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Nonlinear Rayleigh-Taylor Stability with Mass and Heat Transfer

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Abstract

The nonlinear Rayleigh-Taylor stability of a liquid-vapor system is studied using a simplified formulation of the problem. The method of multiple scale expansion is employed for the investigation. It is found that when the heat transfer rate is strong enough, the classically unstable system is stabilized by the nonlinear effect and the effect of mass and heat transfer across the interface. The size of the bubbles detached from the interface can also be estimated when the heat transfer rate is moderately strong. The problem is also studied by the variational method, which corroborates basically the same conclusion.

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I. Introduction

The phenomenon of the classical Rayleigh-Taylor stability is well known [1,2,3]. The essential feature may be stated briefly. When a heavy fluid is on top of a light fluid in a gravitational field, the equilibrium system is unstable, and vice versa. To be more specific, let us take both fluids to be incompressible and inviscid, let the density of the fluid on top be $\rho^{(2)}$, the density of the fluid underneath $\rho^{(1)}$, and let the depths of both fluid layers be infinite, then the dispersion relation for the interfacial wave is given by

$$\omega^2 = [gk(\rho^{(1)}-\rho^{(2)})+\sigma k^3]/(\rho^{(1)}+\rho^{(2)}), \tag{1}$$

where ω is the frequency; k, the wave number; g, the gravitation constant, and σ, the surface tension coefficient. When the surface tension is neglected, this system is stable if $\rho^{(1)} > \rho^{(2)}$, and unstable if $\rho^{(2)} > \rho^{(1)}$. However when $\sigma \neq 0$, then even if $\rho^{(2)} > \rho^{(1)}$, the system is stable if $k > k_c$, where the critical wave number is given by

$$k_c = [g(\rho^{(2)}-\rho^{(1)})/\sigma]^{1/2}. \tag{2}$$

Still when k is small enough, or the wave length is long enough, the equilibrium system is unstable if the heavier fluid is on top of the lighter fluid. These analytical results are generally in agreement with our experience.

There are, however, situations that the configuration of heavier fluid on top of lighter fluid is maintained even though the geometrical boundary conditions permit waves with wave lengths longer than the critical wave length. A notable example is the phenomenon of film boiling. In this case of intense heat transfer, a hot vapor film is apparently capable of supporting a heavier liquid on top. The physical mechanism for the stability of the configuration can be explained intuitively as follows. Although the classical Rayleigh-Taylor instability mechanism tends to disrupt the interface, before the growing protruding liquid fingers can reach the bottom of the vapor film, they are evaporated to become vapor themselves because of the intense heat.

It is evident that in order to analyze such stability problems, the effect of mass and heat transfer across the interface has to be taken into consideration. In a previous paper [4], a general formulation of interfacial flow problem with mass and heat transfer was established and applied to the Rayleigh-Taylor stability problem. From the linearized analysis, it is found that when the vapor region is hotter than the liquid region, as usually so, the effect of mass and heat transfer tends to inhibit the growth of instability. However, the stability criterion remains the same as the classical result [4]. Thus, for the problem of film boiling, the instability would be reduced yet would persist according to the linear analysis. It is clear that a nonlinear analysis is needed to answer the question that whether and how the effect of mass and heat transfer would stabilize the system.

To facilitate the analyses of the nonlinear stability problem, a simplified formulation of the interfacial flow problems with mass and heat transfer has been established [5]. In this formulation, the effect of mass and heat transfer on the dynamics of the system is revealed through an interfacial condition which makes use of the equilibrium temperature distribution of the system, thus bypassing the nonsteady heat equations. It may be termed as a quasi-steady formulation from the thermal perspective. The simplified formulation has been applied to linear Rayleigh-Taylor and Kelvin-Helmholtz stability problems [5]. For the Rayleigh-Taylor stability problems, a comparison of the results with those from the more comprehensive treatment shows that the essential features of the analytical results are retained in the simplified formulation.

In the following, we first present the simplified formulation of the interfacial flow problem with mass and heat transfer. Then the nonlinear Rayleigh-Taylor stability problem of two semi-infinite incompressible inviscid fluids is investigated using the method of multiple scale expansion. For corroboration, a variational study is also made on the same problem. Brief discussion relating to boiling heat transfer is also presented.

II. Formulation of Problem

We shall study the flow of a system of two semi-infinite incompressible inviscid fluids as shown in Fig. 1. The interface is given by

$$s(\underset{\sim}{x},t) = y-\eta(x,z,t) = 0. \tag{3}$$

At equilibrium, the interface is taken to be y=0.

