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Phase 1 Theoretical Description A Geological Formation – Drill String Dynamic Interaction Finite Element Program (GEODYN)

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PHASE 1 THEORETICAL DESCRIPTION

A GEOLOGICAL FORMATION - DRILL STRING DYNAMIC
INTERACTION FINITE-ELEMENT PROGRAM (GEODYN)

SANDIA LABORATORIES CONTRACT NUMBER 68-3061
SAND84-7101

By

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ABSTRACT

The Theoretical Description for the GEODYN Interactive Finite Element Computer Program is presented. The program is capable of performing the analysis of the three-dimensional transient dynamic response of a Polycrystalline Diamond Compact Bit-Bit Sub arising from the intermittent contact of the bit with the downhole rock formations. The program accommodates non-linear, time dependent, loading and boundary conditions.

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PREFACE

The work presented in this document was performed for Sandia National Laboratories, Albuquerque, New Mexico, 87185, under Phase 1 of Contract Number 68-3061, during the period March 15, 1982 through October 15, 1983. The project was sponsored by the Geothermal Technology Development Division - 6241, Sandia National Laboratories; NL Drilling Systems Technology/NL Industries, Inc., P.O. Box 60087, Houston, Texas, 77205; and Superior Oil Company, P.O. Box 51108 OCS, Lafayette, Louisiana, 70505. The Sandia Technical Project Manager for the project was Mr. Bill C. Caskey, with Dr. Donald L. Wesenberg (representing NL Drilling Systems Technology) and Dr. R.E. Martin (representing Superior Oil Company) functioning in a similar capacity for their respective companies. The Sandia Division Manager for the duration of the Phase 1 project was Dr. James R. Kelsey. The work was performed by Jordan, Apostol, Ritter Associates, Inc., Administration Building 7, Davisville, Rhode Island, with Dr. Jerold A. Baird serving as Program Manager and Technical Director.

The authors wish to acknowledge the support and numerous constructive technical comments received from the above cited individuals. This report would be incomplete without the

additional acknowledgment of the technical support and comments provided specifically, by Mr. Charles M. Stone, Mr. John T. Finger, and Dr. Thomas G. Carne, in addition to many of their colleagues at Sandia National Laboratories.

EXECUTIVE SUMMARY

This technical report covers the theoretical formulation and development undertaken by Jordan, Apostol, Ritter Associates, Inc., for Sandia National Laboratories under Phase 1 of contract number 68-3061. The work, which was performed during the period March 15, 1982 through October 15, 1983, included: (1) the development of the finite elements and associated algorithms necessary to provide a Polycrystalline Diamond Compact (PDC) Bit-Bit Sub Model idealization, (2) the characterization of the geometry, boundary conditions and phenomenological description of the PDC Bit-formation interaction, (3) the idealization of the formation response, and (4) the integration of the above mentioned tasks into a Computer Program whose main solution algorithm provides the capability of modeling transient dynamic bit motions accounting for the nonlinear, time dependent, loading and boundary condition response characterizing the stated problem.

The PDC Bit-Bit Sub Finite Element Model idealization, developed for application in this class of problem, incorporates the use of: (1) a 3-D, 8-node isoparametric brick element, (2) a 3-D, linear beam element, (3) a procedure for enforcing the constraint tying required between the eight node brick elements, used to model the bit, and the linear beam elements, implemented in the representation of the Bit Sub, and (4) a set of "spring stiffnesses" attached to the

uppermost Bit Sub beam node which are implemented to approximate upstring boundary conditions. The development and utilization of an analytical artifice, hereafter referred to as a "ghost cutter", permitted the reduction in size (i.e. degrees of freedom, etc.) of the PDC Bit-Bit Sub Finite Element Model without sacrificing the detailed geometric description of the bit.

The characterization of the geometry, boundary conditions and phenomenological description of the PDC Bit-formation interaction is in large part affected by the ghost cutter mechanism. A ghost cutter is essentially treated as a rigid body element (which will be referred to as a Geometric Sub Element) characterized by three unique Geometric Sub Nodes. These geometric sub elements have no mass. Further, although they transfer forces from the formation to the bit without deforming (i.e. they are rigid), a cutter breakoff algorithm has been incorporated to reflect removal due to adverse mechanical (i.e. not thermal) loading. Interaction between the PDC Bit and the formation occurs only via the ghost cutters in the GEODYN Phase 1 Program. As such, the interaction phenomena are characterized by a description of the ghost cutters' sub node positions and velocities, the formation's surface profile, the formations' response if cutter penetration occurs, and the interpolation of the resultant formation forces on the ghost cutters to the PDC Bit Finite Element nodes.

The formation model utilized in the GEODYN Phase 1 Program consists of a collection of discrete points located systematically in space. These formation points, if connected, would map the wellbore's surface. The formation model is comprised of three separate portions (or submodels), i.e. a "Bottom" surface, a "Side-Wall" surface, and an "Incline" surface, for which the direction of ghost cutter penetration varies. A uniform grid pattern is employed for each of the three portions. Each formation point has a penetration coordinate associated with it. This penetration coordinate is used to compute a penetration resistance force if a ghost cutter has penetrated the formation at this point. Further, constitutive formation properties are defined for each formation point. The formation properties currently utilized include a linear stiffness resisting ghost cutter penetration, the maximum attainable coefficient of friction in the rock, and the velocity at which this maximum attainable coefficient of friction is attained. Different constitutive formation properties may be defined for different formation points constituting the model. A "Mesh Generator" has been incorporated in the program to facilitate the description of the formation model.

The solution process utilized in the GEODYN Phase 1 Program embodies the totality of the three models described above to provide the capability of representing the transient dynamic response of the PDC Bit-Bit Sub arising from non-

linear, time dependent loading and formation interaction behavior. A dramatically increased solution efficiency (i.e. time required for solution) has been attained in the GEODYN Phase 1 Program by employing the modal superposition concept to transform the physical degrees of freedom of the PDC Bit-Bit Sub Finite Element Model to a reduced number of equivalent generalized degrees of freedom. The solution algorithm incorporates an iterative solution technique employing a Newton-Raphson approximation within the incremental time step process to obtain a converged equilibrium solution. Newmark's Method is utilized to integrate the modal transformed Equations of Motion of the PDC Bit-Bit Sub System. Intermittent formation interaction forces, externally applied upstring boundary forces (i.e. weight-on-bit, torque, etc.), gravity forces, and centrifugal body forces are all accounted for in the solution process.

I. INTRODUCTION

Prior to detailing the theoretical formulation and development associated with the GEODYN Phase 1 Program, it is important to gain an appreciation of how this effort fits in with an overall goal of numerically modeling the interaction between the wellbore formations and the drill string, the latter being taken from bit to kelly. The following discussion is included both to provide insight as to how Phase 1 fits into a consistent phased developmental effort to achieve the stated objective as well as to identify the principal components of the phased approach itself. The advantages of such a phased development program were previously identified in a feasibility study (see Reference 1.1) prepared by Jordan, Apostol, Ritter Associates, Inc. for Sandia National Laboratories, under Contract Number 62-1219.

The feasibility study concluded that a three phased developmental effort could be undertaken to provide the analytical finite element analysis program necessary to describe the interaction between arbitrary, dynamically loaded down-hole assemblies, utilizing PDC bits, with the bottom-hole formations. Although the scope of the developmental effort has expanded as a result of some encouraging preliminary numerical results obtained during Phase 1, the following discussion will interpret the three encompassing phases as originally defined in Reference 1.1.

Briefly stated, the Phase 1 effort has concentrated on the physical and numerical complexities associated with the three dimensional modeling of the PDC Bit-formation interaction under dynamic impetus. The Phase 2 endeavor will address: the definition of the surveyed wellbore profile; the extension of the PDC Bit-Bit Sub modeling capabilities to provide for the inclusion of stabilizers and longer assembly lengths; enhancement of the formation modeling capabilities consistent with the desire to model bottom hole assembly (BHA) response under dynamic loading; and possibly refinements or additions to formation behavioral idealizations, the solution algorithm, and geometric idealizations. Phase 3 (and potentially subsequent phases) will reflect the integration of the Phase 2 model into a form such that the model will not only extend from the bit to the ground surface, but may also (at perhaps the BHA modeling level) be capable of modeling drill ahead performance. It is anticipated that the enhancement of the program to include modeling capabilities for mud motors, roller cone bits, shock subs, etc. could be accomplished during the yet undefined future phases of development.

Throughout the phased development just outlined, it should be understood that the previously constructed (i.e. previous developmental phase) GEODYN Program will provide the evolutionary framework for enhanced or additional modeling capabilities. It is this evolutionary nature of Program development, and the ability it provides to physi-

cally qualify the analytical modeling procedures, that originally made the phased development program attractive. It has been realized that this approach, coupled with the modular coding format applied to the GEODYN Program, should reduce the total cost of development while allowing the flexibility to pursue promising analytical modeling techniques.

In closing, it should be noted that Section II of this report provides an introductory discussion of the Phase 1 effort. The discussion of Sections III through VI will constitute a review of the theoretical fundamentals underlying the development of the GEODYN Phase 1 Program. A User Instruction Manual for the Phase 1 version of the Program has been published separately (Reference 1.2).

II. A BRIEF OVERVIEW OF THE GENERAL SOLUTION PROCEDURE

As previously indicated, the primary objective of the Phase 1 Project was the development, implementation and verification of an efficient and accurate, large scale, special purpose finite element computer program (GEODYN) to describe the dynamic behavior characteristic of the Polycrystalline Diamond Compact (PDC) Bit-rock formation interaction problem described in Chapter I. This work required the establishment of three major development areas, i.e.,

1. the PDC Bit model idealization,
2. the characterization of the geometry,
boundary condition and phenomenological
description of the PDC Bit-formation
interaction and
3. the idealization of the formation response,

which are brought together in a major solution algorithm which models the dynamic bit motion and which further accounts for the non-linear, time dependent, loading and boundary condition response characterizing the stated problem.

By way of introduction to the remainder of this document, we provide below in Section 2.1 a brief review of the general algorithm used to obtain the dynamic solution to the bit-formation interaction and in Section 2.2 summary reviews of

several of the individual components, sub-algorithms and math models comprising the general solution algorithm of Section 2.1.

The following discussion is meant to provide only an overview of the items referred to above. Detailed discussion of these topics is deferred to later chapters of this document.

2.1 Overview of the General Solution Algorithm

The dynamic equations describing the bit and bit sub response may be written in a finite element formulation as

$$[M] \{\ddot{u}\} + [C] \{\dot{u}\} + [K] \{u\} = \{F_e\} + \{F_f\} + \{F_\mu\} - \{F_{BD}\} \quad (2.1)$$

where

$[M]$ = Structure Mass Matrix
 $[C]$ = Structure Damping Matrix
 $[K]$ = Structure Stiffness Matrix

and where (note that a superimposed dot indicates differentiation with respect to time t)

$\{u\}$ = Nodal Displacement Vector
 $\{\dot{u}\}$ = Nodal Velocity Vector
 $\{\ddot{u}\}$ = Nodal Acceleration Vector

with the applied forces being given by

$\{F_e\}$ = Nodal External Force Vector
 $\{F_f\}$ = Nodal Bit Formation Interaction Contact Force Vector
 $\{F_\mu\}$ = Nodal Bit Formation Interaction Friction Force Vector
 $\{F_{BD}\}$ = Body Forces (Both Inertial and Gravitational)

For the bit-drill string model, the vector $\{u\}$ is the displacement vector containing the degrees of freedom originating from the finite element representation. The finite element representation of the isolated bit and bit sub is linear in form. Thus, directly from this representation, mass $[M]$, damping $[C]$ and stiffness $[K]$ matrices may be derived which, for a specific bit-bit sub geometry, are constant quantities.

The bit-bit sub system is subjected to a set of external forces $\{F_e\}$ which include a prescribed axial load and torque applied at the top of the drill string as well as gravity forces. These external forces are independently specified and may be varied as a function of time. Two types of bit formation interaction forces are incorporated into the model. The formation interaction force $\{F_f\}$ is generated by contact and penetration of the bit and the formation. The force is a function of the displacement vector $\{u\}$ and thus results in a nonlinear relationship when there is intermittent contact. When there is no contact at a bit node degree of freedom, the formation force is zero. When contact occurs, the force is a function of the relative penetration of the formation. The friction force $\{F_\mu\}$ is also generated due to contact between the bit and formation. The force is, in general, a function of both the displacement vector $\{u\}$ and the velocity vector $\{\dot{u}\}$ and thus represents a nonlinear relationship in the bit-bit sub equations which is determined from the bit-formation interaction.

The bit-bit sub equations (Equation 2.1) are in general nonlinear equations because of the formation interaction forces $\{F_f\}$ and $\{F_u\}$. In the cases in which no bit-formation contact occurs, the forces $\{F_f\}$ and $\{F_u\}$ are identically equal to zero and the remaining terms in the equation are linear.

The general flow chart provided in Figure 2.1 outlines the global computational algorithm employed to model the PDC Bit-bit sub/formation interaction mechanism addressed by this study.

Referring to this flow chart, we find that the first task addressed by the general solution algorithm is the definition, in terms of generation of a finite element model, of the bit-bit sub geometry. Long range plans call for the development of a bit, bit sub (and ultimately bottom hole assembly and drill string) mesh generator, but this capability is not currently resident in the Phase 1 Version of GEODYN. As such, the program depends on generation of the bit-bit sub finite element model by an external mesh generation program.

The current element library of the program contains two basic elements, an 8 node isoparametric three-dimensional brick element used for modeling the body of the PDC Bit and a three-dimensional linear beam element used to model the bit sub.

FIGURE 2.1: The Global Computational Algorithm Flow For A PDC Bit-Drill Sub/Formation Interaction

The ghost cutters referred to in the geometry definition phase of the bit model (upper right hand corner, Figure 2.1) are an analytical artifice used to simplify the finite element model of the bit (thus reducing the number of degrees of freedom of the resulting dynamic problem) while maintaining a mechanism for modeling the complex nonlinear dynamic impact effects produced by the non-uniformly distributed cutters on the bit face. Further discussion of the ghost cutter mechanism is provided in Section 2.2.1 below and in Chapter IV.

The wellbore geometry, for the Phase 1 version of GEODYN, is assumed to be adequately represented by a vertical circular (in cross-section) cylinder whose bottom shape can be user defined in any manner compatible with (but not limited to) the continuous surface generated during rotation of the cutters. The structural interaction between the bit's cutters and the formation is accommodated by a nonlinear contact model representing two possibilities. First, the case of no contact, during a time step; between a cutter and the formation (i.e. wellbore surface). Second, the alternative possibility of contact between the cutter and the formation represented by the cutter (i.e. ghost cutter) compressing a piecewise linear formation spring. As indicated through implementation of the dotted arrow in the upper right hand corner of Figure 2.1, the accommodation of more complex geometries (e.g. inclined,

non-circular, wellbores) and more sophisticated lithology description and formation material property characterization are deferred until later phases of the GEODYN program development.

Following the definition of the bit-bit sub finite element model, the algorithm moves on to indicate the initialization of certain primary program variables and sub-algorithms. As indicated in Figure 2.1, the program does not yet account for such real world phenomena as an initial formation wellbore stress state. The primary goals of the Phase 1 development included (see also Chapter I)

- . the design, development, coding and checkout of the overall GEODYN System framework (including ports for inclusion of future Phase 2 and 3 capabilities and sub-algorithms),
- . the formulation, development, coding and checkout of the complex nonlinear (e.g. due to intermittent contact) dynamic behavior of a drilling PDC (or other drag) bit and,
- . the establishment of efficient procedures for the finite element modeling of the complex geometry represented by a PDC bit while keeping the number of degrees of freedom to an absolute minimum.

It was envisioned from the start that the detailed development of other models and capabilities (e.g. initial wellbore stress state definition and more refined tooth, or ghost cutter, failure models) would be deferred for development and implementation later in Phases 2 and 3 (see Figure 2.1).

Following the above described parameter and sub-algorithm initialization phase, the global algorithm (Figure 2.1) indicates entry into two computation loops - an outer time step integration loop used to march the dynamic solution through time and an inner loop, designed to operate within a given time step, which insures that convergence of the numerical process is achieved for each time step. As pictured in Figure 2.1, the outer loop possesses responsibility for definition of all external forces (e.g. weight) and upstring loads as well as responsibility for updating all appropriate variables following the completion of a time step (prior to re-entry into the inner convergence loop for the next time step). The inner loop provides for iteration of the matrix solution to convergence for a time step. This implies that an acceptable equilibrium state, between the deformed elastic PDC Bit-bit sub finite element model and the current dynamic boundary and friction forces acting on it, has been defined. In this iterative attempt at obtaining a converged solution, the algorithm yields a succession of guesses as to the

magnitude of the current values of the displacement $\{u\}$, velocity $\{\dot{u}\}$ and acceleration $\{\ddot{u}\}$ vectors (Equation 2.1) for the finite element model. Not only the explicitly dynamic portions of the equilibrium equations (i.e. the left hand side of Equation 2.1) depend on these quantities, but the magnitudes of the boundary (contact) force vector $\{F_f\}$ and the friction force vector $\{F_\mu\}$ (Equation 2.1) depend on them as well. Thus, for each iteration of the inner iterative convergence loop, the global algorithm provides for redefinition of both these quantities, i.e. $\{F_f\}$ and $\{F_\mu\}$ of Equation 2.1.

2.2 Some Individual Components and Sub-Algorithms of the General Solution Algorithm

In the discussion which follows, we provide brief summary reviews of a few of the more important individual components, sub-algorithms and math models comprising the general solution algorithm of Section 2.1. The discussion provided below addresses such topics as the bit finite element model, the ghost cutters, the formation response mechanism and the dynamic numerical integration scheme. The summaries which follow are intended to be very brief overviews of the topics covered. Detailed discussion of these topics is deferred until later in this report.

2.2.1 The Bit Finite Element Model, the Ghost Cutters and the Formation Response Mechanism

One of the most difficult aspects of the problem being examined is the fact that the analytical problem posed by a rotating PDC Bit in a rigid well bore would be difficult enough to handle; without throwing in the complications associated with trying to broaden the viewpoint by visualizing the formation not to be rigid, but nonlinear elastic or inelastic instead. The most serious source of difficulty posed by the problem is that there are two sources of non-linearities in the problem, both of them related (or coupled): i.e. we have,

- . the intermittent contact to which the
- bit is subjected during rotation and
- . the inelastic response of the formation
- as a result of this intermittent contact
- (or impact).

Because both primary components of the problem evidence complex behavior, it was decided that both these entities should be treated separately during the finite element modeling phase. It has thus been left to the final general solution algorithm incorporated in the program to provide the geometry and force linkup between the separate components (i.e. the rotating PDC Bit and the deforming formation). For the moment, we will concern ourselves only with the characteristics of the geometry of the finite element assembly used to model the rotating PDC Bit-Bit sub system. In the Phase 1 Version of GEODYN, the interaction between the rotating bit and the formation is accommodated solely through implementation of the load vectors $\{F_f\}$ and $\{F_\mu\}$ of Equation 2.1. A complete discussion of these load vectors is deferred until Chapter IV.

