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A FINITE-ELEMENT METHOD FOR ELLIPTIC-TYPE DIFFERENTIAL EQUATIONS

by

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Components Technology Division

May 1978

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ABSTRACT

This report presents several physical problems which can be described by elliptic differential boundary value problems. A finite element solution for such problems is outlined. The mathematical background of converting the differential form into an integral form is discussed. For simplicity, the trial function employed to illustrate the discretizing process and the error source is limited to a linear one. Finally, the solvability of such a discretized system is reviewed.

I. INTRODUCTION

Consider the following second order partial differential equation

$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x_1 \partial x_2} + c \frac{\partial^2 u}{\partial x_2^2} + d = 0, \quad \text{I-1}$$

where a , b , c , and d are assumed to be a function of $\underline{X} = (x_1, x_2)$, u and

$$\underline{\nabla} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right).$$

One normally classifies I-1 in the following manner: the equation is said to be elliptic at $\underline{X}$ and solution u if

$$ca - b^2 > 0 \quad \text{at } \underline{X} \text{ and } u;$$

it is hypobolic if

$$ca - b^2 < 0 \quad \text{at } \underline{X} \text{ and } u;$$

it is parabolic if

$$ca - b^2 = 0 \quad \text{at } \underline{X} \text{ and } u.$$

This classification is critical since the elliptic equations describe completely different physical phenomena than hyperbolic equations. As a consequence, numerical methods that are used for each are also completely different. In this report we deal only with the elliptic type, which describe phenomena such as steady state diffusion, stress distribution of a shaft under torsion, potential flow, and electro and magneto static field problems.

II. GENERAL DESCRIPTION OF THE PHYSICAL PROBLEM

(a) The first physical model for elliptic boundary value problem is the heat diffusion. The context is as follows: for a given body Ω has τ as its boundary. This body is in thermal equilibrium (i.e., the temperature does not depend on time). That is

$$u(x) = \text{temperature at the point } \underline{X} \in \Omega \quad \text{II-1}$$

It is assumed that the body is heated by some outside source denoted by $f(\underline{X})$, a heat balance gives

$$-\nabla(k\nabla u) = f \text{ in } \Omega. \quad \text{II-2}$$

The function k is the diffusion coefficient (so $k\nabla u$ is the heat flux vector), and typically is a function of ∇u ;

$$k = k(|\nabla u|)$$

in such a case II-2 is a non-linear equation. Normally,

$$k(|\nabla u|) = |\nabla u|^{p-2} \quad \text{II-3}$$

for some p , $2 \leq \infty$, the linear case being when $p=2$.

To complete the physical description of the problem, it is necessary to supply boundary conditions. There are, in practice, two types: the first is called a Dirichlet condition and requires that u be specified on τ (i.e., the temperature is specified everywhere on the boundary τ), the other type is a Neumann condition and requires that the normal derivative $\frac{\partial u}{\partial \underline{n}}$ be specified. Physically, $\frac{\partial u}{\partial \underline{n}} = 0$ on τ represents an insulated surface (i.e., there is no heat flux across the boundary τ). For simplicity, τ is broken into two parts:

$$\tau = \tau_1 \cup \tau_2,$$

and a Neumann condition

$$\frac{\partial u}{\partial \underline{n}} = 0 \text{ on } \tau_1 \quad \text{II-4}$$

is used on the first part and a Dirichlet condition

$$u=0 \text{ on } \tau_2 \quad \text{II-5}$$

is used on the other part. The goal is to compute the solution $u(\underline{X})$ of II-2, II-4, and II-5.

There is another type of problem that is of sufficient physical importance to merit comment. Let us suppose Ω consists of parts $\Omega_1, \Omega_2, \dots$, separated by line $c_1, c_2, \dots$. Further, suppose that diffusion coefficient is a constant in each sub-region, $k = k(\underline{X}) = k_i$ if $\underline{X} \in \Omega_i$.