Assuming the flows of the fluids are irrotational, and introducing the velocity potential functions $\phi^{(j)}(\underset{\sim}{x},t)$, we have

$$\nabla^2\phi^{(1)} = 0 \,, \quad -\infty < y < \eta \,, \tag{4}$$

and

$$\nabla^2\phi^{(2)} = 0 \,, \quad \eta < y < \infty \,. \tag{5}$$

The Bernoulli equations are:

$$\frac{p^{(j)}}{\rho^{(j)}} + \frac{1}{2}(\nabla\phi^{(j)})^2 + gy + \frac{\partial\phi^{(j)}}{\partial t} = 0, \quad j=1,2, \tag{6}$$

where $\rho^{(j)}$ are the densities of the fluids, $p^{(j)}$, the pressures, and g is the gravitation constant.

The interfacial conditions which express the conservation of mass and momentum are given by [4,5]:

$$\rho^{(1)}[\frac{\partial s}{\partial t} + (\nabla\phi^{(1)})\cdot(\nabla s)] = \rho^{(2)}[\frac{\partial s}{\partial t} + (\nabla\phi^{(2)})\cdot(\nabla s)], \text{ at } y=\eta, \tag{7}$$

and

$$\rho^{(1)}[(\nabla\phi^{(1)})\cdot(\nabla s)][\frac{\partial s}{\partial t} + (\nabla\phi^{(1)})\cdot(\nabla s)] =$$

$$\rho^{(2)}[(\nabla\phi^{(2)})\cdot(\nabla s)][\frac{\partial s}{\partial t} + (\nabla\phi^{(2)})\cdot(\nabla s)] + [p^{(2)}-p^{(1)}-\sigma(\frac{1}{R_1}+\frac{1}{R_2})]|\nabla s|^2,$$

$$\text{at } y=\eta, \tag{8}$$

where σ is the surface tension coefficient, and R_1 and R_2 are the two principal radii of curvature of the interface. The radius of curvature is taken to be positive if the center of curvature lies on the side of the fluid (2), and negative if otherwise.

The interfacial condition for energy transfer is given by

$$L\rho^{(1)}[\frac{\partial s}{\partial t} + (\nabla\phi^{(1)})\cdot(\nabla s)] = F(\eta), \quad \text{at } y=\eta, \tag{9}$$

where L is the latent heat released when the fluid is transformed from phase (1) to phase (2). The left hand side of (9) represents the net heat flux from the interface into the fluid regions when such phase transformation is taking place. This quantity is taken to be approximately expressible in terms of the balance of heat fluxes in the fluid regions as if the system is instantaneously in dynamic equilibrium. To concentrate on the most salient feature of the issue, we shall take $F(\eta)$ to be linear in η. Thus the condition (9) can be expressed as [5]:

$$\rho^{(1)}[\frac{\partial s}{\partial t} + (\nabla\phi^{(1)})\cdot(\nabla s)] = \alpha\eta \ , \tag{10}$$

where α is a thermal parameter which is positive if the vapor is hotter than the liquid as usually is the case. When the depth of the fluids are finite, say h_1 and h_2, we have [5]:

$$\alpha = \frac{G}{L}(\frac{1}{h_1} + \frac{1}{h_2}),$$

where G is the equilibrium heat flux. Here, we can simply take α to be some physical parameter for theoretical considerations.

Let us further limit our investigation to fluid flows in two space dimension. Making use of equations (3) and (6), the interfacial conditions (7),(8) and (10) at $y=\eta(x,t)$ can be rewritten as

$$\rho^{(1)}[\eta_t+\phi_x^{(1)}\eta_x-\phi_y^{(1)}]=\{(2)\}\ , \tag{11}$$

$$\rho^{(1)}[\phi_t^{(1)}+\frac{1}{2}(\phi_x^{(1)2}+\phi_y^{(1)2})+g\eta-\frac{1}{1+\eta_x^2}(\phi_x^{(1)}\eta_x-\phi_y^{(1)})(\eta_t+\phi_x^{(1)}\eta_x-\phi_y^{(1)})]$$

$$=\{(2)\}+\sigma\eta_{xx}(1+\eta_x^2)^{-3/2}, \tag{12}$$

$$\rho^{(1)}[\eta_t+\phi_x^{(1)}\eta_x-\phi_y^{(1)}]=-\alpha\eta\ , \tag{13}$$

where the notation {(2)} on the right hand sides of the equations is used to denote the same expression as that of the left, except changing the superscript (1) to (2).

Equations (4),(5),(11),(12) and (13), together with some boundedness conditions at infinity constitute the governing equations of the problem.