Since we are primarily concerned with the dynamic effects influencing the behavior of the rotating PDC Bit-Bit sub system, we do not require the same level of stiffness representation which would be necessary if our primary goal were the detailed stress analysis of finite element model repre-

senting the bit. As such, we do not need all the complexity or degrees of freedom introduced by a relatively high order solid element (e.g. a twenty node brick). All that is required is a standard eight node isoparametric brick element. As previously indicated, this element has, in fact, been incorporated into GEODYN for modeling the body of the PDC Bit with a standard three-dimensional linear beam element included to model the attached bit sub.

Though the solid element used to model the bit possesses a fairly simple nature, modeling the geometry of the bit still posed an extremely complex problem. In fact, this was one of the real threats which could prevent ultimate solution of the problem in the first place, i.e. the fact that the bit mesh could require so many degrees of freedom for proper analysis that solving the problem dynamically could prove too costly. It was clear that some procedure was needed to circumvent this problem.

As a first try, however, Jordan, Apostol, Ritter Associates, Inc. considered the worst case of brute forcing the problem. This consisted of attacking the problem head on and producing a finite element model of the bit (including all teeth or cutters). The results of such a total finite element model for a typical bit are shown in Figures 2.2 through 2.8. The figures show profiles and cross-sections of the finite element model and are fairly self-explanatory.

FIGURE 2.2: A Lateral View of the Finite Element Mesh of the Assembled Total Bit Model

FIGURE 2.3: The Finite Element Mesh of the Total Bit Model Looking Directly at the Face of the Bit

FIGURE 2.4: The Finite Element Mesh of the Total Bit Model Looking Directly at the Face of the Bit (with Teeth or Cutter Locations Shaded In)

FIGURE 2.5: A Proposed Finite Element Model of
A Typical Tooth or Cutter

FIGURE 2.6: Finite Element Mesh at Cross-Section A-A
(See Figure 2.2) of the Total Bit Finite
Element Model

FIGURE 2.7: Finite Element Mesh at Cross-Section B-B
(See Figure 2.2) of the Total Bit Finite
Element Model

FIGURE 2.8: Finite Element Mesh at Cross-Section C-C, D-D and E-F (See Figure 2.2) of the Total Bit Finite Element Model

One should, however, note Figures 2.3, 2.4 and 2.5 which demonstrate the extreme complexity of the mesh required to model the location (and inclusion) of all cutters on the surface of the bit in the total model (note especially Figure 2.4 which is a planar view of the mesh on the surface of the bit with all cutter locations shaded in and Figure 2.5 which shows the proposed idealization of a typical tooth or cutter). The total bit model pictured in Figures 2.2 through 2.8 required the number of elements, nodes and degrees of freedom shown in Table 2.1 below for its definition.

Table 2.1: Number of Elements, Nodes and
Degrees of Freedom Required for
Specification of the Total Bit Model
(for an 8 Node Brick Mesh)

Number of Elements	=	324
Number of Nodes	=	651
Number of D.O.F.	=	1,953

Considering the expense of a large scale integration of the dynamic equations of motion of a complex bit finite element model, it was felt necessary to obtain some procedure for reducing the number of degrees of freedom needed to specify the bit model from the 1,953 D.O.F. shown in Table 2.1. The method

of approach for reducing the number of requisite degrees of freedom is contained in the Reduced Finite Element Model of a typical bit pictured in Figures 2.9 and 2.10.

The approach taken in the Reduced as opposed to the Total finite element model is to ignore the cutters in the generation of the mesh. The idea is to assume that compared to the body of the bit, the cutters are rigid structures for which no accounting of the load-deflection (i.e. stiffness behavior) need be made in the finite element formulation. A simplified bit face mesh for the reduced model can then be generated as shown in Figure 2.9. The algorithm assumes that all we need to do is keep track of the deformed orientation of the bit surface pictured in Figure 2.9. A module in the program is designed to "remember" where each rigid cutter is located on the surface (e.g. see Figure 2.9) and thus makes provision for the generation of equivalent forces to be applied to the bit face each time one of these cutters attempt to penetrate (or puncture) the surface of the wellbore.

Because of their physical non-existence in the finite element mesh of the bit, these rigid structures will, herein, be referred to as ghost cutters. The reader should note that these ghost cutters are also used to control the gauge of the bit and are the primary (and only means at this point) of enforcing contact of the bit with the surrounding formation.

FIGURE 2.9: A Simplified Bit Face Mesh for the Reduced Finite Element Model (Note that the Dotted Circles Represent Cutter Locations)

FIGURE 2.10: Finite Element Mesh at Cross-Section A-A
(See Figure 2.2)* of the Reduced Bit Finite
Element Model (Note: Sections B-B, C-C,
D-D and E-F Remain the Same for the Reduced
Model as They Were for the Total Bit Model)

Though, for the purposes of modeling the structural interaction between the bit and the formation, the ghost cutters are treated as rigid structures; provision is made in the Phase 1 Version of GEODYN for material property input for the cutters. These properties are intended for use in the Phase 2 development of failure models for the ghost cutters.

The basic analytical procedure is reserved for discussion later in Chapters IV and VI. Suffice it to say for now, that the use of ghost cutters permits the significant reduction in the complexity of the mesh required to model the bit face. This is an important feature because it is the refinement of the bit face which controls the number of elements required for the entire remaining bit model (e.g. see Figures 2.2 through 2.10). With respect to the reduced bit mesh pictured in Figures 2.9 and 2.10, Table 2.2 below defines the required number of elements, nodes and degrees of freedom necessary for its mathematical definition.

Table 2.2: Number of Elements, Nodes and
Degrees of Freedom Required for
Specification of the Reduced Bit Model
(for an 8 Node Brick Mesh)

Number of Elements	=	117
Number of Nodes	=	218
Number of D.O.F.	=	654

Examining Table 2.2 illustrates an obvious advantage of the reduction procedure. Simply, we have successfully eliminated approximately $2/3$ of the degrees of freedom required to specify the model. This results in a significant reduction in costs necessary for executing the dynamic solution. Further, there is another advantage resulting from the added simplicity apparent when comparing the meshes at the face for the total (Figure 2.3) and reduced (Figure 2.9) models. This advantage arises from the fact that it is much easier to develop and implement an element mesh generator for the latter case than the former.

Eliminating the cutters as structural dynamic quantities makes sense from an analytical accuracy viewpoint as well as an economic viewpoint. Simply, it does not make sense to attempt to integrate the dynamic response of the bit starting from the cutters because the dynamic frequency of the cutters is so much greater than that of the body of the bit itself. The time step to accomplish such an integration is so small as to make the integration of the dynamic equations of motion from this micro-level as too expensive to contemplate. Thus, it makes good sense to ignore the mass effects of the teeth or cutters when defining the finite element model of the bit. This is especially true since what we are really after is definition of the gross dynamic response and deformation of the body of the bit. We are not really trying to determine that information for the various individual cutters anyway.

In closing this discussion of the bit finite element model and the ghost cutters, it should be noted that (see Chapter III)

1. even simpler models than that shown in Figures 2.9 and 2.10 proved to be effective in an analysis of a bit, and
2. a different configuration of bit than that shown in Figures 2.2 through 2.10 was actually used in this study.

The algorithm representing the formation response mechanism is now outlined. A more complete discussion of this algorithm is provided in Chapter IV. Based on the deformed positions of the bit, this algorithm (or module) first determines how far (if at all) each ghost cutter has penetrated the formation. Next, based on the cutter's computed penetration distance and velocity, together with the properties of the formation at the point of contact, the forces being exerted on the cutter are calculated. As indicated in previous discussion (Section 2.1), two types of forces (termed nodal bit formation interaction forces) are computed. The first is a penetration force which is a function of the penetration distance, and the second is a friction force which depends on the magnitude of the penetration force and the velocity of the cutter.

2.2.2 The Numerical Time Integration of the Equations of Motion of the Bit-Bit Sub Finite Element Model

The dynamic response analysis of any finite element model of a PDC Bit involves the time integration of the equations of motion given by Equation 2.1. All practical numerical integration schemes can be described as step-by-step extrapolation procedures. Starting with initial conditions of known displacement and velocity vectors at time t_0 (i.e. $\{u_0\}$ and $\{\dot{u}_0\}$), the numerical processes calculate successively the corresponding vectors $\{u_1\}$ and $\{\dot{u}_1\}$ at time $t_1 = t_0 + \Delta t$, $\{u_2\}$ and $\{\dot{u}_2\}$ at time $t_2 = t_0 + 2\Delta t$, etc., where Δt is a suitably selected time increment.

Clough, in Reference 2.1, defined two basic classifications for extrapolating the solution of the dynamic response equations (Equation 2.1). He classed these as:

- (1) Finite Difference Procedures, in which the acceleration vector is expressed in terms of several successive displacement vectors; the order of the finite difference approximation controlling the refinement of the analytical process and,

- (2) Numerical Integration Procedures, in which the velocity and displacement vectors are obtained for each new time interval by integrating twice the acceleration vector; the order of the numerical integration approximation controlling the refinement of the process.

In each of the above categories, the procedure may be expressed as explicit (open) or implicit (closed). The formulas defining the explicit procedures are found to express the new solution vectors only in terms of vectors at preceding times which are already known (i.e. have previously been solved for). The implicit formulas include the new vectors being sought in the expressions for the new vectors themselves and are, therefore, generally solved for through some iterative process. In some cases, implicit expressions can be reformulated for direct solution of the new vector and are thus converted to explicit procedures.

The primary consideration in the selection of a step-by-step solution procedure is its efficiency (i.e. the accuracy obtained for a given computing cost) and its numerical stability. In general, any desired level of accuracy can be achieved with each of these procedures by merely making the time increment short enough. Shortening the time increment increases the accuracy of the solution at the expense of also

increasing the cost of a solution due to the greater number of total steps required to integrate the equations of motion over a given time interval. For a given length (or magnitude) of time increment, the accuracy generally will be better for the higher order procedures, but at the cost of additional computing operations for each step. Clough (Reference 2.1) identifies three types of errors which can occur in the integrated solution result, i.e.

- . Truncation Errors resulting from the approximation of the actual dynamic behavior by expressions of lower order,
- . Instability Errors resulting when errors from earlier time steps are transmitted with amplification into later steps, and
- . Computational Roundoff Errors

Following a survey of all available candidate numerical procedures, Newmark's Numerical Algorithm was selected for carrying out the time integration of motion of the PDC Bit-Bit Sub Finite Element Model. The selection of this method was primary due to its efficiency and numerical stability.

2.2.2.1 Numerical Integration Formulations

The basic operations of a numerical integration formulation (a class 2 procedure according to Clough, Refer-

ence 2.1) can be expressed as a set of two integrations for each time step, i.e.

$$\{\dot{u}_1\} = \{\dot{u}_0\} + \int_0^{\Delta t} \{\ddot{u}(t)\} dt \quad (a)$$

(2.2)

$$\{u_1\} = \{u_0\} + \int_0^{\Delta t} \{\dot{u}(t)\} dt \quad (b)$$

in which $\{\ddot{u}(t)\}$ is determined from the solution of Equation 2.1, i.e.

$$\{\ddot{u}(t)\} = [M]^{-1} \left\{ \{P(t)\} - [C] \{\dot{u}(t)\} - [K] \{u(t)\} \right\} \quad (2.3)$$

where

$$\{P(t)\} \equiv \{F_e\} + \{F_f\} + \{F_u\} \quad (2.4)$$

The integral over the time interval is generally obtained numerically, using a procedure which assumes the acceleration to vary as some polynomial function of time. The essential difference between most of the popular methods lies in the order of the assumed variation of the acceleration during the time interval.

2.2.2.2 Newmark's Method

In Newmark's Method, Equations 2.2 are assumed to be approximated by the general forms

$$\{\dot{u}_1\} = \{\dot{u}_0\} + (1-\gamma)\Delta t \{\ddot{u}_0\} + \gamma\Delta t \{\ddot{u}_1\} \quad (a)$$

(2.5)

$$\{u_1\} = \{u_0\} + \Delta t\{\dot{u}_0\} + (\frac{1}{2}-\beta)\Delta t^2\{\ddot{u}_0\} + \beta\Delta t^2\{\ddot{u}_1\} \quad (b)$$

where β and γ are fixed parameters that are selected to obtain the best stability and accuracy characteristics in the solution. By selecting specific values of β and γ , the Newmark Method may be reduced to other specific integration algorithms.

Comparing Equations 2.2 and 2.5, we find that Newmark's Method assumes

$$\int_0^{\Delta t} \{\ddot{u}(t)\}dt \approx (1-\gamma)\Delta t \{\ddot{u}_0\} + \gamma\Delta t \{\ddot{u}_1\} \quad (a)$$

(2.6)

$$\int_0^{\Delta t} \{\dot{u}(t)\}dt \approx \Delta t\{\dot{u}_0\} + (\frac{1}{2}-\beta)\Delta t^2 \{\ddot{u}_0\} + \beta\Delta t^2\{\ddot{u}_1\} \quad (b)$$

Upon examining the above relationships, it is clear that the factors γ and β are used to weight (or bias) the approximation of the integrals either backwards towards the initial

acceleration values $\{\ddot{u}_0\}$ or forwards towards the final acceleration values $\{\ddot{u}_1\}$. It is important to note that, since the acceleration vector $\{\ddot{u}_1\}$ appears in the right hand sides of the expressions for $\{u_1\}$ and $\{\dot{u}_1\}$ (see Equation 2.5), Newmark's Method must be classed as an implicit integration procedure.

The details of the application of Newmark's Method to the numerical time integration of the equations of motion of the Bit-Bit Sub Finite Element Model are provided in Chapter VI.

2.2.2.3 Mode Superposition

The number of operations required in the direct integration of the bit-bit sub equations of motion (Equations 2.1 and 2.3) is directly proportional to the number of time steps used in the analysis. Therefore, in general, the use of direct integration can be expected to be effective when the response for a relatively short duration (i.e., for a few time steps) is required. However, if the integration must be carried out for many time steps, it is often more effective to first transform the equilibrium equations (Equation 2.1) into a form in which the step-by-step solution is less costly (Bathe, Reference 2.2). In particular, since the number of operations required is directly proportional to the half-band width of the stiffness matrix, a reduction in this half-band width would decrease proportionally the cost of the step-by-step solution.

It should be recalled that the stiffness $[K]$ and consistent mass $[M]$ matrices of Equation 2.1 have a bandwidth that is determined by the numbering of the finite element nodal points. Therefore, the topology of the finite element mesh determines the order and bandwidth of the system matrices. In order to reduce the bandwidth of the system matrices, we may rearrange the nodal point numbering. However, there is a limit on the minimum bandwidth that can be obtained in this way. As such we chose to follow a different procedure outlined by Bathe in Reference 2.2.

In order to reduce the number of degrees of freedom of the dynamic finite element model being integrated, we implement the Method of Mode Superposition which provides for a change of basis to Modal Generalized Displacements. Specifically, we transform the equilibrium equations (Equation 2.1) into a more effective form for direct integration (using Newmark's Method) by using a transformation of the form

$$\{u\} = [\Phi] \{\eta\} \quad (2.7)$$

where

$$[\Phi] = [\phi_1 \mid \phi_2 \mid \dots \mid \phi_N]$$

$$= \text{Matrix of eigenvectors } \{\phi_i\} \quad (a)$$

$$\{u\} = \text{Nodal Displacement Vector} \quad (b) \quad (2.8)$$

$$\{\eta\} = \text{Vector of Generalized DOF} \quad (c)$$

The application of the foregoing Mode Superposition process to the numerical time integration of the Bit-Bit Sub Equations of Motion is undertaken in Chapter V.

III. THE PDC BIT-BIT SUB FINITE ELEMENT MODEL

The discussion of this Chapter is dedicated solely to consideration of the nature of the finite element model used to represent the PDC Bit-Bit Sub structural system.

Section 3.1 presents the basic ideas behind the formulation of the Eight Node Isoparametric Three-Dimensional Brick Element, which is the basic building block of the PDC Bit finite element model. Included in this discussion are reviews of the formulation of the stiffness, consistent mass and lumped mass matrices characterizing the element as well as examples of the use of the element in constructing two different finite element models of a SMITH TOOL 8 3/4" PDC Bit.

Section 3.2 provides explicit definitions of the stiffness, consistent mass and lumped mass matrices of the Three Dimensional Linear Beam Element used to model the short bit sub collar which is attached to the bit. A simple example is also provided to illustrate the use of this element in modelling a 52" long, 4.25" O.D. bit sub with accompanying 7.345" long, 5.5" O.D. tool joint.

Section 3.3 outlines the development of the constraint equations necessary to enforce the tying required between the eight node brick elements, used to model the bit, and the linear beam elements, implemented in the representation of the Bit Sub. Also presented are the final form (or con-

figuration) of the structure stiffness matrix following incorporation of the constraint equations into the force displacement relations of the tied PDC Bit-Bit Sub Model.

Section 3.4 describes the set of "Spring Stiffnesses" which are attached by GEODYN to the uppermost beam node of the tied finite element model of the PDC Bit-Bit Sub system and which are implemented to approximate upstring boundary conditions.

3.1 The Formulation of the Eight Node Isoparametric Brick Element and Definition of Two Basic PDC Bit Finite Element Models

The most obvious choices for modeling a complex three-dimensional structure such as the bit remain the isoparametric eight node and twenty node brick elements pictured in Figures 3.1 and 3.2 respectively.

By way of definition, isoparametric elements are those for which the same functions used to represent the displacement field are also incorporated to represent the element geometry field. Isoparametric element construction is expressed by the "mapping" of a non-dimensional rectangular brick element with a specified number of nodes into the actual curved boundary element with the same number of nodes. For example, in the case of the eight node and twenty node bricks, the elements are initially defined in a natural ξ , η , ζ coordinate system and are then subjected to the mapping which carries them from their initially cubic shapes (Figures 3.1 (a) and 3.2 (a)) into the distorted shapes which they possess in the physical X, Y, Z rectangular Cartesian coordinate system (Figures 3.1 (b) and 3.2 (b)).

When compared to the eight node brick, the higher order element, the twenty node brick has been shown to be a more rapidly converging element for three-dimensional structural analysis. This is a direct result of the fact that the twenty

(a) The Eight Node Brick Element Pictured In Its Natural ξ, η, ζ Coordinate System

(b) The Eight Node Brick Element Shown Distorted In The Physical Global Rectangular Cartesian X, Y, Z Coordinate System

FIGURE 3.1: The Eight Node Brick Finite Element Configuration

(a) The Twenty Node Brick Element Pictured In Its Natural ξ, η, ζ Coordinate System

(b) The Twenty Node Brick Element Shown Distorted In The Physical Global Rectangular Cartesian X, Y, Z Coordinate System

FIGURE 3.2: The Twenty Node Brick Finite Element Configuration

node brick possesses higher order descriptions of both its displacement and geometry fields. The former condition permits the twenty node brick to more accurately model the strain field in a deforming three-dimensional body while the latter permits the element to more easily represent any curved surfaces on the body.

The foregoing discussion leads us to consideration of an important point. Specifically, the rationalization for employing isoparametric element representations instead of the more conventional element forms. Simply, three-dimensional finite element analyses are notoriously expensive in terms of problem size. If the structure possesses curved boundaries (e.g. a PDC Bit), many conventional elements in a regular element representation are used exclusively to achieve geometric modeling without significant advantages in improved stress or displacement prediction. Thus, isoparametric elements serve to reduce the costs of idealization which are assignable strictly to geometric representation.

