frac{ug(\psi)}{2\pi f(\psi)} \quad (5)$$

The next step is to express the relation between the coordinate systems, that is, to write equations for $X(\psi, \theta)$ and $Z(\psi, \theta)$. The Jacobian relation, Eq. (1), yields

$$X_{\theta} Z_{\psi} - X_{\psi} Z_{\theta} = \psi X / 2\pi R \quad , \quad (6)$$

where subscripts denote derivatives. The equation of force balance, which is equivalent to the Grad-Shafranov equation, is given in this formalism by

$$\begin{aligned} X_{\psi}^2 p_{\psi} + R^2 g g_{\psi} + f f_{\psi} |\nabla \psi|^2 + f^2 \frac{\partial}{\partial \psi} |\nabla \psi|^2 \\ + f^2 \frac{\partial}{\partial \theta} \psi \cdot \nabla \theta = 0 \quad , \end{aligned} \quad (7)$$

where

$$|\nabla \psi|^2 = \frac{4\pi^2 R^2}{\psi^2 X^2} (x_0^2 + z_0^2) \quad ,$$

$$\nabla \psi \cdot \nabla \theta = - \frac{4\pi^2 R^2}{\psi^2 X^2} (x_{\psi} x_0 + z_{\psi} z_0) \quad ,$$

$$|\nabla \theta|^2 = \frac{4\pi^2 R^2}{\psi^2 X^2} (x_{\psi}^2 + z_{\psi}^2)$$

$$= \left[\frac{4\pi^2 R^2}{\psi^2 X^2} + (\nabla \psi \cdot \nabla \theta)^2 \right] / |\nabla \psi|^2 \quad . \quad (8)$$

These differential equations, together with boundary conditions, then determine X and Z . The boundary conditions will be discussed below.

We now consider perturbations around some solution of Eqs. (6) and (7). The unperturbed solution will typically be determined numerically. It will be denoted by $x^{(0)}(\psi, \theta)$, $z^{(0)}(\psi, \theta)$ and the corresponding pressure and magnetic profiles by $p^{(0)}(\psi)$, $f^{(0)}(\psi)$, $g^{(0)}(\psi)$, etc. The pressure profile is then perturbed in a local region centered on ψ_b , so that

$$p = p^{(0)}(\psi) + \mu p^{(1)}(s) + \dots, \quad (9)$$

with

$$s \equiv (\psi - \psi_b)/\mu \quad (10)$$

and μ the parameter of smallness of the perturbation. Thus, the pressure is only slightly disturbed but p_ψ may be altered by a finite amount. Similarly

$$f(\psi) = f^{(0)}(\psi) + \mu f^{(1)}(s) + \dots,$$

$$g(\psi) = g^{(0)}(\psi) + \mu g^{(1)}(s) + \dots,$$

$$x(\psi, \theta) = x^{(0)}(\psi, \theta) + \mu x^{(1)}(s, \theta) + \dots,$$

$$z(\psi, \theta) = z^{(0)}(\psi, \theta) + \mu z^{(1)}(s, \theta) + \dots. \quad (11)$$

Perturbing the equilibrium equation yields differential equations for $x^{(1)}$, $z^{(1)}$, etc. The following boundary conditions will be used with this set. All the perturbed quantities approach their unperturbed values outside the disturbed region, $x^{(1)}$ and $z^{(1)}$ are periodic in ψ with period 2π , and the origin of ψ is chosen so that $z(\psi, 0) = 0$. The latter condition is consistent with mirror symmetric equilibria whose magnetic axis lies on $z = 0$.

Inserting the perturbation expansions in the Jacobian relation, Eq. (6), retaining only terms that are finite in the limit of small μ , and utilizing the fact that the leading order terms are a solution to all the equilibrium equations, yields,

$$x_0^{(0)} z_\psi^{(1)} - z_0^{(0)} x_\psi^{(1)} = 0.$$

This relation between $x^{(1)}$ and $z^{(1)}$ can be expressed as

$$x^{(1)} = D(s, 0) x_0^{(0)},$$

$$z^{(1)} = D(s, 0) z_0^{(0)}, \quad (13)$$

where $D(s, 0)$ is yet undetermined. Note that this equation has been straightforwardly integrated in ψ with the boundary condition $D(s, 0) = 0$ for $|s| \gg 1$ since $D_\psi x_\psi^{(0)} \gg D x_\psi^{(0)}$. The boundary condition on ψ implies $D(s, 0) = 0$. The relation between D and the equilibrium perturbation is elucidated in the Appendix.