The differential equation II-2 does not make sense in all of Ω since $k = k(\underline{X})$ is discontinuous across the line $c_1, c_2, c_3, \dots$ (if the constants k_1, k_2 , are not the same). The governing equation becomes

$$\nabla(k_i \nabla u) = f \quad \text{in } \Omega_i \quad i = 1, 2, 3, \dots$$

Physically, this situation occurs when the body Ω consists of different type of materials, which, therefore, have different diffusion coefficients (e.g. Ω_1 =copper, Ω_2 =iron, etc.). To complete the statement, it is necessary to specify not only the condition in the boundary τ , but also interface conditions across the line $c_1, c_2, \dots$ of discontinuity of k . Physically the right condition is that the heat flux $k \frac{\partial u}{\partial \underline{n}}$ is continuous

$$k_i \frac{\partial u}{\partial \underline{n}} \Big|_{c_j^-} = k_{i+1} \frac{\partial u}{\partial \underline{n}} \Big|_{c_j^+}.$$

This specifies that the temperature u have a jump discontinuity across c_j of a specific type.

Some other physical problems which can be described by elliptic boundary value problems are:

(b) The torsion of a body Ω . Its governing equation is

$$\Delta u = 1 \quad \text{in } \Omega, \text{ with the boundary condition that } u=0 \text{ on } \tau.$$

(c) The potential fluid flow which is governed by

$$\Delta u = 0 \quad \text{in } \Omega, \text{ with the boundary condition that } u=0 \text{ on } \tau.$$

Such equations also arise in electromagnetic theory.

III. PRELIMINARY ANALYSIS

The problem is to find a function $u=u(x_1, x_2)$ satisfying the differential equation

$$-\nabla(k\nabla u) = f \quad \text{in } \Omega \tag{III-1}$$

subject to boundary conditions

$$u=0 \quad \text{on } \tau_2 \tag{III-2}$$

and

$$\frac{\partial u}{\partial \underline{n}}=0 \quad \text{on } \tau_1 \quad \tau=\tau_1 \cup \tau_2 \tag{III-3}$$

where $\underline{n}$ is the outward normal.

The boundary value problems III-1, III-2, and III-3 are usually not a convenient place to start the formulation of approximations. The problem centers around the boundary conditions. It turns out that III-3 is a natural or a free boundary condition and it need not be applied on the approximation. To see this, we must convert III-1, III-2, and III-3 into an equivalent integral law.

The first step is to multiply III-1 by an arbitrary function v and integrate over Ω . This gives

$$-\iint_{\Omega} \nabla(k\nabla u) \cdot v = \iint_{\Omega} f v.$$

Let us integrate by parts to get

$$\iint_{\Omega} k\nabla u \cdot \nabla v - \int_{\tau} k \frac{\partial u}{\partial \underline{n}} v = \iint_{\Omega} f v.$$

We note that III-3 implies that

$$\int_{\tau} k \frac{\partial u}{\partial n} v = \int_{\tau_2} k \frac{\partial u}{\partial n} v.$$

Let us require that v also satisfy III-2 (i.e., $v=0$ on τ_2) to get

$$\iint_{\Omega} k \nabla u \cdot \nabla v = \iint_{\Omega} f v.$$

Observe that for this to make sense, it is required that

$$\iint_{\Omega} k \nabla v \cdot \nabla v < \infty,$$

indeed,

$$\left| \iint_{\Omega} k \nabla u \cdot \nabla v \right| \leq \left(\iint_{\Omega} k |\nabla u|^2 \right)^{\frac{1}{2}} \left(\iint_{\Omega} k |\nabla v|^2 \right)^{\frac{1}{2}}.$$

For simplicity, let us denote by H_E , commonly called the energy space, the space of all functions v satisfying the above conditions. Thus,

$$\begin{aligned} v \text{ is in } H_E &\Leftrightarrow \quad (1) \quad \iint_{\Omega} k |\nabla v|^2 < +\infty \\ &\quad (ii) \quad v=0 \text{ on } \tau_2. \end{aligned} \tag{III-4}$$

With this notion our integral law becomes:

find $u \in H_E$ such that;

$$\iint_{\Omega} k \nabla u \cdot \nabla v = \iint_{\Omega} f v \tag{III-5}$$

for all $v \in H_E$.

In the statement III-5, it has not been required that $\frac{\partial u}{\partial n}$ to vanish on τ_1 , hence the name free or natural boundary condition. Thus, we must verify that if u does satisfy III-5, it will indeed be a solution of III-1, III-2, and III-3 (i.e., verify that the integral law and the boundary value problem are equivalent).