Since we are primarily concerned with the dynamic effects influencing the behavior of the rotating PDC Bit-Bit Sub system, we do not require the same level of stiffness representation which would be necessary if our primary goal were the detailed stress analysis of an exact finite element model representing the bit. As such, we do not need all the complexity or degrees of freedom introduced by the relatively high order twenty node brick solid element. All that is required is the

standard eight node isoparametric brick element pictured in Figure 3.1. As previously indicated, this element has, in fact, been incorporated into GEODYN for modeling the body of the PDC Bit with a standard three-dimensional linear beam element included to model the attached bit sub. The formulations of both the element stiffness matrix $[K_e]$ and the element mass matrix $[M_e]$ for the eight node brick element are briefly reviewed below.

Since it is not possible to express the stiffness and mass matrices of an isoparametric element such as the eight node brick in explicit algebraic form, the objective of the following presentation is only to indicate the general operations involved in doing so numerically via numerical integration. More specific details of the formulation of this element are available from many sources. For example, the reader may consult Zienkiewicz (Reference 3.1).

3.1.1 Development of the Element Stiffness Matrix

Establishing $\{u\}$ as the vector of three displacement components, u , v , w defining the displacement field within the finite element and $\{u_e\}$ as the vector of element nodal displacement components (i.e. degrees of freedom), we can express the displacement field approximation by

$$\begin{aligned}
\{u\} &= [N(\xi, \eta, \zeta)] \{u_e\} & a) \\
\text{with } \{u\}^T &= [u \ v \ w] & b) \\
\{u_e\}^T &= [u_1 v_1 w_1 \mid u_2 v_2 w_2 \mid \dots \mid u_8 v_8 w_8] & c)
\end{aligned} \tag{3.1}$$

and where $[N]$ is the matrix of shape functions used to interpolate the element displacement field. As indicated in Equation 3.1-a, the elements (or components) of the shape function matrix are functions of the natural coordinate system ξ, η, ζ . The element nodal displacement vector $\{u_e\}$ contains 24 degrees of freedom, i.e. the u (x-direction), v (y-direction) and w (z-direction) displacements at each of the elements 8 nodes (see Equation 3.1-c).

The strain vector $\{\epsilon\}$ is obtained from the relationship

$$\{\epsilon\} = [L(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})] \{u\} \tag{3.2}$$

where $[L]$ is the deformation operator matrix. The $[L]$ matrix permits the algebraic expression of the linear strain-displacement relations, of the three-dimensional theory of Elasticity, governing the element's deformational behavior in matrix form.

Substituting Equation 3.1-a into Equation 3.2, we find that

$$\{\epsilon\} = [D] \{u_e\} \tag{3.3}$$

where

$$[D(\frac{\partial [N]}{\partial x}, \frac{\partial [N]}{\partial y}, \frac{\partial [N]}{\partial z})] = [L] [N] \tag{3.4}$$

The strain energy U of the isoparametric finite element is given by

$$U = \frac{1}{2} \int_{V_e} \{\epsilon\}^T [E] \{\epsilon\} dV_{xyz} \quad (3.5)$$

where $[E]$ is the matrix of elastic material constants relating the stress vector $\{\sigma\}$ to the strain vector $\{\epsilon\}$, i.e.

$$\{\sigma\} = [E] \{\epsilon\} \quad (3.6)$$

The quantity V_e in the integral of Equation 3.5 designates the volume of the element in its physical (i.e. X, Y, Z coordinate system), or mapped, configuration while dV_{xyz} represents a differential volume element in the same system.

The substitution of Equation 3.3 into Equation 3.5 yields

$$U = \frac{1}{2} \{u_e\}^T [K_e] \{u_e\} \quad (3.7)$$

where $[K_e]$ is the element stiffness matrix given by

$$[K_e] = \int_{V_e} [H] dV_{xyz} \quad (3.8)$$

with

$$\left[H \left(\frac{\partial [N]}{\partial x}, \frac{\partial [N]}{\partial y}, \frac{\partial [N]}{\partial z} \right) \right] = [D]^T [E] [D] \quad (3.9)$$

Examining Equation 3.9, it is clear that we require the partial derivatives with respect to X, Y, Z of the shape function matrix [N], i.e. we need $\frac{\partial[N]}{\partial x}$, $\frac{\partial[N]}{\partial y}$, $\frac{\partial[N]}{\partial z}$ to complete our formulation of the element stiffness matrix. However, we can see from Equation 3.1 that the shape function matrix [N] is, in reality, a function of the natural coordinates ξ , η , ζ . The most we are able to determine at this point is the partial derivatives represented by $\frac{\partial[N]}{\partial \xi}$, $\frac{\partial[N]}{\partial \eta}$, $\frac{\partial[N]}{\partial \zeta}$. We must, therefore, define the transformation illustrated in Equation 3.10 below before we can carry out the integration of the element indicated by Equation 3.8 above.

$$\begin{bmatrix} \frac{\partial[N]}{\partial \xi} \\ \frac{\partial[N]}{\partial \eta} \\ \frac{\partial[N]}{\partial \zeta} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\partial[N]}{\partial x} \\ \frac{\partial[N]}{\partial y} \\ \frac{\partial[N]}{\partial z} \end{bmatrix} \quad (3.10)$$

We can establish the required transformation through the mapping between the X, Y, Z and ξ , η , ζ coordinates implied by the following definition of the element geometry field

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [N] \begin{Bmatrix} X_e \\ \text{---} \\ Y_e \\ \text{---} \\ Z_e \end{Bmatrix} \quad (3.11)$$

In Equation 3.11 above, the vectors $\{X_e\}$, $\{Y_e\}$ and $\{Z_e\}$ represent, respectively, the vectors of the X, Y and Z coordinates defining the location of each of the element nodes in the global rectangular Cartesian coordinate system (see Figures 3.1 and 3.2).

Defining N_{ij} as being an arbitrary element (or member) of the shape function matrix $[N]$ where

$$N_{ij} = N_{ij}(\xi, \eta, \zeta) \quad (3.12)$$

with i and j indicating, respectively, the row and column numbers locating the element N_{ij} within $[N]$, we can implement the chain rule of partial differentiation to obtain

$$\begin{Bmatrix} \frac{\partial N_{ij}}{\partial \xi} \\ \frac{\partial N_{ij}}{\partial \eta} \\ \frac{\partial N_{ij}}{\partial \zeta} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_{ij}}{\partial x} \\ \frac{\partial N_{ij}}{\partial y} \\ \frac{\partial N_{ij}}{\partial z} \end{Bmatrix} \quad (3.13)$$

where $[J]$ is the Jacobian of the transformation given by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (3.14)$$

Observing that the partial derivatives required for the construction of the Jacobian [J] (Equation 3.14) are easily obtained from Equation 3.11, we can invert the above relationship to obtain the desired transformation

$$\begin{Bmatrix} \frac{\partial N_{ij}}{\partial x} \\ \frac{\partial N_{ij}}{\partial y} \\ \frac{\partial N_{ij}}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_{ij}}{\partial \xi} \\ \frac{\partial N_{ij}}{\partial \eta} \\ \frac{\partial N_{ij}}{\partial \zeta} \end{Bmatrix} \quad (3.15)$$

which was indicated previously in Equation 3.10.

Designating $dV_{\xi\eta\zeta}$ as being a differential volume element in the natural ξ, η, ζ coordinate system, we note that the formulation of an isoparametric element is complete with the establishing of both Equation 3.15 and the transformation between differential volume elements given by

$$dV_{xyz} = \det[J] dV_{\xi\eta\zeta} \quad (3.16)$$

where

$$\begin{aligned} dV_{xyz} &= dx \, dy \, dz & \text{a)} \\ dV_{\xi\eta\zeta} &= d\xi \, d\eta \, d\zeta & \text{b)} \end{aligned} \quad (3.17)$$

As indicated in Figure 3.1, the natural configuration of the eight node brick element is a simple cube, two units on a side, bounded by the surfaces defined below (see also Figure 3.1 a)).

$$\begin{aligned}
 \xi &= \pm 1 \text{ for the two } \eta, \zeta \text{ planes} & \text{a)} \\
 \eta &= \pm 1 \text{ for the two } \xi, \zeta \text{ planes} & \text{b)} \\
 \zeta &= \pm 1 \text{ for the two } \xi, \eta \text{ planes} & \text{c)}
 \end{aligned} \tag{3.18}$$

Implementing Equations 3.16, 3.17 and 3.18 above, we can re-express the integral of Equation 3.8 (for defining the stiffness matrix $[K_e]$) in the form shown below

$$[K_e] = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} [H(\xi, \eta, \zeta)] \det[J(\xi, \eta, \zeta)] d\xi d\eta d\zeta \tag{3.19}$$

The above integral is, of course, much too complicated to obtain in closed (or explicit) algebraic form. The integral is, therefore, approximated numerically through the Gauss quadrature numerical integration procedure. A definition of the shape function N_i for a typical node i of the eight node brick element is given below in Equations 3.20 and 3.21.

$$N_i = 1/8 (1 + \zeta_o)(1 + \eta_o)(1 + \xi_o) \tag{3.20}$$

$$\begin{aligned}
\xi_o &= \xi \xi_i & a) \\
\eta_o &= \eta \eta_i & b) \quad (3.21) \\
\zeta_o &= \zeta \zeta_i & c)
\end{aligned}$$

3.1.2 Development of the Element Mass Matrices

GEODYN provides for the definition of both standard forms of element Mass Matrices, i.e. both consistent and lumped. The formulations of these matrices are summarized below.

Consistent Mass Matrix

The kinetic energy K.E. of a typical eight node isoparametric brick element can be expressed in the matrix form

$$K.E. = \frac{1}{2} \int_{V_e} \{\dot{u}\}^T [\rho_e] \{\dot{u}\} dV_{xyz} \quad (3.22)$$

where $\{\dot{u}\}$ represents the velocity field of the element which (as indicated by the superimposed dot) is derived by differentiation of the element displacement field $\{u\}$ (Equation 3.1) with respect to time t . The matrix $[\rho_e]$, of Equation 3.22, is the element mass density matrix given by

$$[\rho_e] = \rho_e [I] \quad (3.23)$$

where ρ_e is the assumed uniform element mass density and $[I]$ is a (3x3) diagonal unit matrix.

The substitution of Equation 3.23 into Equation 3.22 yields

$$K.E. = \frac{1}{2} \rho_e \int_{V_e} \{\dot{u}\}^T \{\dot{u}\} dV_{xyz} \quad (3.24)$$

In view of the relationship expressed in Equation 3.1, it is reasonable to assume that

$$\{\dot{u}\} = [N] \{\dot{u}_e\} \quad (3.25)$$

The substitution of Equation 3.25 into the right hand side of Equation 3.24 leads to the re-expression of the element kinetic energy K.E. in the form

$$K.E. = \frac{1}{2} \{\dot{u}_e\}^T [M_e^C] \{\dot{u}_e\} \quad (3.26)$$

where $[M_e^C]$ is the element consistent Mass Matrix given by

$$[M_e^C] = \rho_e \int_{V_e} [N]^T [N] dV_{xyz} \quad (3.27)$$

Lumped Mass Matrix

Comparing Equations 3.1 and 3.25 above, we find that we used the same shape functions to interpolate the velocities $\{\dot{u}\}$ as we used to interpolate the displacements $\{u\}$. As a consequence, the kinetic energy K.E. of an element was made to be consistent with the strain energy U (Equation 3.5) of the element. Hence the reason for designating the resulting element Mass Matrix $[M_e^C]$ (Equation 3.27) a consistent Mass Matrix. The consistency resulting from the use of the same shape functions to generate both the stiffness and mass matrices for the eight node brick element implies that the bounding properties of the Ritz process apply.

It is often found not to be desirable to use the consistent mass matrix because excessive computational effort can be required to obtain a desired degree of accuracy. Other element mass matrices can be derived by using different shape functions in Equation 3.25 from those used in Equation 3.1 (i.e. we purposely become inconsistent in the interpolation of our displacement and velocity fields). The simplest results are obtained by letting the shape functions interpolating the velocity field have a unit value over a specified portion of the element volume with a zero value elsewhere (Clough, Reference 3.2). In this case, each diagonal

term of the resulting mass matrix represents the mass associated with the corresponding non-zero area of the shape functions, and the off-diagonal terms are zero if the non-zero areas of the shape functions do not overlap. The great advantage of the lumped mass matrix which results from this formulation, in addition to the obvious ease of evaluating the mass matrix itself, is that the mass matrix of the entire assembled finite element model is diagonal. This leads to great simplifications in the subsequent dynamic analysis.

Expressing the kinetic energy K.E. of our eight node brick element in the alternative form

$$\text{K.E.} = \frac{1}{2} \{\dot{\mathbf{u}}_e\}^T [\mathbf{M}_e^l] \{\dot{\mathbf{u}}_e\} \quad (3.28)$$

where $[\mathbf{M}_e^l]$, the element lumped mass matrix for the eight node brick incorporated into GEODYN, is given by

$$[\mathbf{M}_e^l] = \frac{M_e}{8} [\mathbf{I}] \quad (3.29)$$

In equation 3.29, $[\mathbf{I}]$ is meant to indicate a (24x24) diagonal unit matrix and M_e is the total mass of the element derived from the integral

$$M_e = \rho_e \int_{V_e} dV_{xyz} \quad (3.30)$$

3.1.3 Two Different Eight Node Brick Finite Element Models of a PDC Bit

Figures 3.3 through 3.8 illustrate two different examples of finite element models of a PDC Bit, which have been built up geometrically using the just described Eight Node Brick isoparametric finite element shown in Figure 3.1. The same bit, a SMITH TOOL 8 3/4" PDC, is being modeled in both these cases.

As previously indicated in the discussion of Section 2.2.1, the implementation of the Ghost Cutter concept in the solution algorithm precludes the need of actually physically modeling the bit's cutters using finite elements as shown in Figures 2.4 and 2.5. Thus, the two meshes illustrated in Figures 3.3 through 3.8 model only the body of the bit and ignore the physical presence of the cutters on the bit face.

It should also be noted that the meshes shown in Figures 3.3 through 3.8, as well as the actual plots which comprise these figures were generated using the FEMGEN/FEMVIEW System (Reference 3.3).

Figures 3.3 through 3.5 provide three different views (or perspectives) of the first model, a complex (geometrically detailed) model of the SMITH Bit composed of a total of 192 elements connected at 370 nodes resulting in 1,110 degrees of

FIGURE 3.3: A Hidden Line Plot of a Complex Model
of a Smith Tool 8 3/4" PDC Bit

FIGURE 3.4: A Hidden Line Plot of a Cutaway Section of the Complex Model of the 8 3/4" PDC Bit Shown in Figure 3.3

FIGURE 3.5: A Full Mesh Plot of the Complex Model of the 8 3/4" PDC Bit Shown in Figure 3.3

freedom. Figure 3.3 presents a hidden line plot of a full side view of the complex bit model while a cutaway view of the same representation is shown in Figure 3.4. Figure 3.5 presents the corresponding full mesh (as opposed to hidden line) plot of the complex model of the bit showing the detail suppressed in Figures 3.3 and 3.4.

Figures 3.6 through 3.8 present a different approach to modeling the same bit using a much simpler, dynamically mass equivalent, model characterized by 32 elements, 66 nodes and 198 degrees of freedom. Figure 3.6 presents a hidden line plot of a full side view of the simple bit model while a cutaway (hidden line plot) of the model is shown in Figure 3.7 and a full mesh view of the model is depicted in Figure 3.8.

This simple model, which has been shown to produce satisfactory results for many fewer degrees of freedom, depends on a version of the ghost cutter element (cast in the form of an ordinary gap - spring/slider - contact element) to represent the significant difference in lateral surface geometry the simpler model exhibits when compared to the more complex model of Figures 3.3 through 3.5. The reduced number of degrees of freedom, when compared to the case of the more complex model (i.e. 198 versus 1,110), results in a significant reduction in computational effort and cost necessary for carrying out the numerical integration of the dynamic equations of motion.

FIGURE 3.6: A Hidden Line Plot of A Simple, Dynamically Mass Equivalent, Finite Model of the Smith Tool 8 3/4" PDC Bit

FIGURE 3.7: A Hidden Line Plot of a Cutaway Section of the Simple Model of the 8 3/4" PDC Bit Shown in Figure 3.6

FIGURE 3.8: A Simple, Full Mesh Plot of the Dynamically Mass Equivalent Finite Element Model of the Smith Tool 8 3/4" PDC Bit Shown in Figure 3.6

The reader should note that the dynamics algorithm incorporated into the Phase 1 Version of GEODYN is capable of accommodating either bit model (i.e. Figures 3.5 or 3.8).

3.2 The Definition of the Three-Dimensional Linear Beam Element

The three-dimensional linear beam element defined below is incorporated into the Phase 1 Version of GEODYN solely for the purpose of modeling the short bit sub collar which is attached to the bit. Unlike the eight node isoparametric brick element outlined in Section 3.1, it is possible to express the stiffness and mass matrices of this beam element in explicit algebraic form. The three-dimensional linear beam element has been derived in many sources, so the detailed steps of its formulation will not be repeated here. The discussion provided below is confined solely to the definition of the stiffness and mass matrices characterizing the element. The stiffness and mass matrices which follow are those derived by J.S. Przemieniecki in Reference 3.4 and the reader is directed to that source if a detailed understanding of their development is desired.

The element, as shown in Figure 3.9, possesses a total of 12 degrees of freedom, i.e. six at each node which are the U (x-direction), V (y-direction) and W (z-direction) displacements (or translations) of the node and the θ_x , θ_y , and θ_z rotations taken, respectively, about the X, Y, Z coordinate axes. The element nodal displacement vector $\{u_e\}$ is given by

FIGURE 3.9: Nodal Degrees of Freedom for the Three Dimensional Linear Beam Element

$$\{u_e\}^T = \begin{bmatrix} U_1 & V_1 & W_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} \\ U_2 & V_2 & W_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{bmatrix} \quad (3.31)$$

The material and geometric properties required for the definition of the element stiffness and mass matrices are listed below.

- E = Modulus of Elasticity
- G = Shear Modulus
- ρ = Mass Density per Unit Volume
- A = Cross Sectional Area
- L = Length
- I = Bending Moment of Inertia
- J = Polar Moment of Inertia

Section 3.2.1, which follows, addresses the definition of the element stiffness matrix $[K_e]$ while Section 3.2.2 reviews the consistent $[M_e^c]$ and lumped $[M_e^l]$ mass matrices. An illustration of the use of the beam element in modelling the bit sub is provided in Section 3.2.3.