III. Multi-scale Expansion Near the Critical Wave Number.

When the interface is perturbed from the equilibrium $y=0$ to $y=\eta e^{i(kx-\omega t)}$, the dispersion relation for the linearized problem formulated above is [5]:

$$(\rho^{(1)}+\rho^{(2)})\omega^2 + 2i\alpha\omega + [gk(\rho^{(2)}-\rho^{(1)})-\sigma k^3] = 0. \qquad (14)$$

The critical wave number is again given by

$$k_c = [g(\rho^{(2)}-\rho^{(1)})/\sigma]^{1/2}. \qquad (2)$$

The corresponding critical frequency, ω_c, is zero for this case.

When nonlinear effects are included, it is expected that the critical wave number will be shifted. To investigate the nonlinear effects on the stability of the system, we shall employ first the multi-scale expansion method [6].

Introducing a small parameter ϵ, we assume the following expansions of the variables:

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(X_0,X_1,X_2,T_0,T_1,T_2) + 0(\epsilon^4), \qquad (15)$$

$$\phi^{(j)} = \sum_{n=1}^{3} \epsilon^n \phi_n^{(j)}(X_0,X_1,X_2,T_0,T_1,T_2) + 0(\epsilon^4), \qquad (16)$$

where

$$X_n = \epsilon^n x, \quad T_n = \epsilon^n t.$$

We assume also that $\alpha = 0(1)$.

Substituting (15) and (16) into (4),(5),(11),(12) and (13), and equating like powers of ϵ, we obtain, for n=1,2,3, three sets of equations for $\phi_n^{(j)}$ and η_n. These equations are collected in the Appendix.

To solve these equations in the neighborhood of the linear critical wave number k_c, we assume that the critical wave number, because of the nonlinear effect, will shift to

$$k = k_c + \varepsilon^2 \mu \ . \tag{17}$$

The first order solution is to reproduce the linear result for the critical case. Therefore the solutions are given by

$$\eta_1 = A(X_1, X_2, T_1, T_2)e^{ikX_o} + \bar{A}(X_1, X_2, T_1, T_2)e^{-ikX_o}, \tag{18}$$

$$\phi_1^{(1)} = \frac{\alpha}{\rho^{(1)}k} [A(X_1, X_2, T_1, T_2)e^{ikX_o} + \bar{A}(X_1, X_2, T_1, T_2)e^{-ikX_o}]e^{ky}, \tag{19}$$

$$\phi_1^{(2)} = -\frac{\alpha}{\rho^{(2)}k} [A(X_1, X_2, T_1, T_2)e^{ikX_o} + \bar{A}(X_1, X_2, T_1, T_2)e^{-ikX_o}]e^{-ky}, \tag{20}$$

where a bar denotes complex conjugate quantities. From (A-3) we recover to this order the critical dispersion relation that $k = k_c$.

The second order equations for η_2, $\phi_2^{(1)}$ and $\phi_2^{(2)}$ have nonhomogeneous expressions in terms of the first order solutions. In order to avoid the situation that ratios like η_2/η_1 become unbounded as T_o or X_o goes to infinity, the secular terms are required to vanish. Therefore, we obtain from (A-5)-(A-8),

$$\frac{\partial A}{\partial X_1} = 0, \tag{21}$$

$$\frac{\partial A}{\partial T_1} = 0. \tag{22}$$

Then the solutions for η_2, $\phi_2^{(1)}$ and $\phi_2^{(2)}$ are given by:

$$\eta_2 = \frac{\alpha^2 A^2}{3g\rho^{(1)}\rho^{(2)}} e^{2ikX_o} + CC, \tag{23}$$

$$\phi_2^{(1)} = -\frac{\alpha A^2}{\rho^{(1)}} \left(1 - \frac{\alpha^2}{6gk\rho^{(1)}\rho^{(2)}}\right) e^{2ikX_o} e^{2ky} + CC, \tag{24}$$

$$\phi_2^{(2)} = -\frac{\alpha A^2}{\rho^{(2)}} \left(1 + \frac{\alpha^2}{6gk\rho^{(1)}\rho^{(2)}}\right) e^{2ikX_o} e^{-2ky} + CC, \tag{25}$$

where CC represents the complex conjugate.

Substitute the first and second order solution into the third order equations. In order to avoid the nonuniformity of the expansion, we again impose the condition that secular terms to vanish. Then we found from (A-9) that

$$\frac{\partial A}{\partial X_2} = 0, \tag{26}$$

and after some computation we found from (A-11) that

$$\frac{2\alpha}{k_c}\frac{\partial A}{\partial T_2} + \{2\sigma k_c \mu A + \\ + [2(\frac{1}{\rho^{(1)}} + \frac{1}{\rho^{(2)}})\alpha^2 - \frac{\beta}{3}(\frac{1}{\rho^{(1)}} - \frac{1}{\rho^{(2)}})\alpha^2 - \frac{3}{2}\sigma k_c^3] k_c |A|^2 A\} = 0, \tag{27}$$