Next, the Grad-Shafranov equation is analysed in the same manner as the Jacobian relation, yielding, with the aid of Eq. (13),

$$\begin{aligned} x_{\psi}^2 p_{\psi}^{(1)} + R^2 g_{\psi}^{(0)} g_{\psi}^{(1)} + f_{\psi}^{(0)} f_{\psi}^{(1)} |\nabla\psi|^2 \\ + f_{\psi}^{(0)2} |\nabla\psi|^2 D_{\Theta\psi} = 0 \end{aligned} \quad (14)$$

To obtain the condition that there is a solution, D , that is periodic in Θ , the operator

$$\oint d\Theta / |\nabla\psi|^2 \quad (15)$$

is introduced. This operation on Eq. (14) is an annihilation since it forces the term in D to vanish. Then

$$\begin{aligned} p_{\psi}^{(1)} \oint \frac{x_{\psi}^2}{|\nabla\psi|^2} d\Theta + R^2 g_{\psi}^{(0)} g_{\psi}^{(1)} \oint \frac{d\Theta}{|\nabla\psi|^2} \\ + f_{\psi}^{(0)} f_{\psi}^{(1)} \oint d\Theta = 0 \end{aligned} \quad (16)$$

is a relation among the three quantities, $p_{\psi}^{(1)}$, $g_{\psi}^{(1)}$, and $f_{\psi}^{(1)}$, that must be satisfied by the perturbation. The coefficients depend only on the unperturbed equilibrium. A similar relation holds in the general nonlinear case [14,15], but it is different in detail.

Equation (5) also yields a relation among the perturbed quantities,

$$q_{\psi}^{(1)} = q^{(0)} \left[\frac{g_{\psi}^{(1)}}{g^{(0)}} - \frac{f_{\psi}^{(1)}}{f^{(0)}} \right] \quad (17)$$

Then Eqs. (16) and (17) can be used to eliminate $f_{\psi}^{(1)}$ and $g_{\psi}^{(1)}$ in favor of $p_{\psi}^{(1)}$ and $q_{\psi}^{(1)}$:

$$f_{\psi}^{(1)} = -f \left(\int \frac{x^2 B^2 d\theta}{|\nabla\psi|^2} \right)^{-1} \left[p_{\psi}^{(1)} \int \frac{x^2 d\theta}{|\nabla\psi|^2} + \frac{x^2 q^2}{q} q_{\psi}^{(1)} \int \frac{d\theta}{|\nabla\psi|^2} \right],$$

$$g_{\psi}^{(1)} = g \left(\int \frac{x^2 B^2 d\theta}{|\nabla\psi|^2} \right)^{-1} \left[2\pi \frac{f^2}{q} q_{\psi}^{(1)} - p_{\psi}^{(1)} \int \frac{x^2 d\theta^2}{|\nabla\psi|^2} \right], \quad (18)$$

where superscript 0's have been omitted where the meaning is unambiguous. This form is desirable since much of the intuition of tokamaks is expressed in terms of the global shear, q_{ψ} and pressure gradients, p_{ψ} .

With these relations, the equation for D becomes

$$D_{\psi 0} = \frac{p_{\psi}^{(1)}}{f^2} \left(\int \frac{x^2 B^2}{|\nabla\psi|^2} d\theta \right)^{-1} \left(\frac{x^2 B^2}{|\nabla\psi|^2} \int \frac{x^2 d\theta}{|\nabla\psi|^2} - \frac{x^2}{|\nabla\psi|^2} \int \frac{x^2 B^2}{|\nabla\psi|^2} d\theta \right) - q_{\psi}^{(1)} \frac{q^2}{q} R^2 \left(\int \frac{x^2 B^2}{|\nabla\psi|^2} d\theta \right)^{-1} \\ \times \left[\frac{2\pi}{|\nabla\psi|^2} - \int \frac{d\theta}{|\nabla\psi|^2} \right]. \quad (19)$$

This can be solved in quadratures. Note that the only terms that depend on ψ on the short space scale are D_ψ , $p_\psi^{(1)}$, and $q_\psi^{(1)}$ so that integration with respect to ψ is trivial. The boundary condition that the equilibrium is unperturbed outside the disturbed layer does not lead to any new conditions in this order. However, the results of the Appendix show that higher order corrections require

$$\int_{-\infty}^{\infty} p^{(1)}(s) ds = \int_{-\infty}^{\infty} q^{(1)}(s) ds = 0 \quad (20)$$

Since, as discussed in the Appendix, the quantity D is a measure of the shift and distortion of the magnetic surfaces, Eq. (19) is the analog of the rigid shift of magnetic surfaces that has been introduced in previous theories. Here, however, the distortion of these surfaces is included also. Furthermore, since the shifts are always small and local, the present calculation can be a good approximation even for large values of the pressure gradient and shear.