3.2.1 The Linear Beam Element Stiffness Matrix

Defining the stiffness quantities k_i as given below (Equation 3.32), the element stiffness matrix $[K_e]$ is readily expressed in the form given in Equation 3.33.

$$\begin{aligned}
k_1 &= \frac{12EI}{L^3} & \text{a)} \\
k_2 &= \frac{EA}{L} & \text{b)} \\
k_3 &= \frac{4EI}{L} & \text{c)} \quad (3.32) \\
k_4 &= \frac{GJ}{L} & \text{d)} \\
k_5 &= \frac{6EI}{L^2} & \text{e)} \\
k_6 &= \frac{2EI}{L} & \text{f)}
\end{aligned}$$

3.2.2 The Linear Beam Element Mass Matrices

As in the case of the mass matrices of the eight node isoparametric brick element outlined in Section 3.1.2, we can define both consistent and lumped mass forms for the three dimensional linear beam element.

Consistent Mass Matrix

Defining the mass quantities m_i as shown below (Equations 3.34), the element consistent mass matrix $[M_e^C]$ can be expressed in the form given in Equation 3.35.

$$\begin{bmatrix} k_e \end{bmatrix} = \begin{bmatrix}
 k_1 & 0 & 0 & 0 & k_5 & 0 & -k_1 & 0 & 0 & 0 & k_5 & 0 \\
 k_1 & 0 & -k_5 & 0 & 0 & 0 & 0 & -k_1 & 0 & -k_5 & 0 & 0 \\
 k_2 & 0 & 0 & 0 & 0 & 0 & 0 & -k_2 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & 0 & 0 & k_5 & 0 & k_6 & 0 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & -k_5 & 0 & 0 & 0 & 0 & 0 & k_6 & 0 & 0 & 0 \\
 k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_4 & 0 & 0 \\
 k_1 & 0 & 0 & 0 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_1 & 0 & k_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

SYMMETRIC

EQUATION 3.3: The Stiffness Matrix For The Three Dimensional Linear Beam Element (See Also Equation 3.32)

$$\begin{aligned}
m_1 &= \rho AL \left(\frac{13}{35} + \frac{6I}{5AL^2} \right) & a) \\
m_2 &= \frac{1}{3} \rho AL & b) \\
m_3 &= \rho AL \left(\frac{L^2}{105} + \frac{2I}{15A} \right) & c) \\
m_4 &= \rho AL \left(\frac{J}{3A} \right) & d) \\
m_5 &= \rho AL \left(\frac{11L}{210} + \frac{I}{10AL} \right) & e) \quad (3.34) \\
m_6 &= \rho AL \left(\frac{9}{70} - \frac{6I}{5AL^2} \right) & f) \\
m_7 &= \rho AL \left(-\frac{13L}{420} + \frac{I}{10AL} \right) & g) \\
m_8 &= \frac{1}{6} \rho AL & h) \\
m_9 &= \rho AL \left(\frac{J}{6A} \right) & i) \\
m_{10} &= -\rho AL \left(\frac{L^2}{140} + \frac{I}{30A} \right) & j)
\end{aligned}$$

Lumped Mass Matrix

Defining $\bar{m}$ as the mass per unit length of the linear beam element given by

$$\bar{m} = \rho A \quad (3.36)$$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & m_5 & 0 & m_6 & 0 & 0 & 0 & m_7 & 0 \\ m_1 & 0 & -m_5 & 0 & 0 & 0 & 0 & m_6 & 0 & -m_7 & 0 & 0 \\ m_2 & 0 & 0 & 0 & 0 & 0 & 0 & m_8 & 0 & 0 & 0 & 0 \\ m_3 & 0 & 0 & 0 & 0 & m_7 & 0 & m_{10} & 0 & 0 & 0 & 0 \\ m_3 & 0 & -m_7 & 0 & 0 & 0 & 0 & 0 & m_{10} & 0 & 0 & 0 \\ m_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_9 & 0 & 0 \\ m_1 & 0 & 0 & 0 & -m_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_1 & 0 & m_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SYMMETRIC

EQUATION 3.35: The Consistent Mass Matrix for the Three Dimensional Linear Beam Element (See Also Equation 3.34)

the elements M_{eij}^{ℓ} (i^{th} row and j^{th} column) of the lumped mass matrix $[M_e^{\ell}]$ are given by

$$\begin{aligned}
 M_{eii}^{\ell} &= \frac{\bar{m}L}{2} \text{ for } i = 1, 2, 3, 7, 8, 9 & \text{a)} \\
 M_{eii}^{\ell} &= \frac{\bar{m}L}{2} \left(\frac{I}{A} \right) \text{ for } i = 4, 5, 10, 11 & \text{b)} \quad (3.37) \\
 M_{eii}^{\ell} &= \frac{\bar{m}L}{2} \left(\frac{J}{A} \right) \text{ for } i = 6, 12 & \text{c)} \\
 M_{eij}^{\ell} &= 0 \quad \text{for all } i \neq j & \text{d)}
 \end{aligned}$$

3.2.3 The Use of the Beam Element in Modeling the Bit Sub

Figure 3.10 illustrates the use of the three dimensional linear beam element in modeling both the bit sub and associated tool joint in the overall PDC Bit-Bit Sub System. The model of Figure 3.10 pictures two beam elements (one for representing the tool joint and another for the bit sub body) attached to the simple solid element bit model shown in Figure 3.8. Before tying constraints are applied along the plane or surface A-A, the eight beam elements add 9 nodes and 54 degrees of freedom to those required for the solid element model of the bit (Figure 3.8) for a total of 75 nodes and 252 degrees of freedom. Implementation of the tying procedure, to be out-

FIGURE 3.10: Smith Tool 8 3/4" PDC Bit with Tool Joint and Bit Sub

lined next in Section 3.3, permits the elimination of 48 of these 252 degrees of freedom to yield a final total of 204 degrees of freedom for the assembled, tied, Bit-Bit Sub Model.

3.3 Development of the Constraint Tying Equations

Compared to the Eight Node Isoparametric Brick Element, the Three-Dimensional Linear Beam Element contains three additional degrees of freedom (i.e. the rotations θ_x , θ_y , θ_z). As such, means must be provided to tie both of these types of elements together (i.e. through the aforementioned rotations, θ_x , θ_y , θ_z) in such a way as to insure a continuity of slope, in bending, of the centerline of the PDC Bit-Bit sub finite element model (see Figure 3.11). The following discussion outlines the procedure used by the GEODYN program algorithm to constrain specified nodes on the top of the three-dimensional PDC Bit Model to bend in a plane normal to the centerline of the attached (or tied) beam element (Figure 3.11). Thus, the plane section remains plane assumption inherent in the formulation of the beam element (Reference 3.4) is transferred to a portion of the uppermost (i.e. top) surface of the three-dimensional bit model.

The ultimate objective of the constraint tying operation is therefore to relate displacements, velocities and accelerations of specified nodes on the top surface of the bit to displacements, velocities, and accelerations at the beam node which is connected to the bit in such a way as to preserve the assumptions of Classical Beam Theory at the intersection between the two element types.

Nodes a-b-c-d-e-f-g-h
 represent Section A-A
 Tying Surface (or Plane)
 of Figure 3.10.

Figure 3.11: Illustration of Beam Element - Eight Node Brick
 Degree of Freedom Tying

The problem can be visualized through the example provided in Figures 3.10 and 3.11. In these figures, it can be seen that the connection between the bit sub (i.e. beam element) and the bit (i.e. eight node brick element) model is made between beam node BN_1 and brick element nodes a,b,c,d,e,f,g, and h (Figure 3.11). These eight nodes form a plane (e.g. Plane A-A, Figure 3.10) prior to deformation of the three-dimensional bit model and the constraint relations, which tie these nodes to beam node BN_1 , insure that these nodes will still lie in a common plane throughout the deformation of the bit-bit sub system. Moreover, the plane these nodes define will, at any and all times, remain perpendicular to the current deformed Bit-Bit Sub centerline. The procedure for constructing these tying relations is reviewed immediately below.

In it's initial form, the force-displacement relationship of the assembled finite element model of the PDC Bit-Bit Sub system can be expressed as

$$\{P_o\} = [K_o]\{u_o\} \quad (3.38)$$

Initially, in the un-tied Bit-Bit Sub finite element model, the bit nodes are distributed such that the load and displacement vectors are partitioned as given below where

$$[K_o] = \text{Assembled Stiffness Matrix of the PDC Bit-Bit Sub System}$$

$$\{u_o\} = \begin{Bmatrix} u_o^{3d} \\ - \\ u_o^b \end{Bmatrix} \quad (3.39)$$

$\{u_o^{3d}\}$ = Vector containing all three-dimensional eight node brick element degrees of freedom of the assembled bit model

$\{u_o^b\}$ = Vector containing all beam element degrees of freedom of the assembled bit sub model

and where

$$\{P_o\} = \begin{Bmatrix} P_o^{3d} \\ - \\ P_o^b \end{Bmatrix} \quad (3.40)$$

$\{P_o^{3d}\}$ = Corresponding PDC Bit (Eight Node Brick Element) finite element model load vector

$\{P_o^b\}$ = Corresponding Bit Sub (Linear Beam Element) finite element model load vector

Next, a transformation $[T_r]$ of the form

$$\{u_o\} = [T_r]\{\bar{u}\} \quad (3.41)$$

is implemented which transforms the untied Bit-Sub finite element model degrees of freedom from the partitioned form given below to the original form given in Equation 3.39.

$$\{\bar{u}\} = \begin{Bmatrix} u_r \\ \text{---} \\ u_e \\ \text{---} \\ u_t^b \\ \text{---} \\ u_u^b \end{Bmatrix} \quad (3.42)$$

where (see also Figure 3.11)

$\{u_r\}$ = Vector of Bit Model (Brick Element) degrees of freedom which are to be retained

$\{u_e\}$ = Vector of Bit Model (Brick Element) degrees of freedom which are to be eliminated through tying to the single bit node

$\{u_t^b\}$ = Vector of Bit Sub (Beam Element) degrees of freedom of the single beam node to which the $\{u_e\}$ degrees of freedom are tied (contains 6 D.O.F., i.e. $U, V, W, \theta_x, \theta_y, \theta_z$ at the beam node used in the tying operation)

$\{u_u^b\}$ = Vector of Bit Sub (Beam Element) degrees of freedom for all remaining beam nodes

By way of illustration, vector $\{u_t^b\}$ would contain the six degrees of freedom of beam node BN_1 pictured in Figures 3.10 and 3.11.

Substituting Equation 3.41 into the right hand side of Equation 3.38 we obtain

$$\{P_o\} = [K_o] [T_r] \{\bar{u}\} \quad (3.43)$$

Pre-multiplying both sides of Equation 3.43 above by $[T_r]^T$ yields

$$\{\bar{P}\} = [\bar{K}] \{\bar{u}\} \quad (3.44)$$

where

$$\{\bar{P}\} = [T_r]^T \{P_o\} \quad (3.45)$$

$$\{\bar{P}\} = \begin{Bmatrix} p_r \\ -\frac{p_r}{2} \\ p_e \\ -\frac{p_e}{2} \\ p_t^b \\ -\frac{p_t^b}{2} \\ p_u^b \end{Bmatrix} \quad (3.46)$$

and where

$$[\bar{K}] = [T_r]^T [K_o] [T_r] \quad (3.47)$$

$$[\bar{K}] = \begin{bmatrix} K_{rr} & K_{re} & K_{rt} & K_{ru} \\ K_{er} & K_{ee} & K_{et} & K_{eu} \\ K_{tr} & K_{te} & K_{tt} & K_{tu} \\ K_{ur} & K_{ue} & K_{ut} & K_{uu} \end{bmatrix} \quad (3.48)$$

Equation 3.48 presents the re-ordered stiffness matrix of the un-tied model corresponding to the degree of freedom order specified in Equation 3.42 and Equation 3.46 defines the load vector $\{\bar{P}\}$ which is conjugate in work to the displacement vector $\{\bar{u}\}$ (Equation 3.42).

At this point the constraint, or tying condition, is enforced through the relationship

$$\{u_e\} = [g] \{u_t^b\} \quad (3.49)$$

where

$$\{u_e\} = \left\{ \begin{array}{c} u_{N_r+1} \\ \text{-----} \\ u_{N_r+2} \\ \text{-----} \\ \vdots \\ \text{-----} \\ u_p \\ \text{-----} \\ \vdots \\ \text{-----} \\ u_{N_r+N_e} \end{array} \right\} \quad (3.50)$$

and where

$$[g] = \begin{bmatrix} g_{N_r+1} \\ \text{-----} \\ g_{N_r+2} \\ \text{-----} \\ \vdots \\ \text{-----} \\ g_p \\ \text{-----} \\ \vdots \\ \text{-----} \\ g_{N_r+N_e} \end{bmatrix} \quad (3.51)$$

with

N_r = the number of three-dimensional brick element nodes of the bit model which are to be retained after the tying/condensation procedure

N_e = the number of three-dimensional brick element nodes of the bit model, which are to be tied to a beam node, and are therefore to be eliminated through the tying/condensation procedure

P = designates an arbitrary node number P

$\{u_p\}$ = brick node degrees of freedom (u_p, v_p, w_p) at node P of the bit model

In considering the case of a particular node number P, the relationship of Equation 3.49 yields

$$\{u_p\} = [g_p] \{u_t^b\} \quad (3.52)$$

where, for the previously defined tying (illustrated by Figures 3.10 and 3.11), the matrix $[g_p]$ is given by

$$[g_p] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -Y_p \\ 0 & 1 & 0 & 0 & 0 & X_p \\ 0 & 0 & 1 & Y_p & -X_p & 0 \end{bmatrix} \quad (3.53)$$

and where X_p , Y_p are, respectively, the X,Y coordinates of node number P. Now, combining Equations 3.42 and 3.49 we can write

$$\{\bar{u}\} = [G] \{u\} \quad (3.54)$$

where $[G]$ and $\{u\}$ are given by

$$\{u\} = \left\{ \begin{array}{c} u_r \\ \hline u_t^b \\ \hline u_u^b \end{array} \right\} \quad (3.55)$$

$$[G] = \begin{bmatrix} I & 0 & 0 \\ 0 & g & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (3.56)$$

The substitution of Equation 3.54 into the left hand side of Equation 3.44 yields

$$\{\bar{P}\} = [\bar{K}] [G] \{u\} \quad (3.57)$$

Pre-multiplying both sides of Equation 3.57 above by $[G]^T$ produces

$$\{P\} = [K] \{u\} \quad (3.58)$$

where (in view of Equations 3.45 and 3.47)

$$[K] = [G]^T [\bar{K}] [G] \quad \begin{array}{l} \text{a)} \\ (3.59) \end{array}$$

$$= [G]^T [T_r]^T [K_o] [T_r] [G] \quad \text{b)}$$

$$\{P\} = [G]^T \{\bar{P}\} \quad \begin{array}{l} \text{a)} \\ (3.60) \end{array}$$

$$= [G]^T [T_r]^T \{P_o\} \quad \text{b)}$$

The substitution of Equations 3.48 and 3.56 into Equation 3.59-a yields the more compact expression for $[K]$ given by Equation 3.61.

At this point, accommodation of the constraint tying problem is complete. Equations 3.49, 3.52, 3.53 and 3.54 define the sought after constraint tying equations themselves. Equation 3.61 demonstrates the configuration of the structure stiffness matrix following incorporation of the constraint equations in the force displacement relations of the tied PDC Bit-Bit Sub Model.

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} [K_{rr}] & [K_{rt}] + [K_{re}][g] & [K_{ru}] \\ [k_{tr}] + [g]^T [K_{er}] & \begin{array}{l} [K_{tt}] \\ + [g]^T [K_{et}] + [K_{te}][g] \\ + [g]^T [K_{ee}][g] \end{array} & [K_{tu}] + [g]^T [K_{eu}] \\ [K_{ur}] & [K_{ut}] + [K_{ue}][g] & [K_{uu}] \end{bmatrix}$$

EQUATION 3.61

3.4 Definition of the Uppermost Beam Node Spring Stiffnesses

Displacement boundary conditions are not applied to the PDC Bit-Bit Sub Model per se. Rather, a set of "Spring Stiffnesses" is attached to the uppermost beam node (one for each of the node's 6 degrees of freedom) of the Bit Sub model. For example, in the model pictured in Figure 3.10, the six spring stiffnesses are attached to the six degrees of freedom of Node BN_3 . The springs are provided

- a) to render the bit-bit sub model stiffness matrix non-singular and also
- b) to provide a means of tuning the dynamic response of the finite element model.

In view of (a) above, the spring elements are seen to serve the same purpose as the application of displacement boundary conditions (or restraints) at the uppermost node would serve. Feature (b) above provides an approximate means of accounting for upstring elastic effects which are not directly modeled by the PDC Bit-Bit sub finite element model.

The application of the spring stiffnesses to a linear beam element node is a simple process. First assume that the spring stiffnesses are designated by

k_x = spring stiffness corresponding to the U
(x-direction) displacement of the beam node

k_y = spring stiffness corresponding to the V
(y-direction) displacement of the beam node

k_z = spring stiffness corresponding to the W
(z-direction) displacement of the beam node

κ_x = spring stiffness corresponding to the θ_x
rotation (rotation about the x-axis) of the
beam node

κ_y = spring stiffness corresponding to the θ_y
rotation (rotation about the y-axis) of the
beam node

κ_z = spring stiffness corresponding to the θ_z
rotation (rotation about the z-axis) of the
beam node

Next assume that we have chosen to append the stiffnesses to the six degrees of freedom at Node 1 (i.e. U_1 , V_1 , W_1 , θ_{x1} , θ_{y1} , θ_{z1}) of the Three-Dimensional Linear Beam Element which is pictured in Figure 3.9 and whose stiffness matrix is defined

in Equation 3.33. Following the implementation of these assumptions, the original stiffness matrix of the linear beam element (Equation 3.33) can be modified as shown in Equation 3.62 to account for the six spring stiffness attached to its Node 1 degrees of freedom.

$$\begin{bmatrix} k_e \end{bmatrix} = \begin{bmatrix}
 k_1+k_x & 0 & 0 & 0 & k_5 & 0 & -k_1 & 0 & 0 & 0 & k_5 & 0 \\
 k_1+k_y & 0 & -k_5 & 0 & 0 & 0 & 0 & -k_1 & 0 & -k_5 & 0 & 0 \\
 k_2+k_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_2 & 0 & 0 & 0 \\
 k_3+k_x & 0 & 0 & 0 & 0 & k_5 & 0 & 0 & 0 & k_6 & 0 & 0 \\
 k_3+k_y & 0 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 & k_6 & 0 & 0 \\
 k_4+k_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_4 & 0 \\
 k_1 & 0 & 0 & 0 & 0 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_1 & 0 & k_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

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EQUATION 3.62: The Stiffness Matrix For The Three-Dimensional Linear Beam Element (See Also Equation 3.32) Modified To Include The Spring Stiffnesses

IV. THE GOVERNING EQUATIONS OF EQUILIBRIUM AND THE CONTACT AND FRICTION NODAL FORCE VECTORS

Two different coordinate systems (see Figures 4.1-a, b) were used in order to establish the Global Computational Algorithm, for modeling the PDC Bit-Bit Sub/Formation Interaction, defined in Figure 2.1. These coordinate systems, which are both right hand Rectangular Cartesian in nature, are represented by a Global Fixed Inertial System characterized by the coordinates X_g, Y_g, Z_g and the corresponding unit base vectors $\vec{i}, \vec{j}, \vec{k}$ (Figure 4.1-a) and a Rotating Local System characterized by the coordinates X, Y, Z and unit base vectors $\vec{i}, \vec{j}, \vec{k}$ (Figure 4.1-b).