To do this, let us try to reverse the steps previously presented. Assuming that the solution u of the integral law does have two continuous derivatives in $\Omega \cup \tau$, integrating III-5 by parts we obtain

$$\iint_{\Omega} \{-\nabla(k\nabla u) - f\} v + \int_{\tau} k \frac{\partial u}{\partial \underline{n}} v = 0,$$

and this is true for all $v \in H_E$. The function v is arbitrary and is at our disposal, so let us require that it vanishes everywhere on τ , but, otherwise, arbitrarily. This gives

$$\iint_{\Omega} \{-\nabla(k\nabla u) - f\} v = 0.$$

But since v is arbitrary in Ω , we must admit that

$$-\nabla(k\nabla u) - f = 0 \quad \text{in } \Omega$$

(i.e., III-1 holds). This leaves us with

$$\int_{\tau} k \frac{\partial u}{\partial \underline{n}} v = 0 \quad \text{all } v \in H_E.$$

Again, v is arbitrary on τ_1 , it, of course, is forced to vanish on τ_2 , hence

$$k \frac{\partial u}{\partial \underline{n}} = 0 \quad \text{on } \tau_1$$

with the assumption that $k > 0$ in $\Omega \cup \tau$. We necessarily have III-3, i.e., the natural boundary condition III-3 is implicitly built into the integral law III-5. The Dirichlet condition III-2, on the other hand, is explicit in III-5. Indeed, it is required that u is in the energy space H_E , hence, III-2 is forced to hold: i.e.,

$$u = 0 \quad \text{on } \tau_2$$

for every function u in H_E .

IV. NUMERICAL APPROXIMATION

All of the approximations which are in use today can be derived from the integral law III-5. Each differs in detail, but in particular, each can be characterized by replacing H_E in III-5 with a suitably smaller space

$$S^h \subset H_E; \text{ i.e.,}$$

find $u_h \in S^h$ such that

$$\iint_{\Omega} k \nabla u_h \cdot \nabla v^h = \iint_{\Omega} f v^h \quad \text{IV-1}$$

all v^h in S^h .

We assert that IV-1 is a system of equations. Indeed, suppose $\phi_1, \dots, \phi_N$ is a basis for S^h , i.e., each v^h in S^h can be written:

$$v^h(x, y) = \sum_{j=1}^N c_j \phi_j(x, y),$$

for suitable weight $c_1, \dots, c_N$; in particular the approximation u_h can be written:

$$u_h(x, y) = \sum_{j=1}^N q_j \phi_j(x, y),$$

and we seek to determine the weights $q_1, \dots, q_N$ from IV-1. Substituting the expression for u_h into IV-1 and putting $v^h = \phi_\ell$ ($\ell=1, \dots, N$) gives

$$\sum_{j=1}^N q_j \iint_{\Omega} k \nabla \phi_j \cdot \nabla \phi_\ell = \iint_{\Omega} f \phi_\ell, \quad \ell=1, \dots, N.$$

This is a set of N equations with N unknowns. In the nonlinear case, when the coefficient k depends on the gradient

$$k = k(|\nabla u|)$$

this is a nonlinear system

$$k = k\left(\left|\sum_{n=1}^N q_n \nabla \phi_n\right|\right).$$

The basic idea in approximation is to subdivide Ω into triangles and to approximate the solution $u(x, y)$ in each triangle by a linear

polynomial. Thus the space S^h in IV-1 will consist of piecewise linear function; i.e., functions which are linear polynomials on each triangle.

The first step is to approximate Ω with a polygon which is denoted by Ω^h and its boundary by τ^h . Thus,

$$\tau^h = \tau_1^h \cup \tau_2^h,$$

where τ_j^h is the approximation to τ_j ($j = 1, 2$).

The second step is to subdivide Ω^h into triangles. The following standardized notation is used:

$T_1, \dots, T_m, \dots, T_M$ = All triangles

$\underline{Z}_1, \dots, \underline{Z}_m, \dots, \underline{Z}_N$ = Nodes in $\Omega \cup \tau_1^h$

$\underline{Z}_{N+1}, \dots, \underline{Z}_N$ = Nodes on τ_2^h ,

$$h_m = \text{diameter of } T_m$$

and

$$h = \max_{1 \leq m \leq M} h_m \quad (\text{maximum mesh space}).$$

To approximate u , Ω is replaced by Ω^h in IV-1; i.e., find $u_h \in S^h$ such that:

$$\iint_{\Omega^h} \nabla u_h \cdot \nabla v^h = \iint_{\Omega^h} f v^h \quad \text{IV-2}$$

for all v^h in S^h .