where

$$\beta = \frac{\alpha^2}{gk_c\rho^{(1)}\rho^{(2)}} . \tag{28}$$

There is no loss of generality to treat A as real in (27) since the phase associated with A remains constant. Thus we may rewrite (27) as follows:

$$\frac{dA}{dT_2} + (a_1 + a_2 A^2)A = 0, \tag{29}$$

where

$$a_1 = \frac{\sigma\mu k_c^2}{\alpha}\ , \tag{30}$$

and

$$a_2 = [(\frac{1}{\rho^{(1)}} + \frac{1}{\rho^{(2)}}) - \frac{6}{8}\ (\frac{1}{\rho^{(1)}} - \frac{1}{\rho^{(2)}}) - \frac{3}{4}\ \frac{\sigma k_c^3}{\alpha^2}]\alpha k_c^2. \tag{31}$$

Denote $A(0) = A_o$, we obtain

$$A^2(T_2) = a_1 A_o^2 e^{-2a_1 t} / (a_1 + a_2 A_o^2 - a_2 A_o^2 e^{-2a_1 t}). \tag{32}$$

With finite initial value A_o, A may become infinite when the denominator in (32) vanishes. Otherwise A will be asymptotically bounded. The situation can be summarized as follows:

(I) $a_2 > 0$: stable.

(i) $a_1 > 0$: $A^2 \to 0$, as $T_2 \to \infty$.

(ii) $a_1 < 0$: $A^2 \to -\frac{a_1}{a_2}$, as $T_2 \to \infty$.

(II) $a_2 < 0$:

(i) $a_1 < 0$: unstable.

(ii) $a_1 > 0$, and $A_o^2 > (-\frac{a_1}{a_2})$: unstable.

(iii) $a_1 > 0$, and $A_o^2 < (-\frac{a_1}{a_2})$: stable and

$A^2 \to 0$ as $T_2 \to \infty$.

Thus a sufficient condition for stability is $a_2 > 0$, which is due to the finite amplitude effect. Stability can also be established if $a_1 > 0$ and the initial amplitude is small enough, which is the linear result.

To investigate the sign of a_2, let us consider the case that $\rho^{(1)} << \rho^{(2)}$, which would correspond to the case that the density of the vapor down below is much smaller in comparison with the density of the liquid on top. Then the expression of a_2 can be approximated by

$$a_2 \doteq [1 - \frac{\beta}{6} - \frac{3}{4\beta}] \frac{\alpha}{\rho^{(1)}} k_c^2 . \tag{33}$$

Thus $a_2 > 0$ if $0.88 < \beta < 5.12$. Since the magnitude of α is a measure of the effect of mass and heat transfer due to evaporation, it is puzzling that the system tends to be unstable for large α (or β). To resolve the paradox, we may note that for large α, it should be more appropriate to consider $\alpha = 0(\frac{1}{\epsilon})$ in the original expansion scheme. Then the flow system is indeed stable. Therefore the appropriate criterion for stability as given by (27) is approximately:

$$(\frac{1}{\rho^{(1)}} + \frac{1}{\rho^{(2)}})\alpha^2 > \frac{3}{4} \sigma k_c^2 . \tag{34}$$

When a_2 is positive, the system can be stable even for $a_1 < 0$, thus enlarging the range of spectrum of the stable wave lengths. In that case, the asymptotic amplitude of the interface disturbance is given by $(-\frac{a_1}{a_2})^{1/2}$. From practical consideration, we would expect that as the amplitude exceeds half the wave length, there is a tendency for bubbles to form and detach from the interface. Thus we may obtain an estimate of radius of the bubbles R by

$$R^2 = (-\frac{a_1}{a_2}).$$

In the above equation, we should substitute the value $(\frac{\pi}{R})$ for k in the expression of μ which is $(k-k_c)$. If we approximate the value of a_2 by $(\frac{1}{\rho^{(1)}} + \frac{1}{\rho^{(2)}})\alpha k_c^2$, then an estimate of R is given by

$$R^2 = \frac{\sigma\rho^{(1)}\rho^{(2)}}{\alpha^2(\rho^{(1)}+\rho^{(2)})} (k_c - \frac{\pi}{R}) \quad . \tag{35}$$

From (35), we see that for α small, the relevant root is $R \simeq \frac{\pi}{k_c}$, which is the same as that given by the linear result. As α increases, the size of the bubble will also increase. But when α exceeds certain critical value, no real positive root for R exists, and no bubble will be released from the interface.