As shown in Figure 4.1-b, it is assumed that the origins of the two systems are located at the same point in space and that the Z_g and Z axes coincide which implies both that

$$\vec{k} = \vec{k} \quad (4.1)$$

and that the two pairs of base vectors $\vec{i}, \vec{j}$ and $\vec{i}, \vec{j}$ of Figure 4.1 both lie in (or define) the same plane.

As indicated in Figure 4.1-a, the Global Fixed Inertial System X_g, Y_g, Z_g is used for defining the wellbore geometry and the formation material properties. On the other hand, the Rotating Local X, Y, Z System of Figure 4.1-b is used for the derivation and definition of the finite element

a) The Global Fixed Inertial X_g, Y_g, Z_g Coordinate System Used for Defining the Wellbore Geometry and the Formation Material Properties

b) The Local Rotating X, Y, Z Coordinate System Used for Defining the Finite Element Model of the Bit-Bit Sub System

FIGURE 4.1: Definition of the Two Governing Right Hand Rectangular Cartesian Coordinate Systems of the Global Computational Algorithm for Modeling the PDC Bit-Bit Sub/Formation Interaction

model of the Bit-Bit Sub System (see Figures 3.8, 3.9 and 3.10). The X,Y,Z System is oriented such that the positive Z-axis is lying on the original, undeformed, centerline of the Bit-Bit Sub finite element model and is pointing upstring. It should be noted, however, that the centerline of the Bit-Bit Sub finite element model is not constrained to remain on the Z (or Z_g) axis during rotation and loading, but is free to deform (i.e. translate, rotate, bend or twist) in any way demanded by the applied dynamic load conditions.

It is assumed that the X,Y,Z local coordinate system, and its attached finite element model of the Bit-Bit Sub System, rotates with constant angular velocity ω which implies that

$$\theta = \omega t + \theta_0 \quad \text{a)} \quad (4.2)$$

$$\frac{d\omega}{dt} = \dot{\omega} = 0 \quad \text{b)}$$

In Equations 4.2 above, θ (Figure 4.1b) represents the total angle of rotation of the finite element model about the local Z axis, t represents the total elapsed time of rotation and θ_0 represents the initial rotation reference angle. In the case pictured in Figure 4.1-b, the rotation is measured from the global stationary X_g axis and θ_0 has been taken to be zero.

It is assumed that the resulting motions of the bit (during rotation and impact loading) relative to the X,Y,Z Local Rotating System are not kinematically large. Furthermore, it is assumed in Phase 1 GEODYN that the angular velocity ω is limited enough in magnitude so that Coriolis and gyroscopic forces are small and can be neglected. Also, the finite element model's nodes have mass associated with them but rotary inertia effects are neglected in the Local Rotating X,Y,Z Coordinate System.

As indicated above, the finite element model of the Bit-Bit Sub System is assumed to be derived and defined (i.e. constructed) in the Rotating Local X,Y,Z coordinate system shown in Figure 4.1-b. However, Newton's Second Law (of which Equation 2.1 is an expression) is strictly applicable only to inertial coordinate systems such as the Global X_g , Y_g , Z_g System pictured in Figure 4.1-a. By implementing the related concepts of absolute velocity and absolute acceleration (see Section 4.1 below), Equation 2.1 may be modified so that it is valid for the non-inertial Rotating X,Y,Z System. The ultimate objective is, of course, the embedding of Equation 2.1 into the Rotating System. This modification of the dynamic equations of equilibrium of the rotating finite element model of the Bit-Bit Sub System (i.e. Equation 2.1) is undertaken in Section 4.1 below.

Prior to computing the contact and Friction Force Vectors (acting on the PDC Bit-Bit Sub Model at a given time t) which appear in the modified dynamic equations of equilibrium to be developed in Section 4.1, we must first determine the position of the cutting surface of each PDC Bit Cutter in the Global X_g, Y_g, Z_g System at the same instant. The outline of the Ghost Cutter Definition and Spatial Location Algorithm implemented by GEODYN in the tracking of the Bit's Cutters is presented in Section 4.2 which follows.

Based on the deformed positions of the Ghost Cutter computed through the procedure outlined in Section 4.2, the actual computation of the Contact and Friction Force Vectors is undertaken in GEODYN by the Formation Response Algorithm described in Section 4.3. This algorithm bases the computation of these forces on the cutter's calculated penetration of the formation and the cutter's computed absolute velocity at the instant being considered.

4.1 Modification of the Bit-Bit Sub Model Dynamic Equations of Equilibrium to Accommodate the Non-Inertial X,Y,Z Rotating Coordinate System

The first step in modifying the dynamic equations of equilibrium (Equation 2.1) to accommodate the Non-Inertial X,Y,Z Rotating Coordinate System involves the expression of the absolute velocity and acceleration field vectors for the rotating Bit-Bit sub finite element model in terms of components expressed in the X,Y,Z system.

The absolute velocity and absolute acceleration vectors for an arbitrary node p, designated respectively $\vec{V}_p$ and $\vec{A}_p$, of the finite element model are in turn direct functions of two other nodal vectors. These are (see Figure 4.2) the radial vector $\vec{r}_p$ locating the displaced position of node p (at time t) relative to the origin of the rotating coordinate system and the angular velocity vector $\vec{\omega}$ (derived from Equations 4.2 and 4.3) which can be expressed

$$\vec{\omega} = \omega \vec{K} = \omega \vec{k} \quad (4.3)$$

and which, as shown by Equation 4.3, possesses the same definition in either the X_g, Y_g, Z_g or X,Y,Z System.

Examining Figure 4.2, we find that the radial vector $\vec{r}_p$ can be expressed in the form

FIGURE 4.2: A Geometric Interpretation of Radial Vectors $\vec{r}_p^0$ and $\vec{r}_p$ Expressed in Terms of Components in the Rotating Local X, Y, Z System

$$\vec{r}_p = \vec{r}_p^o + \vec{u}_p \quad (4.4)$$

where $\vec{r}_p^o$ is the radial vector locating the original (undeformed) position of node p in the finite element model and $\vec{u}_p$ is the displacement vector at node p taken at a given time t, i.e. $\vec{u}_p$ is a function of time t. In terms of the rotating X,Y,Z coordinate system, $\vec{r}_p^o$ and $\vec{u}_p$ can be defined as

$$\vec{r}_p^o = X_p \vec{i} + Y_p \vec{j} + Z_p \vec{k} \quad (4.5)$$

$$\vec{u}_p = u_p \vec{i} + v_p \vec{j} + w_p \vec{k} \quad (4.6)$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit base vectors (Figures 4.1-b and 4.2) defining the X,Y,Z axes directions, X_p , Y_p , Z_p are the X,Y,Z coordinates of node p defining its position in the undeformed configuration (Figure 4.2) and where u_p , v_p , w_p are the displacements of node p (at a given time t) in, respectively, the X,Y,Z directions (Figure 4.2). Substituting Equation 4.5 and Equation 4.6 into Equation 4.4 we obtain

$$\vec{r}_p = (X_p + u_p) \vec{i} + (Y_p + v_p) \vec{j} + (Z_p + w_p) \vec{k} \quad (4.7)$$

Since X_p , Y_p , Z_p possess constant values, $\vec{r}_p^o$ is a constant vector with respect to time. We therefore note that

$$\begin{aligned}\dot{\vec{r}}_p^0 &= 0 & \text{a)} \\ \ddot{\vec{r}}_p^0 &= 0 & \text{b)}\end{aligned}\tag{4.8}$$

where each superimposed dot indicates a separate differentiation of the quantity with respect to time. The vector $\vec{u}_p$ and its components u_p, v_p, w_p are, on the other hand, all functions of time. Equation 4.7 thus implies that $\vec{r}_p$ is also a time dependent quantity. In view of Equations 4.4, 4.6 and 4.8, it is clear then that

$$\begin{aligned}\dot{\vec{r}}_p &= \dot{\vec{u}}_p \\ &= \dot{u}_p \vec{i} + \dot{v}_p \vec{j} + \dot{w}_p \vec{k} & \text{a)} \\ \ddot{\vec{r}}_p &= \ddot{\vec{u}}_p & \text{(4.9)} \\ &= \ddot{u}_p \vec{i} + \ddot{v}_p \vec{j} + \ddot{w}_p \vec{k} & \text{b)}\end{aligned}$$

where $\dot{\vec{u}}_p$ and $\ddot{\vec{u}}_p$ are, respectively, the velocity and acceleration vectors of node p relative to the Local Rotating Coordinate System X,Y,Z.

The absolute velocity $\vec{V}_p$ of an arbitrary node p of the Bit-Bit Sub Finite Element Model is given by

$$\vec{V}_p = \dot{\vec{r}}_p + (\vec{\omega} \times \vec{r}_p)\tag{4.10}$$

Implementing Equations 4.3 and 4.7, we obtain

$$\vec{\omega} \times \vec{r}_p = -\omega(Y_p + v_p)\vec{i} + \omega(X_p + u_p)\vec{j} \quad (4.11)$$

Substituting Equations 4.9-a and 4.11 into Equation 4.10, we can re-express the absolute velocity $\vec{V}_p$ in the form

$$\vec{V}_p = v_{xp}\vec{i} + v_{yp}\vec{j} + v_{zp}\vec{k} \quad (4.12)$$

where

$$v_{xp} = \dot{u}_p - \omega(Y_p + v_p) \quad a)$$

$$v_{yp} = \dot{v}_p + \omega(X_p + u_p) \quad b) \quad (4.13)$$

$$v_{zp} = \dot{w}_p \quad c)$$

In matrix form we can define $\{V_p\}$ as

$$\{V_p\} = \begin{Bmatrix} v_{xp} \\ v_{yp} \\ v_{zp} \end{Bmatrix} \quad (4.14)$$

Using the matrix form of the velocity vector provided in Equation 4.14, the velocity vector $\{V\}$ of the assembled finite element model of the Bit-Bit Sub system is given by

$$\{V\}^T = [V_1 \mid V_2 \mid \dots \mid V_p \mid \dots \mid V_N] \quad (4.15)$$

where N represents the total number of nodes in the element mesh.

The absolute acceleration $\vec{A}_p$ of an arbitrary node p is now given by

$$\vec{A}_p = \ddot{\vec{r}}_p + (\dot{\vec{\omega}} \times \vec{r}_p) + (\vec{\omega} \times (\vec{\omega} \times \vec{r}_p)) + (2\vec{\omega} \times \dot{\vec{r}}_p) \quad (4.16)$$

where

$$\dot{\vec{\omega}} = \text{Angular Acceleration}$$

$$\dot{\vec{\omega}} \times \vec{r}_p = \text{Linear Acceleration}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_p) = \text{Centripetal Acceleration}$$

$$2\vec{\omega} \times \dot{\vec{r}}_p = \text{Coriolis Acceleration}$$

In view of Equation 4.2-b and the assumption that the angular velocity ω is limited enough in magnitude so that Coriolis and gyroscopic forces are small and can be neglected; we have

$$\dot{\vec{\omega}} \times \vec{r}_p = 0 \quad (4.17)$$

and

$$2\vec{\omega} \times \dot{\vec{r}}_p \approx 0 \quad (4.18)$$

Substituting Equations 4.9-b, 4.17 and 4.18 into Equation 4.16, we find that the expression for the absolute acceleration at node p reduces to

$$\vec{A}_p = \ddot{\vec{u}}_p + (\vec{\omega} \times (\vec{\omega} \times \vec{r}_p)) \quad (4.19)$$

which leaves only the acceleration vector $\ddot{\vec{u}}_p$, relative to the local rotating system X,Y,Z and the Centripetal Acceleration term to affect the magnitude of the absolute acceleration $\vec{A}_p$. The substitution of Equations 4.3, 4.9-b and 4.11 into Equation 4.19 yields

$$\vec{A}_p = A_{xp}\vec{i} + A_{yp}\vec{j} + A_{zp}\vec{k} \quad (4.20)$$

where

$$A_{xp} = \ddot{u}_p - \omega^2 (X_p + u_p) \quad a)$$

$$A_{yp} = \ddot{v}_p - \omega^2 (Y_p + v_p) \quad b) \quad (4.21)$$

$$A_{zp} = \ddot{w}_p \quad c)$$

In matrix form we can define $\{A_p\}$ as

$$\{A_p\} = \{\ddot{u}_p\} + \{\omega x \omega x r_p\} \quad (4.22)$$

where

$$\{\ddot{u}_p\} = \begin{Bmatrix} \ddot{u}_p \\ \ddot{v}_p \\ \ddot{w}_p \end{Bmatrix} \quad \text{a)} \quad (4.23)$$

$$\{\omega x \omega x r_p\} = \begin{Bmatrix} -\omega^2 (X_p + u_p) \\ -\omega^2 (Y_p + v_p) \\ 0 \end{Bmatrix} \quad \text{b)}$$

Using the matrix form of the acceleration vector provided in Equations 4.22 and 4.23, the absolute acceleration vector $\{A\}$ of the Bit-Bit Sub System can be expressed by

$$\{A\}^T = [A_1 \quad A_2 \quad \dots \quad A_p \quad \dots \quad A_N] \quad (4.24)$$

where

$$\{A\} = \{\ddot{u}\} + \{\omega x \omega x r\} \quad (4.25)$$

and where

$$\begin{aligned} \{\ddot{u}\}^T &= [\ddot{u}_1 \mid \ddot{u}_2 \mid \ddot{u}_p \mid \ddot{u}_N] & \text{a)} \\ & & (4.26) \\ \{\omega x \omega x r\}^T &= [\omega x \omega x r_1 \mid \omega x \omega x r_p \mid \omega x \omega x r_N] & \text{b)} \end{aligned}$$

Implementing the absolute acceleration vector $\{A\}$ provided by Equations 4.24 through 4.26, the dynamic equations of equilibrium (i.e. Equation 2.1) of the assembled finite element model of the Bit-Bit Sub System can be re-expressed in the modified form given in Equation 4.27 below in order to accommodate the fact that the finite element model is defined in terms of the Non-Inertial Rotating X,Y,Z Coordinate System.

$$[M] \{A\} + [C] \{\dot{u}\} + [K] \{u\} = \{F_e\} + \{F_f\} + \{F_\mu\} \quad (4.27)$$

In Equation 4.27 above the Structure Stiffness Matrix $[K]$, Mass Matrix $[M]$ (and ultimately the Damping Matrix $[C]$) are all defined in terms of the Rotating X,Y,Z Coordinate System (Figure 4.1-b) as are the External Force Vector $\{F_e\}$, the bit/formation contact force vector $\{F_f\}$ and the bit/formation friction force vector $\{F_\mu\}$. The external force vector $\{F_e\}$ can be decomposed into

$$\{F_e\} = \{F_x\} - [M] \{g\} \quad (4.28)$$

The first term on the right hand side of Equation 4.28, i.e. $\{F_x\}$, represents the application of concentrated (or direct) loads to arbitrary nodes of the finite element mesh. These include prescribed axial loads, torques and bending moments applied at the top of the Bit Sub section. In the context of the GEODYN System, these forces may be independently specified and can be varied as a function of time. Allowing the vector $\{g\}$ to represent the acceleration due to gravity, the product $[M] \{g\}$ constituting the second term on the right hand side of Equation 4.28 is a body force vector representing the weight of the assembly.

As previously indicated in Chapter II, the contact force vector $\{F_f\}$ is generated by contact and penetration of the bit and foundation. The force is a function of the displacement vector $\{u\}$ (whose components are transformed, in GEODYN, into the Inertial Global X_g, Y_g, Z_g Coordinate System of Figure 4.1-a) and thus results in a nonlinear relationship. When there is no contact at a bit node degree of freedom, the contact force vector $\{F_f\}$ is zero. When contact occurs, the force is a function of the relative penetration of the foundation. The friction force $\{F_u\}$ is also generated due to contact between the bit and foundation.

The force is, in general, a function of both the displacement vector $\{u\}$ and the Absolute Velocity Vector $\{V\}$ and thus represents a nonlinear relationship in the Bit-Bit Sub Equation which is determined from the bit-formation interaction.

Substituting Equations 4.25 and 4.28 into Equation 4.27 and re-arranging terms, we can re-express the dynamic equations of equilibrium in the form

$$\begin{aligned} [M] \{\ddot{u}\} + [C] \{\dot{u}\} + [K] \{u\} \\ = \{F_x\} + \{F_f\} + \{F_u\} - \{F_{BD}\} \end{aligned} \quad (4.29)$$

where $\{F_{BD}\}$ is a body force vector given by

$$\{F_{BD}\} = [M] \{\omega_x \omega_{xr}\} + [M] \{g\} \quad (4.30)$$

and where the first term on the right hand side of Equation 4.30 represents the body force contribution arising from centripetal force effects while the second term represents the above described weight vector for the finite element model.

4.2 Ghost Cutter Definition and Spatial Location Algorithm

Prior to computing the contact $\{F_f\}$ and friction $\{F_\mu\}$ force vectors (Equation 4.29) acting on the PDC Bit-Bit Sub Model at a given time t , we must first determine the position of the cutting surface of each PDC Bit cutter in the Global X_g, Y_g, Z_g System (Figure 4.1-b) at the same instant. In terms of the Global GEODYN Program Algorithm previously flow charted in Figure 2.1, this tracking of the Bit's cutters is monitored on the iterative (inner loop) rather than incremental (outer loop) level (see discussion of Section 2.1, Chapter II).

In the GEODYN System, each Ghost Cutter is dealt with as a rigid body element (which we will term a Geometric Sub Element) characterized by three unique Geometric Sub Nodes (Figure 4.3). The Geometric Sub Element, or Ghost Cutter, is then visualized to be attached, in a manner to be described below, to a particular set of three nodes on the face of the PDC Bit-Bit Sub Finite Element Model. In tracking the positions of the cutters on the bit face, the GEODYN Program Algorithm causes the motions of the Ghost Cutter to be prescribed by the motions of the Bit-Bit Sub Model nodes to which they are attached. Ultimately, it is the global location (or position) of the Sub Nodes at a given time (i.e. iteration within a time step increment) which becomes the basis for determining the forces $\{F_f\}$ and $\{F_\mu\}$ which the formation exerts on the bit.

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FIGURE 4.3: The Locations of the Three Subnodes on Each Ghost-Cutter

In order to enforce Ghost Cutter motions which are consistent with the bit surface deformations, it is not sufficient to simply tie a geometric subnode to a single bit node. The reason is that, since the nodes of a solid element do not have rotational degrees of freedom, the displacement vector of a single node contains no information regarding the rotation of its element surface. It is only in comparing a node's displacement to the displacements of its neighboring nodes, that the orientation of an element surface may be evaluated.

The movements of the Ghost Cutter sub nodes are therefore tied to the translations of a nearby bit node (called the "primary node"), with the qualification that the Ghost Cutter must also move according to the rotation of a plane defined by the primary node and two other nearby bit nodes. This plane is termed, herein, the "characteristic plane".