The local shear [16], S , is another equilibrium quantity that plays an important role in the stability calculations to be described,

$$S \equiv - \frac{\underline{B} \times \underline{\nabla}\psi}{|\underline{\nabla}\psi|^2} \cdot \underline{\nabla} \times \frac{\underline{B} \times \underline{\nabla}\psi}{|\underline{\nabla}\psi|^2} \quad (21)$$

This vanishes when there are surfaces that contain both the magnetic field, $\underline{B}$, and the magnetic flux gradient vector, $\underline{\nabla}\psi$. Thus, it is a measure of the skewness of magnetic field lines on nearby surfaces. Evaluation to lowest order in the perturbed equilibrium yields,

$$S = \frac{2\pi R e^2}{u x^2} \left[q_\psi^{(0)} + q_\psi^{(1)} + q \frac{\partial}{\partial \theta} \left(\frac{\underline{\nabla}\psi \cdot \underline{\nabla}\theta}{|\underline{\nabla}\psi|^2} \right) - q^D \psi_0 \right] \quad (22)$$

The integral of this quantity along a field line also plays a role in the theory. Thus we define

$$\begin{aligned}
 I &\equiv \frac{1}{f} \int S \frac{dl}{|B|} \\
 &= \frac{1}{f} \int S \frac{d\theta}{\underline{B} \cdot \underline{\nabla} \theta} \\
 &= (q_{\psi}^{(0)} + q_{\psi}^{(1)}) (\theta - \theta_k) + q \left(\frac{\underline{\nabla} \psi \cdot \underline{\nabla} \theta}{|\underline{\nabla} \psi|^2} - D_{\psi} \right) , \quad (23)
 \end{aligned}$$

where θ_k is a constant of integration. In the unperturbed equilibrium

$$I = q_{\psi} (\theta - \theta_k) + q \frac{\underline{\nabla} \psi \cdot \underline{\nabla} \theta}{|\underline{\nabla} \psi|^2} . \quad (24)$$

III. LOCAL STABILITY

The Mercier [17,18] condition for stability of localized modes with low to moderate toroidal mode numbers is given in terms of [15]

$$D_I \equiv E + F + H - 1/4 , \quad (25)$$

$$\begin{aligned}
 E &\equiv - \frac{v_{\psi}^2}{(2\pi f)^4 R^2 q_{\psi}^2} \oint \frac{x_B^2}{|\underline{\nabla} \psi|^2} d\theta \left(\oint \frac{\partial x^2}{\partial \psi} d\theta - \frac{f}{f} \oint x^2 d\theta \right) \\
 &\quad + \frac{q_{\psi}^2}{2\pi f^2 q_{\psi}} \oint \frac{x^2 d\theta}{x_B^2} \oint \frac{x_B^2}{|\underline{\nabla} \psi|^2} d\theta ,
 \end{aligned}$$

$$F \equiv \frac{v^2 p_\psi^2}{(2\pi)^4 f R^2 q_\psi^2} \left[\int \frac{x_B^2}{|\underline{v}\psi|^2} d\theta \int \frac{x^4}{|\underline{v}\psi|^2} d\theta - R^2 g^2 \left(\int \frac{x^2}{|\underline{v}\psi|^2} \right)^2 \right],$$

$$H \equiv \frac{q p_\psi}{2\pi f^2 q_\psi} \left(\int \frac{x^2}{|\underline{v}\psi|^2} d\theta - \frac{\int x^2 d\theta}{\int x_B^2 d\theta} \int \frac{x_B^2}{|\underline{v}\psi|^2} d\theta \right). \quad (26)$$

There are localized instabilities on surfaces for which

$$D_I(\psi) > 0 \quad . \quad . \quad (27)$$

This can be evaluated to lowest order in the perturbed equilibrium. There are some remarkable cancellations yielding, with the aid of Eqs. (13), (18), and (19),

$$D_I = - \frac{p_\psi^{(0)} + p_\psi^{(1)}}{[q_\psi^{(0)} + q_\psi^{(1)}]^2} v^{++} - \frac{1}{4}, \quad (28)$$

where

$$v^{++} \equiv \frac{v^2}{R^2 (2\pi f)^4} \left\{ \int \frac{x_B^2}{|\underline{v}\psi|^2} d\theta \left[\int \frac{\partial x^{(0)}}{\partial \psi} d\theta - \frac{f_\psi^{(0)}}{f} \int x^2 d\theta \right] \right.$$