v^h is a suitable class of piecewise linear function which satisfies the essential boundary condition

$$v^h = 0 \quad \text{in } \tau_2,$$

and the compatibility condition

$$\iint_{\Omega^h} k(|\nabla v^h|) |\nabla v^h|^2 < \infty. \quad \text{IV-3}$$

For all the methods considered, the essential step is to construct a basis for S^h , starting with a qualitative description of the space. In particular:

S^h = all functions $v^h(x,y)$ such that

- (i) In each triangle T_m , v^h is linear
 $v^h(x,y) = a + bx + cy$,
- (ii) v^h vanishes on τ_2^h and
- (iii) v^h is continuous in $\Omega_{U\tau}^h$.

To construct a basis for S^h note that any piecewise linear function v^h is uniquely determined by 3 conditions on each triangle, e.g., the coefficients a , b and c in (i). Suppose the values of v^h are specified at each node $\underline{z}_j$, thus, three conditions on each triangle can be obtained. Moreover, this specification insures that v^h is continuous in $\Omega \cup \tau$, thus, taking care of (i) and (iii). For exactly the same reason, $v^h = 0$ on τ_2^h if v^h vanishes at the nodes $\underline{z}_{N+1}, \dots, \underline{z}_N$ on τ_2 .

In summary:

- (a) A continuous piecewise linear function is uniquely determined by specifying its values at the nodes $\underline{z}_1, \underline{z}_2, \dots$
- (b) $v^h = 0$ on τ_2^h is true for any continuous piecewise linear function v^h if and only if v^h vanishes at the nodes $\underline{z}_{N+1}, \dots, \underline{z}_N$ on τ_2^h .

There are many possibilities for a basis $\phi_1, \dots, \phi_N$ for S^h , but the one which is the most natural is to let ϕ be 1 at $\underline{z}_j$ and zero at the other nodes:

ϕ_j = "tent" or "hill" centered at the node $\underline{z}_j$.

Recalling that we have agreed to let $\underline{z}_1, \dots, \underline{z}_N$ be the nodes in $\Omega \cup \tau_1^h$, we have as a consequence of (a) and (b) that

- (c) each v^h in S^h can be written as:

$$v^h(x,y) = \sum_{j=1}^N c_j \phi_j(x,y) ,$$

and the weights $c_1, \dots, c_N$ are given by

$$c_j = v^h(\underline{z}_j). \quad \text{IV-5}$$

The description of ϕ_i by mathematical formulas is best done with area coordinates on each triangle. Let T be a triangle,

then each point $p(x, y)$ in T can be written

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 \\ L_1 + L_2 + L_3 &= 1 \end{aligned} \quad \text{IV-6}$$

The coordinate (L_1, L_2, L_3) are called area coordinates and have the following expression:

$$L_i = (a_i + b_i x + c_i y) / \Delta$$

where

$$\Delta = \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \quad \text{IV-7}$$

and the values of a_i , b_i , and c_i are given in the following table:

i	1	2	3
a_i	$x_2 y_3 - x_3 y_2$	$x_3 y_1 - x_1 y_3$	$x_1 y_2 - x_2 y_1$
b_i	$y_2 - y_3$	$y_3 - y_1$	$y_1 - y_2$
c_i	$x_3 - x_2$	$x_1 - x_3$	$x_2 - x_1$

The "hills" $\phi_j(x,y)$ have a concise representation in area coordinates. It is zero in any triangle which does not contain $\underline{Z}_j$. If on the other hand, $\underline{Z}_j \in T$, then $\phi_j(x,y) = L_1$.

L_1 = area coordinate associated with $\underline{Z}_j = \frac{\text{shaded area}}{\text{area of } T}$

L_1 is computed from IV-6, IV-7 where $\underline{Z}_1 \rightarrow \underline{Z}_j$, $\underline{Z}_2 \rightarrow \underline{Z}_k$, and $\underline{Z}_3 \rightarrow \underline{Z}_\ell$. Indeed, note that L_1 is a linear function of x and y , which is equal to 1 at $\underline{Z}_j$ and zero at $\underline{Z}_k$ and $\underline{Z}_\ell$.