The above mentioned characteristic plane can be easily defined in terms of the Local r,s,t Ghost Cutter Coordinate System pictured in Figure 4.4. Examining Figure 4.4 we find that the r,s,t coordinate system is defined in terms of the orthonormal triad of right hand unit base vectors designated by $\vec{I}_r, \vec{I}_s, \vec{I}_t$ with the unit vectors $\vec{I}_r$ and $\vec{I}_s$ defining the orientation of the characteristic plane in space and the unit vector $\vec{I}_t$ defining the normal to the plane. The labels N1, N2 and N3 (Figure 4.4) represent the numbers of three adjacent nodes lying on the face of the three dimensional bit model to which the Ghost Cutter is attached (or related).

FIGURE 4.4: Ghost Cutter Local Coordinate System

Nodes N1, N2 and N3 lie in and, indeed, are used to define the characteristic plane to which the cutter's motions are related. It should be noted that, in the algorithm incorporated into GEODYN, the nodes N1, N2 and N3 do not have to belong to the same element.

Referring to Figure 4.4, we find that the distance d_1 is measured along side N1-N2 in the local r-direction, the distance d_2 is measured in the local s-direction (in the r-s plane) and is taken perpendicular to the side N1-N2 and the distance d_3 is the perpendicular distance from the characteristic (i.e. N1, N2, N3) plane to the geometric sub node (measured in the t direction). The distances, or lengths, d_1 , d_2 and d_3 may be physically interpreted as "rigid links" which do not change length or relative angles. Except in the case of cutter break off, the values of d_1 , d_2 and d_3 never change for a given geometric sub node. The unit base vectors $\vec{i}_r$, $\vec{i}_s$ and $\vec{i}_t$ can be defined mathematically utilizing the following simple process.

Reaching back to Equation 4.7 for the definition

$$\begin{aligned} \vec{r}_{Ni} &= (X_{Ni} + u_{Ni})\vec{i} + (Y_{Ni} + v_{Ni})\vec{j} + (Z_{ni} + w_{Ni})\vec{k} \\ i &= 1, 2, 3 \end{aligned} \tag{4.31}$$

We construct the unit vector $\vec{I}_r$ from

$$\vec{I}_r = \frac{(\vec{r}_{N2} - \vec{r}_{N1})}{|\vec{r}_{N2} - \vec{r}_{N1}|} \quad (4.32)$$

Where the symbol $|\vec{C}|$ indicates the magnitude of the included vector $\vec{C}$ and where the definition of Equation 4.32 orients $\vec{I}_r$ along a vector pointing from Node N1 to Node N2 (Figure 4.4). Defining $\vec{V}_{13}$ as a vector pointing from Node N1 to Node N3

$$\vec{V}_{13} = \vec{r}_{N3} - \vec{r}_{N1} \quad (4.33)$$

the unit vector $\vec{I}_t$ normal to the characteristic r-s plane can be constructed from the cross (vector) product of $\vec{I}_r$ and $\vec{V}_{13}$ in the following manner.

$$\vec{I}_t = \frac{\vec{I}_r \times \vec{V}_{13}}{|\vec{I}_r \times \vec{V}_{13}|} \quad (4.34)$$

The remaining vector $\vec{I}_s$, which together with $\vec{I}_r$ is used to define the orientation of the r-s plane, is derived from the vector product taken between $\vec{I}_t$ and $\vec{I}_r$ as shown below.

$$\vec{I}_s = \vec{I}_t \times \vec{I}_r \quad (4.35)$$

Substituting Equation 4.31 into Equations 4.32 through 4.35, we can express the base vectors $\vec{i}_r, \vec{i}_s, \vec{i}_t$ of the Local r,s,t Ghost Cutter Coordinate System (Figure 4.4) in terms of the base vectors $\vec{i}, \vec{j}, \vec{k}$ of the Rotating Non-Inertial X,Y,Z Coordinate System (Figure 4.1-b) as follows

$$\begin{aligned}\vec{i}_r &= T_{11} \vec{i} + T_{12} \vec{j} + T_{13} \vec{k} \\ \vec{i}_s &= T_{21} \vec{i} + T_{22} \vec{j} + T_{23} \vec{k} \\ \vec{i}_t &= T_{31} \vec{i} + T_{32} \vec{j} + T_{33} \vec{k}\end{aligned}\tag{4.36}$$

In view of the definition of $\vec{r}_{N1}$ provided by Equation 4.31 above, it is clear that the coefficients T_{ij} of the above relations are functions of the displacements of bit nodes N1, N2, N3; i.e. (see also Equation 4.6)

$$T_{ij} = T_{ij} (\vec{u}_{N1}, \vec{u}_{N2}, \vec{u}_{N3})\tag{4.37}$$

The transformation of Equation 4.36 can be re-constructed in matrix form where

and where

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad (4.39)$$

Defining the vector $\vec{D}$ to be the position vector locating a Ghost Cutter Geometric Sub Node with respect to the primary bit Node N1 and defining the vector $\vec{R}$ to be the Position Vector locating the same sub node with respect to the origin of the Rotating X,Y,Z Coordinate System, we can relate $\vec{R}$ and $\vec{D}$ through the expression (see Figure 4.5)

$$\vec{R} = \vec{D} + \vec{r}_{N1} \quad (4.40)$$

where $\vec{r}_{N1}$ (see Equation 4.31) is the position vector locating the primary Node N1 with respect to the origin of the Rotating X,Y,Z Coordinate System. The vector $\vec{D}$ can be expressed as (see Figure 4.5)

$$\vec{D} = d_1 \vec{i}_r + d_2 \vec{i}_s + d_3 \vec{i}_t \quad (4.41)$$

in terms of the Local r,s,t Coordinate System and can be expressed in the alternative form

$$\vec{D} = d_x \vec{i} + d_y \vec{j} + d_z \vec{k} \quad (4.42)$$

FIGURE 4.5: Ghost Cutter Spatial Description

to accommodate the Rotating X,Y,Z System. In matrix form, we can express the transformation between the two sets of components expressed in Equations 4.41 and 4.42 by (see also Equation 4.39)

$$\begin{Bmatrix} d_x \\ d_y \\ d_z \end{Bmatrix} = [T]^T \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \quad (4.43)$$

Observing that the vector $\vec{D}$ possesses a constant magnitude, we must have

$$\sqrt{d_1^2 + d_2^2 + d_3^2} = \sqrt{d_x^2 + d_y^2 + d_z^2} \quad (4.44)$$

In view of Equations 4.31, 4.42 and 4.43, we find that the expression for $\vec{R}$ resulting from Equation 4.40 provides us the means of computing the position, at a given time t , of a Ghost Cutter Geometric Sub Node in terms of coordinates expressed in the Rotating X,Y,Z Coordinate System. As previously indicated, prior to computing the contact $\{F_f\}$ and friction $\{F_\mu\}$ force vectors acting on the Bit-Bit Sub System at a given time t , we must first determine the position of the cutting surface (i.e. a Ghost Cutter Geometric Sub Node) in the Global Fixed X_g, Y_g, Z_g System of Figure 4.1-a. As indicated above, all calculations so far have led to the

deformed positions of the sub nodes in the Rotating X,Y,Z System. In order to interact with the formation, we must transform the X,Y,Z coordinates locating a given sub node in the Rotating Coordinate System (calculated from Equation 4.40) to their corresponding position in the Global X_g, Y_g, Z_g System.

Since the Rotating Coordinate System has a constant angular velocity ω with respect to the Global Fixed Coordinate System (see Figures 4.1-a, b) and since the Local Z and Global Z_g axes coincide, the transformation from the X,Y,Z coordinates in the Rotating System to the X_g, Y_g, Z_g coordinates in the Fixed Inertial System is simply

$$\begin{Bmatrix} X_g \\ Y_g \\ Z_g \end{Bmatrix} = [T_r] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad (4.45)$$

where

$$[T_r] = \begin{bmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.46)$$

and where the t in Equation 4.46 indicates the elapsed time of rotation.

As related in the introductory discussion to this section, the tracking of the positions of the PDC Bit's Cutters (i.e. Ghost Cutter Geometric Sub Nodes) is monitored on the iterative rather than the incremental level. Given the assumed displacement state at the end of a particular iteration within a time increment (or step), it is a simple matter to compute the Global Position of each Geometric Sub Node being tracked via the implementation of the relationships expressed by Equations 4.31 through 4.45. The global positions of the Geometric Sub Nodes are now available to compute the contact $\{F_f\}$ and Friction $\{F_\mu\}$ force vectors for the PDC Bit-Bit Sub Finite Element Model.

4.3 The Formation Response Algorithm and the Computation of the Contact and Friction Force Vectors

Based on the deformed positions of the Ghost Cutters computed through the procedure outlined in Section 4.2, the Formation Response Algorithm is first concerned with determining how far (if at all) each cutter has penetrated the formation. Based on the cutter's penetration and absolute velocity (i.e. in the X_g, Y_g, Z_g System), and the properties of the formation at the point of contact, the forces being exerted on the cutter are calculated. As indicated in Equations 2.1 and 4.29, two types of forces are computed. The first is a penetration force which is a function of the penetration distance and the second is a friction force which depends on the magnitude of the penetration force and the absolute velocity of the cutter.

4.3.1 Idealization of the Formation

A collection of discrete points (called "formation points") is used to model the formation surfaces with which the bit may come in contact. Formation properties are assigned to each of these points. Once it is determined that a cutter is closest to a specific formation point, all geometric and physical properties of the formation at the true point of contact are said to be equal to those at that formation point.

The formation model has three separate portions (or sub-models) for which the direction of penetration varies: There is the "Bottom" surface, the "Side-wall" surface, and the "Incline" surface (see Figure 4.6). The direction of penetration for the bottom surface is always the negative Z direction (regardless of whether the bottom is flat), and the penetration directions for the side-wall and incline surfaces are always perpendicular to the formation surface in question.

At every formation point, a coordinate is stored which is used to determine the penetration distance. This is termed the "penetration coordinate". For example, on the bottom surface a point's penetration coordinate is its Z coordinate.

The following formation properties are identified for every formation point:

k_{pen} = A linear stiffness (analogous to a spring) resisting penetration. It is used to calculate the penetration force.

μ_o = The maximum attainable coefficient of friction in the rock. The actual friction coefficient varies with the cutter velocity, and will approach μ_o as the velocity approaches V_{max} .

Hole Centerline

→ = Penetration Direction

FIGURE 4.6: The Three Surface Portions of the Formation Model

$V_{\max}$ = The velocity at which the rock's coefficient of friction is equal to μ_0 .

The actual friction coefficient μ is computed from the relationship

$$\mu = \mu_0 \tanh \left(\frac{|V|}{V_{\max}} \right) \quad (4.47)$$

The friction coefficient value μ provided above is utilized by the GEODYN Program Formation Response Algorithm in the computation of the velocity dependent Friction Force Vector $\{F_\mu\}$ of Equation 4.29.

4.3.2 The General Algorithm Implemented in the Computation of the Contact and Friction Force Vectors

The general algorithm characterizing the formation's response to an impacting cutter can be broken into the following four steps:

1. The deformed (i.e. displaced) position of the ghost cutter is retrieved via the Spatial Location Algorithm outlined in Section 4.2.

2. The Formation Response Algorithm determines whether the cutter is closest to contacting the Bottom, Side-wall or Incline (Figure 4.6) portion of the Formation Surface.
3. Based on the result of Step 2, the algorithm calls the appropriate routine to:
 - a. Find the number (i.e. location) of the formation point closest to the ghost cutter.
 - b. Retrieve the penetration coordinate and formation properties (k_{pen} , μ_o , V_{max}) at that specific formation point.
 - c. Determine the amount of Penetration (i.e. the penetration Distance D_{pen}) which has occurred.
 - d. Calculate the forces on the ghost cutter.
4. The force vector (determined in Step 3-d above) for the ghost cutter is stored in a Cutter Force Array by the GEODYN program for use later by the Cutter Failure and Nodal Load Interpolation Sub-Modules (i.e. Sub-Algorithms) of the GEODYN System Geometry Interpreter. Cutter Forces are stored in the Rotating X,Y,Z System (Figure 4.1-b).

Step 3-a above implies that we must "search" for the number of the formation point (since the formation points are stored by number in GEODYN) closest to the cutter at the time step (i.e. iteration) being considered. In the GEODYN Program Algorithm, this "searching" process actually reduces to a direct calculation of the sought after formation point number by first computing the position where the closest point must fall and then implementing a simple algorithm to secure the actual number of the formation point in that location. This convenient application of the searching process is a result of the arrangement of the formation points in a way which simplifies the retrieval of the point numbers. There are two important aspects of this approach to understand.

First, uniform grid patterns are employed for each of the three portions (see Figure 4.6) of the formation model. The formation points are arranged so that a simple two-dimensional "search" may be performed, always looking normal to the formation surface in question. Because of the different orientations of the three formation sub-models, the two coordinate directions searched are different on each.

Second, the formation point numbers for each of the formation sub-models are stored in arrays such that a point's row and column location is directly analogous to its position on the formation surface.

With regard to Steps 3-b,c above, the definition of the penetration coordinate and the computation of the penetration distance D_{pen} is a special case for each of the three portions of the Formation Surface pictured in Figure 4.6.

Bottom Formation Surface:

For all formation points on the bottom surface, "Z" is the penetration coordinate (see Figure 4.7). This means that once the Z coordinates of the formation point and the cutter are known, the penetration distance may be found from

$$D_{pen} = Z_f - Z_c \quad (4.48)$$

where Z_f is the Z coordinate of the formation point and Z_c is the Z coordinate of the ghost cutter (remember that penetration will occur in the negative Z direction). The Z coordinates of the bottom formation points are stored along with the formation properties.

Side-Wall Formation Surface:

For all points on the side-wall surface, "R" is the penetration coordinate (see Figure 4.8). This means that once the R coordinate of the formation point is known (stored

FIGURE 4.7: Penetration Distance on the Bottom Surface of the Formation

FIGURE 4.8: Penetration Distance on the Side Wall Surface of the Formation

with the formation properties), the penetration distance may be found from

$$D_{\text{pen}} = R_c - R_f \quad (4.49)$$

where R_c is the R coordinate of the ghost cutter and where R_f is the R coordinate of the Formation Point.

Incline Formation Surface:

For the incline surface, the penetration coordinate is called "P". P is a function of R and Z, and is a measure of distance perpendicular to the incline (see Figure 4.9). The P coordinate of the cutter is calculated from

$$P_c = [(R_c - R_1)\tan\phi - (Z_c - Z_o)] \cos\phi \quad (4.50)$$

where the P coordinates of the Incline formation points (P_f) are stored along with the formation properties. The penetration distance is then calculated from

$$D_{\text{pen}} = P_c - P_f \quad (4.51)$$

FIGURE 4.9: Penetration Distance on the Incline Surface of the Formation

Calculation of the Cutter Forces:

Once the penetration distance D_{pen} is calculated, and the formation properties (K_{pen} , μ_o , and V_{max}) are known, the following steps are performed:

- A. The cutter's velocity in the rotating X,Y,Z Coordinate System is obtained from the absolute velocity vector (Equation 4.15). (The cutter velocity is set equal to the velocity of the bit node closest to the cutter.)
- B. The penetration force F_{pen} is calculated from

$$F_{pen} = K_{pen} \times D_{pen} \quad (4.52)$$

where F_{pen} is the magnitude of the penetration force. At this point, its direction has not yet been specified.

- C. The magnitude of the velocity dependent friction force (F_{mu}) is then computed from

$$F_{mu} = F_{pen} \times \mu_o \tanh \left(\frac{|V|}{V_{max}} \right) \quad (4.53)$$

D. The components of the friction force on the rotating X,Y,Z System $F_{fr}^{(i)}$ are then found by multiplying F_{mu} by a unit vector oriented in the direction of the cutter velocity, i.e.

$$F_{fr}^{(i)} = F_{mu} \times \left(\frac{V^{(i)}}{|V|} \right) \quad (4.54)$$

For the bottom surface, the penetration force is applied in the Z direction. For the incline and sidewall surfaces, the global components of the penetration force must be calculated based on the θ (Figure 4.1) coordinate of the cutter (and the angle ϕ (Figure 4.9) if on the incline). The global components of the penetration force must then be transformed into the rotating system before they may be combined with the friction forces.

As implied in Step 4 above of the General Formation Response Algorithm, a nodal load interpolation capability has been implemented in the GEODYN System to re-distribute the formation forces acting on the cutter (which are calculated above) back to the appropriate nodes of the finite element model of the PDC Bit. In some ways, this process reverses the effects of the Ghost Cutter Spatial Tracking Algorithm outlined in Section 4.2 which takes PDC Bit Model nodal displacements and converts them to Ghost Cutter Motions.

V. TRANSFORMATION OF THE DYNAMIC EQUATIONS OF EQUILIBRIUM
OF THE BIT-BIT SUB SYSTEM TO A MODAL BASIS

In Section 4.1, the dynamic equations of equilibrium of the Bit-Bit Sub System were modified to accommodate their description in the Non-Inertial X,Y,Z Rotating Coordinate System of Figure 4.1-b. The modified equations of motion, as defined previously in Equations 4.29 and 4.30, are given by

$$\begin{aligned} [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} \\ = \{F_x\} + \{F_f\} + \{F_\mu\} - \{F_{BD}\} \end{aligned} \quad (5.1)$$

where $\{F_{BD}\}$ is a body force vector given by

$$\{F_{BD}\} = [M] \{\omega \times \omega \times r\} + [M] \{g\} \quad (5.2)$$

and where the first term on the right hand side of Equation 5.2 represents the body force contribution arising from centrifugal force effects while the second term represents the previously described weight vector for the finite element model (see Section 4.1 discussion).

In Equation 5.1, the matrices $[M]$, $[C]$ and $[K]$ are, respectively, the Assembled Mass, Damping and Stiffness Matrices for the finite element model while the vectors $\{F_x\}$, $\{F_f\}$ and $\{F_\mu\}$ are the model's concentrated Load, Contact Force and Friction Force Vectors.

In order to reduce the number of degrees of freedom of the dynamic finite element model being integrated, we follow a line of thinking established in Section 2.2.2.3 and turn now toward implementing the Method of Mode Superposition to provide a change of basis of our System Degrees of Freedom $\{u\}$ to Modal Generalized Displacements $\{\eta\}$. Specifically, we transform the equilibrium equations (Equation 5.1) into a more effective (i.e. economical) form for direct integration by using a transformation of the form

$$\{u\} = [\Phi]\{\eta\} \quad (5.3)$$

where

$$[\Phi] = [\phi_1 | \phi_2 | \dots | \phi_{\bar{N}}] \quad (5.4)$$

and where $\bar{N}$ indicates the total number of system degrees of freedom in $\{u\}$. In Equations 5.3 and 5.4 above,

- $\{u\}$ = Vector of Original System Degrees of Freedom
- $\{\eta\}$ = Vector of Generalized Degrees of Freedom to which we are Transforming
- $[\Phi]$ = Matrix of Eigenvectors $\{\phi_i\}$
- $\{\phi_i\}$ = Eigenvector Corresponding to Degree of Freedom Number i of $\{u\}$, i.e. u_i of the Assembled System Model

The application of the above indicated Mode Superposition process to the numerical time integration of the Bit-Bit Sub Equations of Motion is undertaken in the discussion which follows.