$$- \frac{p_\psi^{(0)}}{f^2} \left[\int \frac{x_B^2}{|\underline{v}\psi|^2} d\theta \int \frac{x^4 d\theta}{|\underline{v}\psi|^2} - R^2 g^2 \left(\int \frac{x^2 d\theta}{|\underline{v}\psi|^2} \right)^2 \right]$$

$$\left. - \frac{2\pi R^2 g^2 q_\psi^{(0)}}{q} \int \frac{x^2 d\theta}{|\underline{v}\psi|^2} \right\} \quad (29)$$

depends only on unperturbed equilibrium quantities. All the terms that are quadratic in the perturbed pressure cancel when the pressure induced equilibrium shifts are included. This is reminiscent of cancellations seen previously near the magnetic axis [19], but here it occurs on every surface.

IV. BALLOONING EQUATION

As is well known, in the limit of large toroidal mode number there are localized modes whose stability is determined by the ordinary differential equation [2-4],

$$\frac{\partial}{\partial \theta} \alpha(\theta) \frac{\partial}{\partial \theta} \xi - [K(\theta) - \omega^2] \xi = 0 \quad . \quad (30)$$

Instabilities exist when this equation has negative eigenvalues,

$\omega^2 < 0$, with square integrable eigenfunctions in the interval $-\infty < \theta < \infty$. The coefficients of this equation will be discussed individually in the absence of the short scale profile variations, and then the effects of these perturbations will be considered. In the unperturbed PEST coordinates,

$$\alpha(\theta) \equiv \frac{4\pi^2 f^4}{x_B^2} |\nabla[\phi - q(\theta - \theta_k)]|^2 \quad . \quad (31)$$

This factor is associated with the stretching of magnetic field lines. In the ballooning approximation the instability is an interchange of magnetic field lines on nearby surfaces. Since the local shear is a measure of the skewness of field lines on adjacent surfaces, it plays a dominant role in determining the amount of stretching that is produced by the mode. The position along a magnetic field line where the integrated local shear I vanishes can be chosen,

through θ_k , to be the point at which the interchanging field lines are directly opposite on their nearby surfaces. Then the constant of integration of Eq. (23) can be identified with the quantity θ_k of Eq. (31). The explicit algebraic identification will be made below. With the aid of Eqs. (3), (5), and (8) α can be expressed in terms of the unperturbed integrated local shear, I of Eq. (24),

$$\begin{aligned}\alpha &= \frac{4\pi^2 f^4}{x^2 B^2} \left[\frac{1}{x^2} + q^2 |\nabla\theta|^2 + 2(\theta - \theta_k) q q_\psi \nabla\psi \cdot \nabla\theta \right. \\ &\quad \left. + (\theta - \theta_k)^2 q_\psi^2 |\nabla\psi|^2 \right] \\ &= \frac{4\pi^2 f^2}{x^2 |\nabla\psi|^2} + \frac{4\pi^2 f^4 |\nabla\psi|^2}{x^2 B^2} I^2.\end{aligned}\quad (32)$$

The other coefficient, K , of Eq. (30) is related to the curvature of the magnetic field lines,

$$K(\theta) \equiv \frac{x^2 v^2}{R^2 B^4} f p_\psi \frac{\partial}{\partial \psi} \nabla[\phi - q(\theta - \theta_k)] \cdot \nabla(2p + B^2) \quad (33)$$

$$= -p_\psi (\kappa_n + \kappa_g) \quad (34)$$

Here, κ_n and κ_g are respectively proportional to the normal and geodesic curvature, the latter of which can also be written in terms of I ,

$$\begin{aligned}\kappa_n &\equiv \frac{v^2 x^2}{R^2 B^2} \left[\frac{\partial}{\partial \psi} (2p + B^2) + \frac{\nabla\psi \cdot \nabla\theta}{|\nabla\psi|^2} \frac{\partial}{\partial \theta} (2p + B^2) \right] \\ \kappa_g &\equiv -\frac{v^2 q^2}{q B^4} I \frac{\partial}{\partial \theta} (2p + B^2)\end{aligned}\quad (35)$$

In regions of negative K the curvature of the eigenfunctions of Eq. (30) has a sign opposite to that of the eigenfunction itself. As a result, the modes have their maxima here, and are at least somewhat localized. Since negative K promotes instability, these will be called regions of unfavorable curvature.

The last term in Eq. (30) defines an eigenvalue problem. Ballooning instabilities are predicted for those configurations that yield a negative eigenvalue, $\omega^2 < 0$, for some value of Q_k . That is, Q_k must also be varied to obtain a complete test. Since the calculations described in this paper are concerned with finding the conditions for marginal stability, $\omega^2 = 0$, the particular form of the term containing the eigenvalue is unimportant.