Having defined a basis, an approximation solution can be constructed. The latter can be written as:

$$u_h(x,y) = \sum_{j=1}^N q_j \phi_j(x,y). \quad \text{IV-8}$$

Noting that the j^{th} weight q_j is equal to the value of u_h at $\underline{Z}_j$, i.e.,

$$q_j = u_h(\underline{Z}_j). \quad \text{IV-9}$$

Substituting this into the approximated integral law IV-1 and setting $v^h = \phi_j$, $1 \leq j \leq N$, the following is obtained:

$$\sum_{i=1}^N q_i \iint_{\Omega^h} k \nabla \phi_i \nabla \phi_j = \iint_{\Omega^h} f \phi_j \quad 1 \leq j \leq N \quad \text{IV-10}$$

This is a system of equations for the weights $q_1, \dots, q_N$ and in the nonlinear case, k depends on x, y , and $|\nabla u^h|$; i.e.,

$$k = k(x, y, \left| \sum_{j=1}^N q_j \nabla \phi_j \right|).$$

Thus, IV-10 can be written as:

$$K(\underline{q}) \underline{q} = \underline{Q}. \quad \text{IV-11}$$

The entries of the matrix $K(\underline{q})$ are:

$$K(\underline{q})_{ij} = \iint_{\Omega^h} k \nabla \phi_i \nabla \phi_j \quad i, j = 1, \dots, N \quad \text{IV-12}$$

and the entries of the vector $\underline{Q}$ are:

$$\underline{Q}_j = \iint_{\Omega^h} f \phi_j \quad j = 1, \dots, N. \quad \text{IV-13}$$

In general, the integrals IV-12 and IV-13 can not be evaluated exactly; the coefficient $k(x, y, |\nabla v|)$ and $f(x, y)$ are typically too complicated to permit this. As a consequence, the use of quadrature rule is necessary. For linear elements, it is not difficult to show that the optimal choice is mid-point quadrature. In particular, the integrals over the triangles T_m is replaced with a mid-point rule over each triangle. More precisely,

$$\iint_{\Omega^h} g = \sum_{m=1}^M \iint_{T_m} g$$

and the integral is replaced over T_m with:

$$I_m(g) \equiv g(\xi_m)\omega_m$$

$$\omega_m = \text{area of triangle } T_m \quad \text{IV-14}$$

$$\xi_m = \text{centroid of } T_m ,$$

Returning to System IV-11, IV-12 is replaced with

$$\tilde{K}_{ij} = \sum_{m=1}^M I_m(k\nabla\phi_i \cdot \nabla\phi_j) \quad \text{IV-15}$$

and IV-13 with

$$\tilde{Q}_i = \sum_{m=1}^M I_m(f\phi_i). \quad \text{IV-16}$$

This gives a new algebraic system

$$\tilde{K}(\underline{\tilde{q}}) \underline{\tilde{q}} = \underline{\tilde{Q}}, \quad \text{IV-17}$$

for a new set of weights $\tilde{q}_1, \dots, \tilde{q}_N$ and a new approximation

$$u_h(x,y) = \sum_{j=1}^M \tilde{q}_j \phi_j(x,y)$$

where $\tilde{q}_j$ is the value of $\tilde{u}_h$ at $\underline{z}_j$. The final step is to solve IV-17 for weight $\tilde{q}_1, \dots, \tilde{q}_N$, e.g. by Newton's method in the nonlinear case and by Gaussian elimination in the linear cases.

V. DISCUSSION

Let L be the differential operator, (Lu,v) be the inner product of Lu and v , and $a(u,v)$ be the result of (Lu,v) after the proper boundary condition is applied. To summarize the present analysis, the search of a solution u for a differential equation $Lu = f$ with proper boundary conditions is finding:

$$u \in H_E \text{ such that}$$

$$a(u,v) = (f,v) \quad \text{V-1}$$

$$\text{for all } v \in H_E$$

The approximation becomes:

(i) choose $S^h \subset H_E$

(ii) find $u_h \in S^h$ such that

$$a(u_h, v^h) = (f, v^h)$$

all v^h in S^h .