5.1 Definition of the Standard Eigenproblem

Following Bathe in Reference 5.1, we establish the Standard Eigenproblem to be given by

$$[K] - \lambda[I] \{\phi\} = \{0\} \quad (5.5)$$

where $[K]$ is the Stiffness Matrix of the Bit-Bit Sub Model, $[I]$ is a diagonal unit matrix, $\{0\}$ is a null vector, $\{\phi\}$ is an arbitrary eigenvector of the system and λ is the corresponding eigenvalue. The Stiffness Matrix $[K]$ is of order $(\bar{N} \times \bar{N})$ and is also assumed to be positive semi-definite or positive definite. There are $\bar{N}$ eigenvalues and corresponding eigenvectors satisfying Equation 5.5 with the i^{th} eigenpair denoted by $(\lambda_i, \{\phi_i\})$ and where the eigenvalues are ordered according to their magnitudes (Reference 5.1):

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots \leq \lambda_{\bar{N}} \quad (5.6)$$

5.2 Definition of the Vibration Mode Eigenproblem

In free vibration of a structure, the only forces acting on the mass of the system are those imposed by the internal elastic and damping forces generated during motion of the structure as it vibrates. Thus, Equation 5.1 can be reduced to

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{0\} \quad (5.7)$$

in order to represent this free vibration condition for the Bit-Bit Sub Model. Assuming that the damping influence on the mode shapes of the vibrating structure is negligible, we can further reduce Equation 5.7 to obtain

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \quad (5.8)$$

Equations 5.8 form a set of linear second-order differential equations whose solution is of the form

$$\{u\} = \{\phi\} \sin \Omega (t - t_0) \quad a) \quad (5.9)$$

$$\{\ddot{u}\} = -\Omega^2 \{\phi\} \sin \Omega (t - t_0) \quad b)$$

where Ω is the angular natural frequency of vibration of the structural system in any one of its natural modes. Substituting Equation 5.9 into 5.8, we obtain

$$[K] - \lambda[M] \{\phi\} = \{0\} \quad (5.10)$$

where

$$\lambda = \Omega^2 \quad (5.11)$$

Equation 5.10 above defines the basic eigenproblem which is to be solved in vibration mode superposition analysis. In the $\bar{N}$ eigenpairs $(\lambda_i, \{\phi_i\})$ representing the solutions to Equation 5.10, the eigenvalues λ_i are Ω_i^2 (Equation 5.11), the free vibration frequencies squared, and the eigenvectors $\{\phi_i\}$ are the corresponding mode shape vectors. We should note that Equation 5.6 implies that

$$0 \leq \Omega_1^2 \leq \dots \leq \Omega_i^2 \leq \dots \leq \Omega_{\bar{N}}^2 \quad (5.12)$$

The properties of $[K]$ in Equation 5.10 are the same as those discussed in Section 5.1 above. If the element mass matrices are consistent matrices (see Equations 3.27 and 3.35), the structure mass matrix $[M]$ will be banded and will always be positive definite in nature. If the element mass matrices

are lumped mass matrices (see Equations 3.29 and 3.37), the assembled mass matrix $[M]$ is positive definite only if all its diagonal elements are greater than zero (which is the case for the Bit-Bit Sub Models generated by GEODYN). In other instances, however, Lumped Mass Matrices can possess a positive semi-definite nature.

The eigenvalues λ_i ($i = 1, \bar{N}$) of Equation 5.10 are all real and positive with the restrictions that

- o some λ_i 's may be approximately equal to zero indicating rigid body modes of deformation and,
- o some λ_i 's may be repeated due to symmetry conditions.

With respect to the eigenvector $\{\lambda_i\}$ ($i = 1, \bar{N}$), we observe that

- o if the λ_i 's are distinct, then $\{\phi_1\} \cdots \{\phi_{\bar{N}}\}$ are each orthogonal to each other directly from the basic theorems of Linear Algebra and,
- o if the λ_i 's are repeated, then $\{\phi_1\} \cdots \{\phi_{\bar{N}}\}$ are not necessarily orthogonal in nature but special procedures can be involved to enforce such orthogonality in Finite Element Analysis.

We should also note that if the eigenvectors $\{\phi_1\} \dots \{\phi_N\}$ represent an orthogonal system, then we can show that

$$\begin{aligned}
 \{\phi_i\}^T [M] \{\phi_j\} &\neq 0 && \text{for } i=j \\
 &= 0 && \text{for } i \neq j
 \end{aligned}
 \tag{a}$$

$$\begin{aligned}
 \{\phi_i\}^T [K] \{\phi_j\} &\neq 0 && \text{for } i=j \\
 &= 0 && \text{for } i \neq j
 \end{aligned}
 \tag{b}$$

(5.13)

In view of Equation 5.13, we find that the matrix products $[\phi]^T [M] [\phi]$ and $[\phi]^T [K] [\phi]$, see Equation 5.3, possess diagonal forms. Implementing Equation 5.10 to write,

$$\left[[K] - \lambda_i [M] \right] \{\phi_i\} = \{0\}
 \tag{5.14}$$

we can pre-multiply both sides of Equation 5.14 by $\{\phi_i\}^T$ and solve the resulting expression for λ_i to obtain

$$\lambda_i = \frac{\{\phi_i\}^T [K] \{\phi_i\}}{\{\phi_i\}^T [M] \{\phi_i\}}
 \tag{5.15}$$

by which expression we obtain a direct relationship between a given eigenvalue λ_i and its eigenvector $\{\phi_i\}$.

5.3 Transformation of the Generalized Eigenproblem of Equation 5.10 to the Standard Form of Equation 5.5

In the brief discussion which follows, we undertake the transformation of the Generalized Eigenproblem of Equation 5.10 to the Standard Form of Equation 5.5. The reasons for this transformation are twofold. First, because the transformation is possible, use can be made of the various solution algorithms available for standard eigenproblems. For example, the EISPACK System of Programs (developed by B.T. Smith et al, Reference 5.2, at Argonne National Laboratory) which can be used in the computation of the eigenvectors and eigenvalues required by GEODYN to complete a dynamic analysis of the Bit-Bit Sub Assembly.

Second (Reference 5.1), by transforming the Vibration Mode Eigenproblem of Equation 5.10 to the Standard Eigenproblem of Equation 5.5, the properties of the eigenvalues and eigenvectors of the Vibration Mode Eigenproblem can be deduced from the properties of the corresponding quantities of the Standard Eigenproblem.

Assuming that the Structure Mass Matrix $[M]$ is positive definite (i.e. GEODYN always yields this result), we can initiate the transformation of Equation 5.10 by employing a decomposition of $[M]$ of the form

$$[M] = [L][L]^T \quad (5.16)$$

where $[L]$ is any non-singular matrix.

The two most popular methods of achieving the decomposition of Equation 5.16 are through Cholesky Factorization or Spectral Decomposition. Though arguments can be put forth (Bathe, Reference 5.1) to support the latter approach as being slightly more accurate, the former technique (i.e. Cholesky Factorization) has been incorporated into GEODYN because it is generally recognized as being the most computationally efficient of the two methods.

In the case where $[M]$ is a Lumped Mass Matrix (see Sections 3.1.2 and 3.2.2), i.e. $[M]$ can be expressed in the form

$$[M] = \begin{bmatrix} m_{11} & & & \\ & \ddots & & \\ & & m_{ii} & \\ & & & \ddots \\ & & & & m_{\overline{NN}} \end{bmatrix} \quad (5.17)$$

where M_{ii} is the diagonal lumped mass value associated with degree of freedom i , the computation of $[L]$ is a trivial process and reduces to

$$[L] = \begin{bmatrix} \sqrt{m_{11}} & & & \\ & \ddots & & \\ & & \sqrt{m_{ii}} & \\ & & & \ddots \\ & & & & \sqrt{m_{NN}} \end{bmatrix} \quad (5.18)$$

In the interest of economy of effort, the GEODYN Program Algorithm bypasses the Cholesky Factorization Routines and computes the $[L]$ matrix of Equation 5.16 directly from Equation 5.18 for this special case.

We now define $\{\bar{\phi}\}$ as an alternative eigenvector (or mode shape vector) resulting from the transformation

$$\{\bar{\phi}\} = [L]^T \{\phi\} \quad (5.19)$$

which we can invert to obtain

$$\{\phi\} = [L]^{-1} {}^T \{\bar{\phi}\} \quad (5.20)$$

The substitution of Equations 5.16 and 5.20 into Equation 5.10 can be made to yield

$$\left[[K][L]^{-T} - \lambda [L] \right] \{\bar{\phi}\} = \{0\} \quad (5.21)$$

Pre-multiplying both sides of Equation 5.21 by $[L]^{-T}$, we obtain

$$[A] - \lambda[I] \{\bar{\phi}\} = \{0\} \quad (5.22)$$

where $[A]$ is symmetric and is given by

$$[A] = [L]^{-1}[K][L]^{-T} \quad (5.23)$$

Noting that the matrix $[A]$ of Equation 5.23 has the properties (i.e. symmetry and positive definiteness) ascribed to the matrix $[K]$ of Equation 5.5, we find that our transformation of the Generalized Eigenproblem of Equation 5.10 to the Standard Form of Equation 5.5 is complete with the development of Equation 5.22.

Referring back to the original decomposition proposed by Equation 5.16, we should note that while $[M]$ and $[L]$ are both banded for a consistent mass formulation, the resulting $[L]^{-1}$ is a full matrix. Thus $[A]$ (through Equation 5.23) turns out to be a full matrix which can make the proposed transformation (described above) ineffective for very large finite element analyses. On the other hand, in the case of a lumped mass matrix $[M]$, both $[M]$ and $[L]$ turn out to be diagonal matrices

(through Equations 5.17 and 5.18) which results in $[A]$ possessing the same bandwidth as $[K]$. Thus, the implementation of a lumped mass approach can significantly reduce numerical problems in the case of Large Order Modal Analysis.

Solving Equation 5.24 below for $\{\bar{\phi}_i\}$, i.e. the transformed eigenvector (or mode shape) corresponding to degree of freedom i ,

$$\left[[A] - \lambda_i [I] \right] \{\bar{\phi}_i\} = \{0\} \quad (5.24)$$

we can implement Equation 5.20 to obtain

$$\{\phi_i\} = [L]^{-T} \{\bar{\phi}_i\} \quad (5.25)$$

which provides the required transformation from $\{\bar{\phi}_i\}$ to $\{\phi_i\}$, i.e. the eigenvector of the original system which represents the solution to Equation 5.14.

5.4 Changing the Basis of the Dynamic Equations of Equilibrium from Physical Displacements and Rotations to Modal Generalized Degrees of Freedom

In the development which follows, our intent is to transform the dynamic equations of equilibrium of the Bit-Bit Sub Finite Element Model (i.e. Equation 5.1) into a more effective form for direct integration by using a transformation of the form

$$\{u(t)\} = [\Gamma]\{\eta(t)\} \quad (5.26)$$

where $[\Gamma]$ is a square matrix characterizing the nature of the transformation, $\{u\}$ is the vector of physical degrees of freedom in terms of which the assembled finite element model of the PDC Bit-Bit Sub System was originally formulated and $\{\eta\}$ is the vector of generalized degrees of freedom to which we are transforming. Equation 5.26 indicates that both $\{u\}$ and $\{\eta\}$ are time dependent quantities (i.e. functions of time t) while the matrix $[\Gamma]$ is not. As implied by the fact that $[\Gamma]$ is (at this point) a square matrix, the number of degrees of freedom represented by $\{u\}$ and $\{\eta\}$ are the same and are taken to be equal to $\bar{N}$ (the number of degrees of freedom of the original System).

We should note that the primary objective of the transformation implied by Equation 5.26 is the development of Stiffness $[K_\eta]$, Damping $[C_\eta]$ and Mass $[M_\eta]$ matrices which correspond to the generalized degrees of freedom $\{\eta\}$ and which have a smaller bandwidth than the original system matrices $[K]$, $[C]$ and $[M]$ of Equation 5.1. The key to reaching this goal is careful selection of the transformation matrix $[\Gamma]$. Theoretically there is a wide range of possibilities available for defining $[\Gamma]$ in such a way as to reduce the bandwidth of the system matrices. Following a particularly effective approach outlined by Bathe in Reference 5.1, we establish the transformation matrix $[\Gamma]$ by using the displacement (or modal) solution of the free vibration equilibrium equations with damping neglected (see Equations 5.8 and 5.10) outlined in Sections 5.2 and 5.3; i.e. we establish $[\Gamma]$ through the equality

$$[\Gamma] \equiv [\Phi] \tag{5.27}$$

where $[\Phi]$ is the matrix of the $\bar{N}$ system eigenvectors $\{\phi_i\}$ defined previously in Equation 5.4.

The substitution of Equation 5.27 into Equation 5.26 leads to the observation that

$$\begin{aligned}
\{u\} &= [\Phi]\{\eta\} & \text{a)} \\
\{\dot{u}\} &= [\Phi]\{\dot{\eta}\} & \text{b)} \\
\{\ddot{u}\} &= [\Phi]\{\ddot{\eta}\} & \text{c)}
\end{aligned}
\tag{5.28}$$

The first step in transforming the Equations of Equilibrium of the Bit-Bit Sub System from a physical to modal basis involves the substitution of Equations 5.28 into the left hand side of Equation 5.1 to obtain

$$\begin{aligned}
[M][\Phi]\{\ddot{\eta}\} + [C][\Phi]\{\dot{\eta}\} + [K][\Phi]\{\eta\} \\
= \{F_x\} + \{F_f\} + \{F_u\} - \{F_{BD}\}
\end{aligned}
\tag{5.29}$$

Pre-multiplying both sides of Equation 5.29 by $[\Phi]^T$, we obtain the desired transformed equilibrium equations given below

$$\begin{aligned}
[M_\eta]\{\ddot{\eta}\} + [C_\eta]\{\dot{\eta}\} + [K_\eta]\{\eta\} \\
= \{F_{x\eta}\} + \{F_{f\eta}\} + \{F_{u\eta}\} - \{F_{BD\eta}\}
\end{aligned}
\tag{5.30}$$

where the transformed System Modal Mass $[M_\eta]$, Damping $[C_\eta]$ and Stiffness $[K_\eta]$ matrices are given by

$$\begin{aligned}
[M_\eta] &= [\phi]^T [M] [\phi] & \text{a)} \\
[C_\eta] &= [\phi]^T [C] [\phi] & \text{b)} \\
[K_\eta] &= [\phi]^T [K] [\phi] & \text{c)}
\end{aligned} \tag{5.31}$$

and where the corresponding Modal Load Vectors $\{F_{x\eta}\}$, $\{F_{f\eta}\}$, $\{F_{\mu\eta}\}$ and $\{F_{BD\eta}\}$ appear as

$$\begin{aligned}
\{F_{x\eta}\} &= [\phi]^T \{F_x\} & \text{a)} \\
\{F_{f\eta}\} &= [\phi]^T \{F_f\} & \text{b)} \\
\{F_{\mu\eta}\} &= [\phi]^T \{F_\mu\} & \text{c)} \\
\{F_{BD\eta}\} &= [\phi]^T \{F_{BD}\} & \text{d)}
\end{aligned} \tag{5.32}$$

Comparing Equations 5.13-a and 5.31-a and Equations 5.13-b and 5.31-c, we find that the Modal Mass $[M_\eta]$ and Stiffness $[K_\eta]$ matrices are both diagonal matrices in nature and we have achieved our desired result of reducing the bandwidth of the original system matrices $[M]$ and $[K]$ through the transformation imposed by Equation 5.31.

The primary function of the Damping Matrix $[C]$ in Equations 5.1 and 5.31-b is to approximate the overall energy dissipation during the Bit-Bit Sub Model dynamic response. In general, it is very difficult to formulate the $[C]$ matrix in the direct manner in which the Element Stiffness $[K_e]$ and

Mass $[M_e]$ matrices were developed in Section 3.1 and 3.2 of Chapter III. Consequently, in GEODYN, we do not compute the Modal Damping Matrix $[C_\eta]$ in the manner implied by Equation 5.31-b. Rather, we take another approach which assumes that damping is proportional to the Mass Matrix. The Modal Damping Matrix $[C_\eta]$ is defined by

$$\begin{aligned} C_{nij} &= \{\phi_i\}^T [C] \{\phi_j\} \quad i, j = 1, \bar{N} \\ &= 2\Omega_i \xi_i M_{nij} \end{aligned} \quad (5.33)$$

where C_{nij} is an arbitrary element of $[C_\eta]$ (located at the intersection of the i^{th} row and j^{th} column of $[C_\eta]$), ξ_i is the ratio of damping to critical damping and Ω_i is the modal frequency corresponding to the i^{th} response mode approximated by

$$\Omega_i \approx \sqrt{\frac{K_{nii}}{M_{nii}}}, \quad i = 1, \bar{N} \quad (5.34)$$

which implies

$$K_{nii} \approx \Omega_i^2 M_{nii} \quad (5.35)$$

and ξ_i is the Modal Damping Ratio also corresponding to the i^{th} mode. In Equation 5.34, $K_{\eta ii}$ and $M_{\eta ii}$ are the diagonal elements (corresponding to the i^{th} degree of freedom) of, respectively, the Diagonal Modal Stiffness $[K_\eta]$ and Mass $[M_\eta]$ matrices.

Upon examining Equation 5.33, we find that our definition of the Modal Damping Matrix $[C_\eta]$ only obliquely infers the true nature of the System Damping Matrix $[C]$. An important implication of using Equation 5.33 is that we have assumed that the total damping in the structure is the sum of the individual damping contributions of each mode.

The value of the Modal Damping Ratio is usually established on the basis of past experience. For example, in the case of steel structures, ξ_i has been found to fall in the range

$$.001 \leq \xi_i \leq .02 \quad (5.36)$$

In implementing the GEODYN System in PDC Bit-Bit Sub Dynamic Qualification Analyses, a value of $\xi_i = .001$ has generally been found to be a reasonable estimate of the dissipation for all the modes of response (i.e. $i = 1, \bar{N}$).

We should note that the definition of the Modal Damping Matrix $[C_\eta]$ provided by Equation 5.33 causes $[C_\eta]$ to be a diagonal matrix, which insures that it conforms with the diagonal nature of the Modal Stiffness and Mass Matrices

$[K_\eta]$ and $[M_\eta]$. In view of the fact that $[M_\eta]$, $[C_\eta]$ and $[K_\eta]$ are all diagonal matrices, the transformed Dynamic Equations of Equilibrium of the PDC Bit-Bit Sub System (i.e. Equation 5.30) turn out to be an uncoupled set of equations which can be expressed in the component form

$$\begin{aligned} M_{\eta ii} \ddot{\eta}_i + C_{\eta ii} \dot{\eta}_i + K_{\eta ii} \eta_i & \quad i = 1, \bar{N} \\ & = F_{x\eta i} + F_{f\eta i} + F_{\mu\eta i} - F_{BD\eta i} \end{aligned} \quad (5.37)$$

where $\ddot{\eta}_i$ is the i^{th} component of the generalized acceleration vector $\{\ddot{\eta}\}$, $F_{f\eta i}$ is the i^{th} component of the generalized contact force vector and so on.