The modification of these coefficients due to the perturbation of the equilibrium are relatively minor. The only changes are in the integrated local shear, and the coefficient of the curvature terms in Eq. (34). Thus

$$K \rightarrow - (p_{\psi}^{(0)} + p_{\psi}^{(1)}) (\kappa_n \cdot \cdot) \quad , \quad (36)$$

and wherever the term I occurs, it is replaced by Eq. (23). All other coefficients retain their unperturbed values.

Since modes with large toroidal mode number, n , are localized in ψ in a region of width $[10,11] n^{-1/2}$, Eq. (30) is a good approximation when

$$1 \gg \mu \gg n^{-1/2} \quad . \quad (37)$$

Then, even in the perturbed region, the equilibrium is slowly varying on the scale of the variation of the ballooning modes. Thus, the net effect of the perturbation on the ballooning equation is the introduction of the two

parameters, $p_{\psi}^{(1)}$ and $q_{\psi}^{(1)}$, that can be varied independently of the other terms in the equation, with equilibrium consistency guaranteed by Eqs. (18) and (19). The flexibility gained by the use of these two parameters to arbitrarily vary the pressure and safety factor profiles enables us to gain a better understanding of the ballooning mechanism. The results of numerically integrating Eq. (30) are given in the next section.

V. NUMERICAL RESULTS

The numerically determined equilibrium that we chose to study is shown in Fig. 1. The details of the equilibrium are unimportant but it should be noted that the moderate pressure, with an average β of about $4\frac{1}{2}\%$, gave no problems in the determination of the equilibrium. In fact this equilibrium lies in the low-pressure region of stability against ballooning modes.

The particular magnetic surface, ψ_b , chosen for this calculation was the 30th in an equally spaced grid of 97 surfaces. This surface is shown in Fig. 1. Here, the safety factor, q , is 1.51.

The effect of changes in the profiles at that surface is shown in Fig. 2, where regions of stability and instability are given in the plane

$p_{\psi} = p_{\psi}^{(0)} + p_{\psi}^{(1)}$, $q_{\psi} = q_{\psi}^{(0)} + q_{\psi}^{(1)}$. This is similar to previous plots used to illustrate the second region of stability, but it differs in two respects. First, all four quadrants are shown. This makes sense in the present approximation since the imposed local profile perturbation can easily reverse the global shear and pressure gradient. Secondly, previous plots have

used a parameter proportional to the negative pressure gradient, so that the normal operating regime, $p_\psi < 0$ and $q_\psi > 0$, would lie in the first quadrant, whereas it lies in the second quadrant in Fig. 2. The unperturbed equilibrium point in this figure is at $p_\psi = -0.064$, $q_\psi = 1.525$.

The region where the Mercier condition for localized instabilities is satisfied is labeled M in Fig. 2. According to Eq. (28), it is bounded by a simple parabola. On the magnetic surface chosen for this calculation, these instabilities can only be driven by pressure gradients reversed from the usual. It has been shown that Mercier-type instabilities are a restricted subset of ballooning instabilities, so that the latter would be expected to be found in a region around, and including, the former. This relationship for the present case is shown in Fig. 3, where a more extensive portion of the (p_ψ, q_ψ) plane is shown for the same parameters as Fig. 2. The regions where ballooning instabilities are found are labeled U.

One outstanding feature of this figure is the near absence of ballooning modes in the lower half plane. The penumbral region is very thin or absent along the lower boundary of the M region, at least up to very large values of the pressure gradient.

Another major feature that appears in Figs. 2 and 3 is the usual ballooning unstable region, which appears as a disconnected branch in the second quadrant. This unstable region lies in a wedge, with stability for either larger or smaller values of the pressure gradient. These are known, respectively, as the first and second stability regions. Further, Fig. 2 shows that, for this equilibrium, there is stability for low positive values of the global shear.

A final gross feature of the calculation that produced these figures is the variation of the parameter Q_k that was required to maximize the area of the instability region. For small pressure gradients, both positive and negative, $Q_k \approx 0$ was optimal for instability. Further, the corresponding ballooning modes achieved their maximum amplitude near $\theta = 0$, which is also the point at which the normal curvature of Eq. (35) is most unfavorable. Along the marginal stability boundaries at larger pressure gradients, however, the minimizing Q_k is finite, and the modes were localized either near the top or bottom of the magnetic surface shown in Fig. 1. This is a clue to the nature of these portions of the instability boundary, and thus to origin of the second region of stability.