IV. Variational Method

For interfacial flow problems with two immiscible fluids, the governing equations can be shown to be equivalent to some variational principle [7]. For the simpler case that the fluids are both incompressible and irrotational, the variational principle can be stated as follows: the flow field of the system and the motion of the interface are such that the functional

$$J = \int_{t_1}^{t_2} dt \int_G - \rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right]d^3\underset{\sim}{x} - \int_{t_1}^{t_2} dt \int_A \sigma dA, \qquad (36)$$

is a extremum, where $G = G^{(1)}+G^{(2)}$ denotes the entire space occupied by the fluids and A denotes the interface between $G^{(1)}$ and $G^{(2)}$. Thus in each respective region $G^{(i)}$, the variables ρ and ϕ should take their respective designations $\rho^{(i)}$ and $\phi^{(i)}$. The variations with respect to $\phi^{(1)}$ and $\phi^{(2)}$ and the variation of the interface lead to the governing equations and the kinematic and dynamic interfacial conditions as well [7]. Since the fluids are assumed to be immiscible, they are indeed physically two distinct materials. Hence the variations with respect to $\phi^{(1)}$ and $\phi^{(2)}$ are completely independent. As a result, the kinematic interfacial conditions are

$$\frac{\partial s}{\partial t} + (\nabla\phi^{(i)})\cdot(\nabla s) = 0, \quad \text{on} \quad s(\underset{\sim}{x},t) = 0, \quad i=1,2. \qquad (37)$$

When there is mass transfer across the interface, it turns out that the variational principle using the same J as in (36) is also applicable. However, since one fluid can change into the other through phase transformation, these two fluids are physically really not distinct, although the flow variables are discontinuous across the interface. Therefore the variations $\delta\phi^{(1)}$ and $\delta\phi^{(2)}$ are the same at the interface rather than independent of each other. As a result, we obtain one single interfacial condition (7) instead of the two conditions as given by (37). Taking into consideration of the mass flow across the interface as given by (7), we also obtain the dynamical interfacial condition (8), as a result from the variation of the interface. Together with the condition of interfacial energy transfer (9) or (10), the variational formulation is then equivalent to the governing equations (4)-(10). The condition of interfacial energy transfer can be obtained from the variation with respect to ϕ of the following functional:

$$I = \int_{t_1}^{t_2} dt \int_{G^{(1)}} \rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right] d^3\underset{\sim}{x} - \int_{t_1}^{t_2} dt \int_{A} \alpha\eta dA. \qquad (38)$$

Following the approach as presented elsewhere [8], let us take the following trial function in the process of variation:

$$\eta = 2A \cos(kx+\omega t), \qquad (39)$$

$$\phi^{(1)} = [B \cos(kx+\omega t) + B' \sin(kx+\omega t)]e^{ky} , \qquad (40)$$

$$\phi^{(2)} = [C \cos(kx+\omega t) + C' \sin(kx+\omega t)]e^{-ky}, \qquad (41)$$

and substitute directly into the functional J and I. Retaining only the secular terms and after straightforward but somewhat lengthy calculations, we found that, up to $0(A^4)$:

$$J = -\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx\{\rho^{(1)}[\omega AB'(1 + \frac{1}{2} k^2A^2)$$

$$+ \frac{k}{4}(B^2+B'^2)(1+4k^2A^2) + gA^2]$$

$$+ \rho^{(2)}[-\omega AC'(1 + \frac{1}{2} k^2A^2) + \frac{k}{4}(C^2+C'^2)(1+4k^2A^2)-gA^2]$$

$$+ \sigma[1+k^2A^2 - \frac{3}{4} k^4A^4]\}, \tag{42}$$

and

$$I = \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx\{\rho^{(1)}[\omega AB'(1 + \frac{1}{2} k^2A^2)+ \frac{k}{4}(B^2+B'^2)(1+4k^2A^2)+gA^2]$$

$$- \alpha AB[1+2k^2A^2]\}. \tag{43}$$

Now we shall take the variation of J with respect to A,B,B',C and C'. Since $\delta\phi^{(1)}$ and $\delta\phi^{(2)}$ are the same at the interface in this variational formulation, therefore we have to take $\delta B = \delta C$ and $\delta B' = \delta C'$. From the variation of J, we thus obtain:

$$\delta B = \delta C: \quad \frac{1}{2} \rho^{(1)}kB(1+4k^2A^2) + \frac{1}{2} \rho^{(2)}kC(1+4k^2A^2) = 0, \tag{44}$$

$$\delta B' = \delta C': \quad \rho^{(1)}[\omega A(1 + \frac{1}{2} k^2A^2) + \frac{k}{2} B'(1+4k^2A^2)]$$

$$+ \rho^{(2)}[-\omega A(1 + \frac{1}{2} k^2A^2) + \frac{k}{2} C'(1+4k^2A^2)] = 0, \tag{45}$$