V-2

The essential point is that any differential equation $Lu=f$ can be put into the form V-1. One simply multiplies both sides of the equation by v and then integrates:

$$\iint_{\Omega} vLu = \iint_{\Omega} f v = (f, v).$$

One then attempts to use the boundary data by integration by parts. This will produce a form $a(u, v)$ and a space H_E , for which V-1 is true. In addition, most approximations can be viewed as V-2. We are, thus, led to ask for which problem the approximation works, and the answer is when the form a is positive definite in the sense that:

$$a(v, v) > 0 \quad \text{all nontrivial } v \in H_E.$$

V-3

In the model problem, we have assumed that:

(i) $k > 0$

(ii) τ_2 , the boundary arc length where Dirichlet boundary condition applies, is not trivial.

(iii) $\iint_{\Omega} (f)^2 < \infty$, i.e., f is square integrable.

Then, $a(v, v)$ becomes:

$$\iint_{\Omega} k \nabla v \nabla v = \iint_{\Omega} k \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right];$$

it has a simple interpretation. It is the thermal energy associated with the

temperature distribution v . In this regard, H_E is the space of temperature distribution, which has finite energy. Let us check V-3 for the problem. Clearly,

$$a(v,v) \geq 0 \quad \text{all } v \in H_E,$$

because of assumption (i), we must show that:

$$a(v,v) = 0, \text{ if and only if } v \equiv 0.$$

Indeed, the formula gives:

$$k|\nabla v|^2 = 0 \quad \text{in } \Omega.$$

Since k is positive in Ω , we learn that:

$$v = \text{constant in } \Omega.$$

It is at this point that assumption (ii) enters. Indeed, since $\tau_2 \neq \emptyset$ (τ_2 is nontrivial) and $v = 0$ on τ_2 (since $v \in H_E$), we have $v \equiv 0$ in Ω as desired.

Actually, for the model problems, a slightly stronger condition exists:

For all $v \in H_E$,

$$a(v,v) \geq \lambda_0 \|v\|_1^2$$

where λ_0 is a constant depending only on Ω and τ_2 (it goes to zero with the arc length of τ_2) and

$$\|v\|_1^2 = \left(\iint_{\Omega} [|\nabla v|^2 + v^2] \right)^2 \quad (L_2 \text{ norm})$$

moreover,

$$|a(v,w)| \leq k_{\max} \|v\|_1 \|w\|_1.$$

After establishing those, one concludes that the ellipticity property V-3 of the model problem exists and the approximate solution can be obtained by V-1. The immediate question becomes, "Can the matrix problem associated with the approximation V-1 always be solved?"

To answer this question, let us recall that we selected a basis $\phi_1, \dots, \phi_N$

for S^h , and V-1 (with this choice of basis) becomes the linear system:

$$\sum_{j=1}^N q_j a(\phi_j, \phi_\ell) = (f, \phi_\ell) \quad \ell=1, \dots, N. \quad V-4$$

We write V-4 as:

$$Kq = Q$$

where the (ℓ, j) entry of K is

$$K_{\ell j} = a(\phi_j, \phi_\ell);$$

the ℓ^{th} entry of Q is

$$Q_\ell = (f, \phi_\ell).$$

It can be easily shown that if the problem is elliptic in the sense that V-3 holds as we have shown for the model problems, then K is invertible. Moreover, K is a real symmetric and positive definite matrix. The matrix problem associated with the approximation V-1 can always be solved.

There are several interesting questions:

- (a) estimate the "size" (in a suitable norm) of the error

$$\ell_h = u - u_h.$$

- (b) estimate the effect of the crimes

- (i) replacing Ω with Ω^h

- (ii) approximating the integrals by quadrature.

To pose the questions physically, it is equivalent to estimated:

- (i) boundary errors - the domain Ω is replaced and approximated by Ω^h (e.g. a polygon for linear elements, as we stated in the beginning of Section IV). We, therefore, compute not u_h as we mentioned in Eq. IV-1, but some other $\hat{u}_h$ and, hence, $u - \hat{u}_h$.

- (ii) quadrature error - when the inner products are evaluated by quadrature rules, we solve not $Kq = Q$, but some nearby system $\tilde{K}\tilde{q} = \tilde{Q}$. The discussion of the error estimation is beyond the scope of this report and can be found in a book by Strang and Fix^[1].

REFERENCES

1. Strang, Gilbert and Fix, George J., *An Analysis of the Finite Element Method*, Prentice-Hall, 1973.