To understand better these features of the ballooning instabilities, the nature of the quantities I , α , and K of Eqs. (23), (32), (34), and (35), is considered in more detail. To this end it is convenient to rewrite the ballooning equation, Eq. (30), as

$$\hat{\xi}_{00} = \frac{1}{\alpha} \left[K + \frac{1}{2} \alpha^{1/2} (\alpha^{-1/2} \alpha_0)_0 - \omega^2 \right] \hat{\xi} \quad , \quad (38)$$

where

$$\hat{\xi} = \alpha^{1/2} \xi \quad .$$

As described in the previous section, the eigenfunction of the ballooning equation tends to be localized near where K has its minimum value. It is clear from Eq. (38) that variations in α can play an equally important role, with the most unstable cases having small values of α where K is large and negative. Furthermore, since the term in I that is linear in θ dominates the

behavior of α , at least for moderate to large values of Θ , the maximum destabilizing value of Θ_k must tend to make I vanish in a region of negative K .

An example of the trend of these functions near the low pressure boundary of the ballooning instabilities is shown in Fig. 4. Here $\Theta_k = 0$ yields a quadratic minimum for α at the point where the curvature K is most unfavorable. Even this relatively low value of pressure is sufficient to reverse the local shear, as is seen in the negative slope of I at small values of Θ .

For larger negative pressure gradients, the reverse slope of I increases in magnitude. That is, the local shear, S , becomes increasingly negative, and the quadratic minimum of α becomes narrower. This variation of α is stabilizing in both the multiplicative factor on the right of Eq. (38), and in the term added to K inside the bracket. Note that if, for example

$$\alpha \approx a_0 + a_1 \Theta^2$$

where a_0 and a_1 are both positive, then

$$\frac{1}{2} \alpha^{-1/2} (\alpha^{-1/2} a_0)_0 = \frac{a_0 a_1}{a_0 + a_1 \Theta^2}$$

is positive, and thus stabilizing. Thus, pressure gradients have stabilizing effects through α that compete with the destabilizing effects acting through K , shown in Eq. (36). At some point, these tendencies stabilize modes with $\Theta_k = 0$.

At these large values of the pressure gradient, however, the most destabilizing choice for Θ_k is that which makes I small where S vanishes. Then α has an approximately quartic minimum so that its stabilizing effect is

diminished. This strategy yields instabilities so long as the points of vanishing S lie in regions of negative K . However, these points migrate farther around the plasma surface with increasing pressure gradient, and eventually there is no way for α to be small over an extended region of negative K . Thus, the boundary of the second region of stability is met. Typical behaviors of I and α yielding marginal stability at this boundary are shown in Fig. 5. The marginal mode was localized around the broad minimum of α near $\Theta = \pi/2$.

The value of Θ at which the local shear vanishes depends on p_ψ and q_ψ , and thus this parameter can be plotted on the (p_ψ, q_ψ) plane of Figs. 2 and 3. The condition that S vanishes at a given value of Θ , $\Theta = \Theta_s$, can be expressed, with the aid of Eqs. (19) and (22),

$$\begin{aligned}
 & (q_\psi^{(0)} + q_\psi^{(1)}) g^2 R^2 \left(\frac{2\pi}{|\nabla\psi|^2} - \int \frac{d\Theta}{|\nabla\psi|^2} + \frac{1}{g^2 R^2} \int \frac{x_B^2}{|\nabla\psi|^2} d\Theta \right) \\
 & - (p_\psi^{(0)} + p_\psi^{(1)}) \frac{q}{f^2} \left(\frac{x_B^2}{|\nabla\psi|^2} \int \frac{x_B^2 d\Theta}{|\nabla\psi|^2} - \frac{x^2}{|\nabla\psi|^2} \int \frac{x_B^2}{|\nabla\psi|^2} d\Theta \right) \\
 & = q_\psi^{(0)} g^2 R^2 \left(\frac{2\pi}{|\nabla\psi|^2} - \int \frac{d\Theta}{|\nabla\psi|^2} \right) - p_\psi^{(0)} \frac{q}{f^2} \left(\frac{x_B^2}{|\nabla\psi|^2} \int \frac{x_B^2 d\Theta}{|\nabla\psi|^2} \right. \\
 & \quad \left. - \frac{x^2}{|\nabla\psi|^2} \int \frac{x_B^2}{|\nabla\psi|^2} d\Theta \right) - q \frac{\partial}{\partial \Theta} \left(\frac{\nabla\psi \cdot \nabla\Theta}{|\nabla\psi|^2} \right) \int \frac{x_B^2}{|\nabla\psi|^2} d\Theta . \quad (39)
 \end{aligned}$$

This is a straight line in the (p_ψ, q_ψ) plane, with coefficients that depend in a complicated way on Θ_s . Three such lines, for different values of Θ_s , are

shown in Fig. 6. The slope of the lines increases monotonically with Θ_s from a large negative value at $\Theta_s = 0$ to a large positive value at $\Theta_s = \pi$.