$$\delta A: \quad \rho^{(1)}[\omega B'(1 + \frac{3}{2} k^2A^2) + 2k^3AB'^2 + 2k^3AB^2 + 2gA]$$

$$+ \rho^{(2)}[-\omega C'(1 + \frac{3}{2} k^2A^2) + 2k^3AC'^2 + 2k^3AC^2 - 2gA]$$

$$+ 2\sigma k^2A(1 - \frac{3}{2} k^2A^2) = 0. \tag{46}$$

The variation with respect to B and B' of I leads to:

$$\delta B:\quad \frac{1}{2}\rho^{(1)}kB(1+4k^2A^2) - \alpha A(1+2k^2A^2) = 0, \qquad (47)$$

$$\delta B':\quad \rho^{(1)}[\omega A(1+\frac{1}{2}k^2A^2) + \frac{k}{2}B'(1+4k^2A^2)] = 0. \qquad (48)$$

It is illuminating to consider the linear problem first. It may be seen then that the equations (45),(46) and (48) form an independent set, and the dispersion relation (1) is obtained, just like the case when the two fluids are immiscible. The mode associated with B and C plays a passive role through the equations (44) and (47). The critical condition is achieved when we set $\omega=0$. Then the mode associated with B' and C' becomes vanishingly small. This is the situation when the effect of mass and heat transfer enters to play the active and dominant role. Now the amplitudes of both the surface disturbance and the flow field can be kept finite even at the critical condition.

Setting $\omega=0$, then we obtain from (45),(48),(46) and (47):

$$B' = C' = 0. \qquad (49)$$

$$\rho^{(1)}B = -\rho^{(2)}C = \frac{2\alpha A}{k}\left(\frac{1+2k^2A^2}{1+4k^2A^2}\right). \qquad (50)$$

Equation (46) then yields a nonlinear dispersion relation:

$$\sigma k^2-(\rho^{(2)}-\rho^{(1)})g+[4(\frac{1}{\rho^{(1)}}+\frac{1}{\rho^{(2)}})\frac{\alpha^2}{k}-\frac{3}{2}\sigma k^2]k^2A^2+O(A^4) = 0. \qquad (51)$$

Thus, when

$$(\frac{1}{\rho^{(1)}}+\frac{1}{\rho^{(2)}})\alpha^2 > \frac{3}{8}\sigma k^3, \qquad (52)$$

the system may reach stability for finite amplitude waves.

The variational method gives the same result as the previous approach for the linear problem. The nonlinear results as given by (51) and (52) do not agree completely with those given in (29), although they tend to corroborate qualitatively with each other.

V. Discussion

We have presented an analysis of nonlinear Rayleigh-Taylor stability with mass and heat transfer based on a simplified formulation. The simplified formulation reduces greatly the complexity of the problem, making it easier to keep track of the mechanism involved. In this simplified formulation, the effect of mass and heat transfer is revealed through a single parameter α. It is found that when α is large enough, the system, which would be unstable classically, can be stabilized for finite amplitude disturbances. For moderately large α, the size of the vapor bubbles detached from the interface can also be estimated. Detailed applications to real problems of boiling heat transfer will be discussed in another paper.

We have presented two mutually corroborating analyses of the problem, i.e. the multiple-scale expansion method and the variational method. In the multi-scale expansion method, the scale assigned to α is of crucial importance. The main results were obtained by choosing $\alpha = O(1)$. It may be shown that if we have chosen $\alpha = O(\epsilon)$, then we would have obtained essentially the classical result with only slight modification due to the effect of mass and heat transfer. On the other hand, if we had chosen $\alpha = O(\frac{1}{\epsilon})$, then we would have the solution corresponding to the quiescence of the interface. These results again tend to confirm our general expectation that the larger is α the more stable is the system.

The variational method has the potential to be applicable beyond the perturbation approach. It also offers another perspective on the problem. It is encouraging that a variational formulation of the problem can be established and the results of the linear analysis agree completely with those of the multi-scale expansion method. The nonlinear results of these two methods are qualitatively corroborating. But the lack of complete quantitative agreement does indicate that a fuller understanding on the variational method for the study of nonlinear stability problems is still needed.

Although great analytical advantage was gained by using the simplified formulation, it is clear we should go back to the general formulation to get a full picture of the problem. What we have now learned from the simplified problem can serve as a useful guide for that purpose.