The line for $\Theta_s = 0$ in Fig. 6 strikes at an angle across the plane. Along the portion of this line in the upper half plane the pressure gradient that forces the local shear to vanish at $\Theta = 0$ combines with the normal curvature at that point to produce destabilization. In the lower half plane, the pressure gradient that is required for vanishing local shear has the opposite sign, and thus is stabilizing. Considerations for $\Theta_s = \pi$ yield similar conclusions. Thus, ballooning instabilities are concentrated in the upper half of the (p_ψ, q_ψ) plane because that is where the local shear can vanish where K is negative.

A rough estimate for the lower boundary of ballooning modes is given by the line Θ_c in Fig. 6. The quantity Θ_c is defined as the angle at which the normal curvature changes sign. Since the curvature is dominated by the toroidal field, Θ_c lies near the top of the magnetic surface. Thus, when the local shear vanishes at an angle $\Theta_s < \Theta_c$, negative pressure gradients are destabilizing in the vicinity. On the other hand, when the local shear vanishes for $\Theta_s > \Theta_c$, the opposite is true. The result is that ballooning instabilities lie above the line Θ_c . This behavior has also been found in other models [20].

VI. CONCLUSIONS

The addition of small-amplitude, short scale-length variations to the pressure and safety factor profiles has clarified several aspects of ballooning modes. The technique allows the gradients to be varied independently of the pressure and safety factors themselves, so their effects

can be studied separately. In particular, the first and second regions of stability against ballooning modes is shown clearly in Fig. 3. This is accomplished in an approximation scheme in which all neglected terms are genuinely small, and which involves no loss of numerical accuracy.

The importance of shear in stabilizing ballooning modes has been demonstrated in this paper, particularly the importance of local shear. These modes are governed by a competition between the destabilizing combination of magnetic field line curvature and pressure gradient, and the stabilizing effects arising from the local shear. The scale of these modes along the magnetic field lines is known as the connection length, and will be called L . In these terms Eq. (30) can be written roughly

$$B^2/L^2 - \kappa |\nabla p| = 0 \quad , \quad (40)$$

where κ is a magnetic field line curvature. The connection length is generally taken to be the distance over which the field line curvature is unfavorable. The results of this paper show that in general local shear plays a very important role in determining this quantity. This is shown most directly in Eq. (38), together with the variations in α shown in Figs. 4 and 5.

Near the boundary of the first region of stability the local shear is small, the quantity α does not show extreme variations, as shown in Fig. 2, and the connection length is appropriately given as the region of unfavorable field line curvature. However, at the boundary of the second region of stability, as shown in Fig. 3, the effect of local shear is dominant in determining the connection length.

The origin of these variations in the local shear can be seen in the following way. The global shear is positive in a tokamak, so outer lines of force, on the average, rotate around the magnetic axis more slowly than inner ones. However, large pressures can alter this locally along a magnetic surface. One way to see this is to note that a tokamak discharge tends to push outward, in a direction that would increase R in Fig. 1. This must be counterbalanced by strengthening the poloidal field on the outside of the discharge. In regions of strong pressure gradients the necessary poloidal field increases outwardly, so on outer magnetic surfaces the magnetic field lines whip around faster than inner ones. This leads to a reversal of the local shear around $\theta = 0$ shown in Figs. 4 and 5. Thus, the effect of the pressure in distorting the equilibrium is as important as its interaction with magnetic field line curvature in determining the stability of ballooning modes.

The fact that the ballooning mode branch is stable at low values of the shear, as shown in Fig. 2, has considerable promise toward developing configurations stable against ballooning modes. Comparison with previous work strongly suggests that positive stability against localized modes, D_1 of Eq. (28) negative, contributes to this phenomenon. In particular, computations with a simple model that contains the features of Eq. (30) as modified here but for which $v^{++} = 0$, showed some region of instability for all positive values of the shear [7].

The results given in this paper lead to the conclusion that the most favorable configurations for stability against ballooning modes have all large changes in the pressure gradient concentrated in regions of small global shear, and positive stability against localized modes.

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APPENDIX

Here the nature of the equilibrium perturbation is explored, and particularly the meaning of the quantity D , introduced in Eq. (13), is discussed.

The changes in the pressure and safety factor profiles lead to changes in the shift of the magnetic surfaces, and also to changes of their shape. In the PEST coordinates, these changes in magnetic surfaces are accompanied by shifts in the lines of constant θ in order to maintain the condition on the Jacobian, Eq. (1). Because of the short space scale structure of the perturbations, the shifts of the θ coordinates are of first order in μ , the parameter of smallness, while the shifts of the surfaces themselves are second order. These results will now be expressed in algebra.

First, the change in θ at a given point in space will be calculated, and that point will be specified in the unperturbed coordinate system, $(\psi^{(0)}, \theta^{(0)})$. This can be obtained by solving to first order

$$X(\psi, \theta) = X^{(0)}(\psi^{(0)}, \theta^{(0)}) ,$$

$$Z(\psi, \theta) = Z^{(0)}(\psi^{(0)}, \theta^{(0)}) , \quad (A1)$$

where

$$\psi = \psi^{(0)} + \mu^2 \psi^{(2)} + \dots , \quad (A2)$$

$$\theta = \theta^{(0)} + \mu \theta^{(1)} + \dots ,$$

and (ψ, θ) is the perturbed coordinate system. Then

$$x_{\theta}^{(0)} \theta^{(1)} + x^{(1)} = 0 \quad ,$$

$$z_{\theta}^{(0)} \theta^{(1)} + z^{(1)} = 0 \quad , \tag{A3}$$

yields, with the aid of Eq. (13),

$$\theta^{(1)} = -D \quad . \tag{A4}$$

Now, by the construction of the coordinate system, the Jacobian at a given point is constant under the perturbation,

$$\frac{\partial \psi}{\partial \psi^{(0)}} \frac{\partial \theta}{\partial \theta^{(0)}} - \frac{\partial \psi}{\partial \theta^{(0)}} \frac{\partial \theta}{\partial \psi^{(0)}} = 1 \quad . \quad (A5)$$

To first order this yields

$$\frac{\partial \psi^{(2)}}{\partial \psi^{(0)}} + \frac{\partial \theta^{(1)}}{\partial \theta^{(0)}} = 0 \quad , \quad (A6)$$

or

$$\psi^{(2)} = \mu \int \theta_0 d\xi \quad . \quad (A7)$$

Thus, D measures the shift in the θ coordinate, and is directly related to the shift in the magnetic surfaces. Note that the effect of the perturbations vanishes outside the perturbed region when the integral in Eq. (A7) vanishes when taken across the perturbed layer. This yields Eq. (20).

An illustrative case is given in Fig. 7 where the region around surface 30 of Fig. 1. is perturbed by $\mu p_\psi^{(1)} = -0.2 \cos[10\pi(\psi - \psi_b)]$, $\mu q_\psi^{(1)} = 0$, for $|\psi - \psi_b| \leq 0.1$. The distortion of the constant θ surfaces is clearly seen whereas the second order distortion of the ψ surfaces is barely discernible in the figure.

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(PPPL-802297)

Fig. 1. The equilibrium used in this paper, showing the major radius, R , the (X, ϕ, Z) cylindrical coordinate system, and an equally spaced grid in the (ψ, θ, ϕ) flux coordinates. The 30th surface at $\psi = \psi_b$, which was used for the numerical study of the ballooning equation, is shown.

(PPPL-802295)

Fig. 2. Regions of instability in the (p_ψ, q_ψ) plane. The region marked M is unstable against modes with low toroidal mode number; the regions marked U are unstable to ballooning modes.

(PPPL-802294)

Fig. 3. Similar to Fig. 2, but plotted on a smaller scale.

(PPPL-802125)

Fig. 4. The angular variation of the coefficients α , I , and K of the ballooning equation, Eq. (30), for $p_\psi = -0.064$, $q_\psi = 1.525$, near the low pressure boundary of the ballooning instabilities.

(PPPL-802124)

Fig. 5. The angular variation of the coefficients of the ballooning equation, α and I , for $p_\psi = -1.125$, $q_\psi = 11.525$, near the high pressure boundary of the ballooning instabilities.

(PPPL-802293)

Fig. 6. Lines of the vanishing of the local shear, superposed on Fig. 3. Along the line marked θ^0 , the local shear vanishes at $\theta = 0$; along θ_c it vanishes at the highest point along surface 30 in Fig. 1; and along π it vanishes at $\theta = \pi$.

(PPPL-802296)

Fig. 7. The distortion of the region around surface 30 of the equilibrium for the perturbation, $\mu p_{\psi}^{(1)} = -0.2 \cos[10\pi(\psi - \psi_b)]$, $\mu q_{\psi}^{(1)} \approx 0$, for $|\psi - \psi_b| \leq 0.1$.