Appendix

Order ϵ

$$\frac{\partial^2\phi_1^{(j)}}{\partial X_o^2} + \frac{\partial^2\phi_1^{(j)}}{\partial y^2} = 0 , \quad j=1,2; \tag{A-1}$$

$$\rho^{(1)}[\frac{\partial\eta_1}{\partial T_o} - \frac{\partial\phi_1^{(1)}}{\partial y}] = \{(2)\} , \quad \text{on} \quad y=0; \tag{A-2}$$

$$\rho^{(1)}[\frac{\partial\phi_1^{(1)}}{\partial T_o} + g\eta_1] = \{(2)\} + \sigma \frac{\partial^2\eta_1}{\partial X_o^2} , \quad \text{on} \quad y=0; \tag{A-3}$$

$$\rho^{(1)}[\frac{\partial\eta_1}{\partial T_o} - \frac{\partial\phi_1^{(1)}}{\partial y}] = -\alpha\eta_1 , \quad \text{on } y=0. \tag{A-4}$$

Order ϵ^2

$$\frac{\partial^2\phi_2^{(j)}}{\partial X_o^2} + \frac{\partial^2\phi_2^{(j)}}{\partial y^2} = -2 \frac{\partial^2\phi_1^{(j)}}{\partial X_o\partial X_1} , \quad j=1,2; \tag{A-5}$$

$$\rho^{(1)}[\frac{\partial\eta_2}{\partial T_o} - \frac{\partial\phi_2^{(1)}}{\partial y} + (\frac{\partial\eta_1}{\partial T_1} + \frac{\partial\phi_1^{(1)}}{\partial X_o}\frac{\partial\eta_1}{\partial X_o} - \eta_1 \frac{\partial^2\phi_1^{(1)}}{\partial y^2})] = \{(2)\}, \ y=0; \tag{A-6}$$

$$\rho^{(1)}\{\frac{\partial\phi_2^{(1)}}{\partial T_o} + g\eta_2 + [\frac{\partial\phi_1^{(1)}}{\partial T_1} + \frac{\partial^2\phi_1^{(1)}}{\partial T_o\partial y}\eta_1 + \frac{1}{2}(\frac{\partial\phi_1^{(1)}}{\partial y})^2 + \frac{1}{2}(\frac{\partial\phi_1^{(1)}}{\partial X_o})^2$$

$$- \frac{\partial\phi_1^{(1)}}{\partial y}(\frac{\partial\phi_1^{(1)}}{\partial y} - \frac{\partial\eta_1}{\partial T_o})]\} = \{(2)\} + \sigma(\frac{\partial^2\eta_2}{\partial X_o^2} + 2\frac{\partial^2\eta_1}{\partial X_o\partial X_1}), \ \text{on } y=0; \tag{A-7}$$

$$\rho^{(1)}(\frac{\partial\eta_2}{\partial T_o} - \frac{\partial\phi_2^{(1)}}{\partial y} + (\frac{\partial\eta_1}{\partial T_1} + \frac{\partial\phi_1^{(1)}}{\partial X_o}\frac{\partial\eta_1}{\partial X_o} - \eta_1 \frac{\partial^2\phi_1^{(1)}}{\partial y^2})] = -\alpha\eta_2, \ \text{on } y=0. \tag{A-8}$$

Order ϵ^3

$$\frac{\partial^2\phi_3^{(j)}}{\partial X_o^2} + \frac{\partial^2\phi_3^{(j)}}{\partial y^2} = -2\,\frac{\partial^2\phi_2^{(j)}}{\partial X_o\partial X_1} - \frac{\partial^2\phi_1^{(j)}}{\partial X_1^2} - 2\,\frac{\partial^2\phi_1^{(j)}}{\partial X_o\partial X_2},\quad j=1,2; \qquad \text{(A-9)}$$

$$\rho^{(1)}\Big\{\frac{\partial\eta_3}{\partial T_o} - \frac{\partial\phi_3^{(1)}}{\partial y} + \Big[\frac{\partial\eta_2}{\partial T_1} + \frac{\partial\eta_1}{\partial T_2} + \frac{\partial\phi_1^{(1)}}{\partial X_o}\Big(\frac{\partial\eta_1}{\partial X_1} + \frac{\partial\eta_2}{\partial X_o}\Big)$$

$$+ \frac{\partial\eta_1}{\partial X_o}\Big(\frac{\partial\phi_1^{(1)}}{\partial X_1} + \frac{\partial\phi_2^{(1)}}{\partial X_o} + \frac{\partial^2\phi_1^{(1)}}{\partial X_o\partial y}\,\eta_1\Big)$$

$$- \Big(\frac{\partial^2\phi_1^{(1)}}{\partial y^2}\,\eta_2 + \frac{\partial^2\phi_2^{(1)}}{\partial y^2}\,\eta_1 + \frac{\eta_1^2}{2}\,\frac{\partial^3\phi_1^{(1)}}{\partial y^3}\Big)\Big]\Big\} = \{(2)\},$$

$$\text{on } y=0; \qquad \text{(A-10)}$$

$$\rho^{(1)}\Big\{\frac{\partial\phi_3^{(1)}}{\partial T_o} + g\eta_3 + \Big[\frac{\partial\phi_2^{(1)}}{\partial T_1} + \frac{\partial\phi_1^{(1)}}{\partial T_2} + \frac{\partial^2\phi_1^{(1)}}{\partial T_o\partial y}\,\eta_2$$

$$+ \Big(\frac{\partial^2\phi_2^{(1)}}{\partial T_o\partial y} + \frac{\partial^2\phi_1^{(1)}}{\partial T_1\partial y}\Big)\eta_1 + \frac{1}{2}\,\frac{\partial^3\phi_1^{(1)}}{\partial T_o\partial y^2}\,\eta_1^2\Big]$$

$$+ \frac{1}{2}\Big[2\,\frac{\partial\phi_1^{(1)}}{\partial X_o}\,\frac{\partial\phi_2^{(1)}}{\partial T_o} + 2\,\frac{\partial\phi_1^{(1)}}{\partial X_o}\,\frac{\partial\phi_1^{(1)}}{\partial T_1} + \eta_1\,\frac{\partial}{\partial y}\Big(\frac{\partial\phi_1^{(1)}}{\partial X_o}\Big)^2\Big]$$

$$+ \frac{1}{2}\Big[2\,\frac{\partial\phi_1^{(1)}}{\partial y}\,\frac{\partial\phi_2^{(1)}}{\partial y} + \eta_1\,\frac{\partial}{\partial y}\Big(\frac{\partial\phi_1^{(1)}}{\partial y}\Big)^2\Big]$$

$$- \Big[\Big(\frac{\partial\eta_1}{\partial T_o} - \frac{\partial\phi_1^{(1)}}{\partial y}\Big)\Big(\frac{\partial\phi_1^{(1)}}{\partial X_o}\,\frac{\partial\eta_1}{\partial X_o} - \frac{\partial\phi_2^{(1)}}{\partial y}\Big)$$

$$- \Big(\frac{\partial\phi_1^{(1)}}{\partial y}\Big)\Big(\frac{\partial\eta_2}{\partial T_o} + \frac{\partial\eta_1}{\partial T_1} - \frac{\partial\phi_2^{(1)}}{\partial y} + \frac{\partial\phi_1^{(1)}}{\partial X_o}\,\frac{\partial\eta_1}{\partial X_o}\Big)$$

$$+ \eta_1\,\frac{\partial}{\partial y}\Big(\frac{\partial\phi_1^{(1)}}{\partial y}\,\Big\{\frac{\partial\phi_1^{(1)}}{\partial y} - \frac{\partial\eta_1}{\partial T_o}\Big\}\Big)\Big]\Big\}$$

$$= \{(2)\} + \sigma[\frac{\partial^2\eta_3}{\partial X_o^2} + \frac{\partial^2\eta_1}{\partial X_1^2} + 2\frac{\partial^2\eta_1}{\partial X_o\partial X_2} + 2\frac{\partial^2\eta_2}{\partial X_o\partial X_1}$$

$$- \frac{3}{2}(\frac{\partial^2\eta_1}{\partial X_o^2})(\frac{\partial\eta_1}{\partial X_o})^2], \quad \text{on } y=0; \tag{A-11}$$

$$\text{Left Hand Side of (A-10)} = -\alpha\eta_3, \quad \text{on } y=0. \tag{A-12}$$

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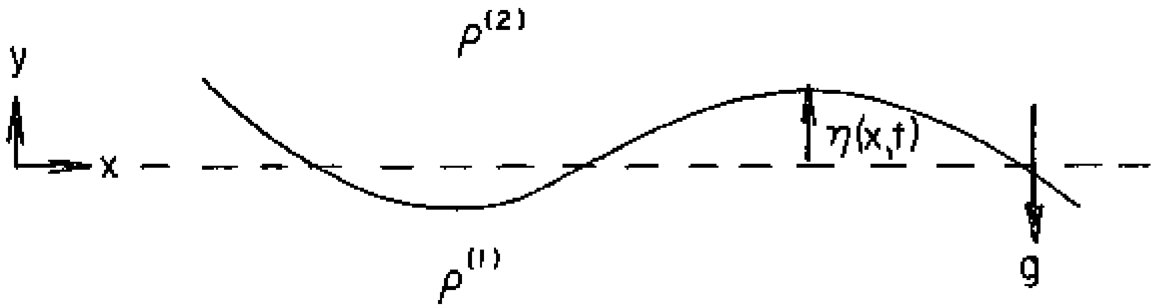

FIG. I CONFIGURATION OF THE FLOW SYSTEM