Prior to concluding our discussion of the transformation of the Dynamic Equations of Equilibrium of the PDC Bit-Bit Sub System to a modal basis, it serves us to dwell a moment on a useful approximation which has been incorporated into GEODYN. This approximation is based on the hierarchy of eigenvalues established through Equation 5.6 and introduces further economy into the integration of the transformed equations of motion (i.e. Equation 5.30) by reducing the number of Generalized Degrees of Freedom $\{\eta\}$ used in characterizing the dynamic response of the Bit-Bit Sub Finite Element Model. The approximation is developed in the following manner.

Designating u_i , η_i and $\phi_i^{(j)}$ respectively as elements of the vectors $\{u\}$, $\{\eta\}$ and $\{\phi_j\}$, i.e. (see Equation 5.4)

$$\begin{aligned}
\{u\}^T &= [u_1 \ u_2 \ \cdots \ u_{\bar{N}}] & a) \\
\{\eta\}^T &= [\eta_1 \ \eta_2 \ \cdots \ \eta_{\bar{N}}] & b) \\
\{\phi_j\}^T &= [\phi_1^{(j)} \ \phi_2^{(j)} \ \cdots \ \phi_{\bar{N}}^{(j)}] & c)
\end{aligned}
\tag{5.38}$$

We can express Equations 5.3 and 5.28-a in the series form

$$u_i = \sum_{j=1}^{\bar{N}} \phi_i^{(j)} \eta_j \quad i = 1, \bar{N} \tag{5.39}$$

where $\bar{N}$ indicates the total number of degrees of freedom of the Bit-Bit Sub Finite Element Model being analyzed.

Incorporated into GEODYN is the assumption that the expansion represented by Equation 5.39 can be approximated by

$$u_i \approx \sum_{j=1}^{\hat{N}} \phi_i^{(j)} \eta_j \quad i = 1, \hat{N} \tag{5.40}$$

where $\hat{N}$ is a user specified quantity which satisfies the inequality

$$\hat{N} \leq \bar{N} \tag{5.41}$$

Truncating $[\phi]$ and $\{\eta\}$ of Equations 5.4 and 5.38-a to obtain

$$[\hat{\phi}] = [\phi_1 \ | \ \phi_2 \ | \ \cdots \ | \ \phi_{\hat{N}}] \tag{5.42}$$

$$\{\hat{\eta}\}^T = [\eta_1 \ \eta_2 \ \cdots \ \eta_{\hat{N}}] \quad (5.43)$$

Equations 5.3, 5.39 and 5.40 imply that we can approximate the vector of Original System Degrees of Freedom $\{u\}$ by

$$\{u\} \approx [\hat{\Phi}]\{\hat{\eta}\} \quad (5.44)$$

where $[\hat{\Phi}]$ is the rectangular Truncated Matrix of Eigenvectors (which is of order $\bar{N} \times \hat{N}$) and $\{\hat{\eta}\}$ is the reduced generalized degree of freedom vector of order $(\hat{N} \times 1)$.

Inherent in the approximation expressed by Equation 5.44 above is the assumption that the nodal displacement field $\{u\}$, and subsequently the corresponding nodal velocity and acceleration fields $\{\dot{u}\}$ and $\{\ddot{u}\}$, of the Bit-Bit Sub Model can be adequately defined in terms of the first $\hat{N}$ eigenvectors (of the total of $\bar{N}$ possible eigenvectors) characterizing the free vibration response of the Finite Element Model. The value of $\hat{N}$ is established from the hierarchy of eigenvalues expressed by Equation 5.6 and is based on past experience garnered in running GEODYN in PDC Bit-Bit Sub Dynamic Analyses.

In many practical situations, $\hat{N}$ can be taken to be much less than $\bar{N}$ which results in a large reduction in the number of generalized degrees of freedom (i.e. where we reduce $\{\eta\}$ to $\{\hat{\eta}\}$) characterizing the dynamic problem. These cases result in a corresponding significant reduction in the effort needed to integrate the Dynamic Equations of Motion of the PDC Bit-Bit Sub Finite Element Model.

In view of the reduced (or truncated) transformation expressed by Equation 5.44, we can follow the procedure established previously in Equations 5.28 through 5.32 to transform the Equations of Equilibrium (Equation 5.1) to the form

$$\begin{aligned} [\hat{M}_\eta] \{\ddot{\hat{\eta}}\} + [\hat{C}_\eta] \{\dot{\hat{\eta}}\} + [\hat{K}_\eta] \{\hat{\eta}\} \\ = \{\hat{F}_{x\eta}\} + \{\hat{F}_{f\eta}\} + \{\hat{F}_{\mu\eta}\} - \{\hat{F}_{BD\eta}\} \end{aligned} \quad (5.45)$$

where the Reduced Modal Mass $[\hat{M}_\eta]$ and Stiffness $[\hat{K}_\eta]$ Matrices are given by

$$\begin{aligned} [\hat{M}_\eta] &= [\hat{\phi}]^T [M] [\hat{\phi}] & \text{a)} \\ [\hat{K}_\eta] &= [\hat{\phi}]^T [K] [\hat{\phi}] & \text{b)} \end{aligned} \quad (5.46)$$

and where the corresponding Reduced Modal Load Vectors $\{\hat{F}_{x\eta}\}$, $\{\hat{F}_{f\eta}\}$, $\{\hat{F}_{\mu\eta}\}$ and $\{\hat{F}_{BD\eta}\}$ appear as

$$\{\hat{F}_{x\eta}\} = [\hat{\Phi}]^T \{F_x\} \quad a)$$

$$\{\hat{F}_{f\eta}\} = [\hat{\Phi}]^T \{F_f\} \quad b)$$

$$\{\hat{F}_{\mu\eta}\} = [\hat{\Phi}]^T \{F_\mu\} \quad c)$$

$$\{\hat{F}_{BD\eta}\} = [\hat{\Phi}]^T \{F_{BD}\} \quad d)$$

(5.47)

Following Equation 5.33, the terms of the Reduced Modal Damping Matrix $[\hat{C}_\eta]$ are set independently for each of the $\hat{N}$ Modes through the relationship

$$\hat{C}_{nij} = 2\hat{\Omega}_i \xi_i M_{nij} \quad (5.48)$$

where

$$\hat{\Omega}_i = \sqrt{\frac{\hat{K}_{nii}}{M_{nii}}} \quad (5.49)$$

Within the context of the solution algorithm incorporated into the Phase 1 Version of GEODYN, we are now able to undertake the integration of the reduced set of transformed equilibrium equations (containing $\hat{N}$ generalized degrees of freedom) represented by Equation 5.45 instead of having to deal with the

original full set of equilibrium equations (containing the $\bar{N}$ original physical degrees of freedom) represented by Equation 5.1 which existed prior to the introduction of the truncated transformation of Equation 5.44.

VI. THE GENERAL SOLUTION ALGORITHM

The discussion of this chapter is primarily concerned with the mathematical definition of the General Solution Algorithm (Figure 2.1) utilized by GEODYN in the time integration of the transformed generalized Equations of Equilibrium (Equation 5.45) of the PDC Bit-Bit Sub Finite Element Model. In beginning the development of our algorithm, we assume that we initially have available the reduced generalized nodal displacement $\{\hat{\eta}\}$, velocity $\{\dot{\hat{\eta}}\}$ and acceleration $\{\ddot{\hat{\eta}}\}$ vectors which represent the solution to Equation 5.45 for a previous time t . Following the execution of the next time step Δt , our desire is to determine the counterparts of these vectors for the time $t + \Delta t$.

Implementing Equation 5.45, the reduced transformed Equations of Equilibrium evaluated at time $t + \Delta t$ can be expressed

$$\begin{aligned} [\hat{M}_\eta] \{\ddot{\hat{\eta}}(t+\Delta t)\} + [\hat{C}_\eta] \{\dot{\hat{\eta}}(t+\Delta t)\} + [\hat{K}_\eta] \{\hat{\eta}(t+\Delta t)\} \\ = \{\hat{F}_{x\eta}(t+\Delta t)\} + \{\hat{F}_{FR\eta}(t+\Delta t)\} - \{\hat{F}_{BD\eta}(t+\Delta t)\} \end{aligned} \quad (6.1)$$

where

$$\{\hat{F}_{FR\eta}(t+\Delta t)\} = \{\hat{F}_{f\eta}(t+\Delta t)\} + \{\hat{F}_{\mu\eta}(t+\Delta t)\} \quad (6.2)$$

As indicated in Equation 6.2 above, the nodal force vector $\{\hat{F}_{FR\eta}\}$ is a composite generalized formation force vector composed of both reduced generalized contact force $\{\hat{F}_{f\eta}\}$ and friction force $\{\hat{F}_{\mu\eta}\}$ components.

The primary function of the General Solution Algorithm can now be seen to be the time integration of Equation 6.1. Following a survey of several possible candidate numerical integration procedures (Section 2.2.2), Newmark's Method was selected as the most appropriate means of achieving these ends. The details of the application of Newmark's Method to the numerical time integration of the transformed Equations of Motion of the PDC Bit-Bit Sub Finite Element Model are provided in the discussion which follows. This discussion is concerned first (Section 6.1) with the modification of Equation 6.1 to account for the introduction of the Newmark Integration Scheme into the transformed Equations of Equilibrium of the Bit-Bit Sub System and then (Section 6.2) with the details of the iterative solution of the resulting modified equilibrium equations, which possess a nonlinear nature, through implementation of the well known Newton-Raphson numerical solution process.

6.1 The Introduction of Newmark's Numerical Integration Scheme

In Newmark's Method (Section 2.2.2.2), the reduced generalized nodal displacement $\{\hat{h}(t+\Delta t)\}$ and velocity $\{\dot{\hat{h}}(t+\Delta t)\}$ vectors for time $t + \Delta t$ are approximated by (Equation 2.5)

$$\begin{aligned} \{\dot{\hat{h}}(t+\Delta t)\} &= \{\dot{\hat{h}}(t)\} + (1-\gamma)\Delta t\{\ddot{\hat{h}}(t)\} + \gamma \Delta t\{\ddot{\hat{h}}(t+\Delta t)\} & \text{a)} \\ \{\hat{h}(t+\Delta t)\} &= \{\hat{h}(t)\} + \Delta t\{\dot{\hat{h}}(t)\} + (\frac{1}{2}-\beta)\Delta t^2\{\ddot{\hat{h}}(t)\} & \\ &+ \beta \Delta t^2 \{\ddot{\hat{h}}(t+\Delta t)\} & \text{b)} \end{aligned} \tag{6.3}$$

where β and γ are fixed parameters that are selected to obtain the best stability and accuracy characteristics in the solution. The algorithm expressed by Equations 6.3 essentially represents a weighted average method of computing the acceleration over the time period t and $t + \Delta t$ (i.e. over the time step Δt).

When implementing the Newmark algorithm in the integration of the transformed Equations of Motion of the Bit-Bit Sub System, it is convenient to re-express Equation 6.3 in the equivalent form

$$\{\ddot{\hat{\eta}}(t+\Delta t)\} = a_0 \left\{ \{\hat{\eta}(t+\Delta t)\} - \{\hat{\eta}(t)\} \right\} - a_2 \{\dot{\hat{\eta}}(t)\}$$

a)

$$- a_3 \{\ddot{\hat{\eta}}(t)\}$$

(6.4)

$$\{\dot{\hat{\eta}}(t+\Delta t)\} = a_1 \left\{ \{\hat{\eta}(t+\Delta t)\} - \{\hat{\eta}(t)\} \right\} - a_4 \{\dot{\hat{\eta}}(t)\}$$

b)

$$- a_5 \{\ddot{\hat{\eta}}(t)\}$$

where

$$a_0 = \frac{1}{\beta \Delta t^2}$$

a)

$$a_1 = \frac{\gamma}{\beta \Delta t}$$

b)

$$a_2 = \frac{1}{\beta \Delta t}$$

c)

(6.5)

$$a_3 = \frac{1}{2\beta} - 1$$

d)

$$a_4 = \frac{\gamma}{\beta} - 1$$

e)

$$a_5 = \frac{\Delta t}{2} \left(\frac{\gamma}{\beta} - 2 \right)$$

f)

In Equations 6.4, the generalized velocity $\{\dot{\hat{h}}(t+\Delta t)\}$ and acceleration $\{\ddot{\hat{h}}(t+\Delta t)\}$ vectors at time $t + \Delta t$ are related both to their counterparts $\{\dot{\hat{h}}(t)\}$ and $\{\ddot{\hat{h}}(t)\}$ at time t and to the incremental displacement, represented by the quantity $\{\hat{h}(t+\Delta t)\} - \{\hat{h}(t)\}$, occurring during the time step Δt .

Next we substitute the Newmark approximations, represented by Equation 6.4, into the transformed Equations of Equilibrium, given by Equation 6.1, and manipulate and rearrange the resulting expression to obtain

$$\begin{aligned} & \left(a_0 [\hat{M}_\eta] + a_1 [\hat{C}_\eta] + [\hat{K}_\eta] \right) \{\hat{h}(t+\Delta t)\} - \{\hat{h}(t)\} \\ & - \{\hat{F}_{FR\eta}(t+\Delta t)\} = - [\hat{K}_\eta] \{\hat{h}(t)\} + \left(a_2 [\hat{M}_\eta] + a_4 [\hat{C}_\eta] \right) \{\dot{\hat{h}}(t)\} \quad (6.6) \\ & + \left(a_3 [\hat{M}_\eta] + a_5 [\hat{C}_\eta] \right) \{\ddot{\hat{h}}(t)\} + \{\hat{F}_{x\eta}(t+\Delta t)\} - \{\hat{F}_{BD\eta}(t+\Delta t)\} \end{aligned}$$

Defining the Effective Stiffness Matrix $[\tilde{K}_\eta]$ and the Effective Force Vector $\{\tilde{F}_\eta\}$ to be given by

$$[\tilde{K}_\eta] = a_0 [\hat{M}_\eta] + a_1 [\hat{C}_\eta] + [\hat{K}_\eta] \quad (6.7)$$

$$\begin{aligned}
\{\tilde{F}_\eta(t+\Delta t)\} &= \{\hat{F}_{x\eta}(t+\Delta t)\} - \{\hat{F}_{BD\eta}(t+\Delta t)\} - [\hat{K}_\eta]\{\hat{\eta}(t)\} \\
&+ \left(a_2[\hat{M}_\eta] + a_4[\hat{C}_\eta]\right)\{\dot{\hat{\eta}}(t)\} \\
&+ \left(a_3[\hat{M}_\eta] + a_5[\hat{C}_\eta]\right)\{\ddot{\hat{\eta}}(t)\}
\end{aligned} \tag{6.8}$$

We can substitute Equations 6.7 and 6.8 into Equation 6.6 to derive the following modified form of equilibrium statement which reflects the Newmark numerical integration approximation.

$$[\hat{K}_\eta] \left\{ \{\hat{\eta}(t+\Delta t)\} - \{\hat{\eta}(t)\} \right\} - \{\hat{F}_{FR\eta}(t+\Delta t)\} = \{\tilde{F}_\eta(t+\Delta t)\} \tag{6.9}$$

Equation 6.9 above represents the dynamic Equations of Motion of the Bit-Bit Sub Finite Element Model which must be satisfied at time $t + \Delta t$. The generalized body force vector $\{\hat{F}_{BD\eta}(t+\Delta t)\}$ on the right hand side of Equation 6.8 is assumed, in GEODYN, to be computed based on the displaced state of the body (through the $\{\omega x \omega x r\}$ vector of Equation 4.26-b) existing at time t and consequently is found to remain constant over the time step Δt . Further, the generalized direct force vector $\{\hat{F}_{x\eta}(t+\Delta t)\}$ is based on forces $\{\hat{F}_x(t+\Delta t)\}$ (see Equations 4.28 and 5.47-a) at time $t + \Delta t$ which are themselves assumed to already be known at time t and are taken to remain constant over the time step Δt . All the remaining terms

on the right hand side of Equation 6.8 are constructed from quantities which are constant in nature and do not depend on time in any way (e.g. $[\hat{K}_\eta]$, $[\hat{M}_\eta]$, $[\hat{C}_\eta]$) or are time dependent quantities for which we only require values at time t (i.e. $\hat{n}(t)$, $\dot{\hat{n}}(t)$, $\ddot{\hat{n}}(t)$). As a result, the Effective Force Vector $\{\hat{F}_\eta\}$ itself is completely specified at the start of the time interval (i.e. at time t) and, from the standpoint of its computation, is found to remain constant (and thus known) over time time step Δt .

The generalized formation force vector $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ (Equation 6.2) appearing in the left hand side of Equation 6.9 is dependent on both the nature and degree of current contact between the bit's ghost cutters and the formation. Noting from Equation 6.2 that the force is a composite of both generalized contact and friction force components, $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ is found to be a function of both the relative penetration of the formation by the bit's cutters as well as the absolute velocity of the bit's cutters with respect to the formation. Thus, the force $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ is not constant in magnitude and, further, cannot be completely specified at the start of a time step. At best, only an average value of $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ can be determined for the time step, and that can only be obtained through an iterative process which is outlined below.

6.2 The Iterative Solution Technique

Designating $\{\Psi(t+\Delta t)\}$ to be the increment of generalized nodal displacement, occurring over the time step Δt , given by

$$\{\Psi(t+\Delta t)\} \equiv \{\hat{n}(t+\Delta t)\} - \{\hat{n}(t)\} \quad (6.10)$$

We can substitute Equation 6.10 into the left hand side of Equation 6.9 to obtain

$$[\tilde{K}_n] \{\Psi(t+\Delta t)\} - \{\hat{F}_{FRn}(t+\Delta t)\} = \{\tilde{F}_n(t+\Delta t)\} \quad (6.11)$$

To completely determine the state of the system at time $t + \Delta t$, Equation 6.11 must be solved for the generalized displacement increment $\{\Psi(t+\Delta t)\}$. In the special case in which the bit (i.e. through the bit's ghost cutters) has no contact with the formation, the generalized formation force vector $\{\hat{F}_{FRn}(t+\Delta t)\}$ is zero and Equation 6.11 can be solved directly to obtain

$$\{\Psi(t+\Delta t)\} = [\tilde{K}_n]^{-1} \{\tilde{F}_n(t+\Delta t)\} \quad (6.12)$$

However, in the more usual case where the PDC Bit has at least minimal contact and, therefore, interaction with the formation during a time step Δt , Equation 6.11 above must be solved iteratively since the force vector $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ depends on the value of $\{\Psi(t+\Delta t)\}$. The velocity dependence of $\{\hat{F}_{FR\eta}(t+\Delta t)\}$ is, of course, accommodated implicitly through the relationships expressed between $\{\dot{\hat{h}}(t+\Delta t)\}$, $\{\hat{h}(t+\Delta t)\}$, $\{\hat{h}(t)\}$, $\{\dot{\hat{h}}(t)\}$, $\{\ddot{\hat{h}}(t)\}$ and $\{\Psi(t+\Delta t)\}$ given by Equations 6.4-b and 6.10 above. Thus, values of $\{\Psi(t+\Delta t)\}$ are successively established by the iterative process until the procedure finally leads to a value of $\{\Psi(t+\Delta t)\}$ which satisfies Equation 6.11. Then, implementing Equations 6.4 and 6.10, all remaining quantities may be computed at time $t + \Delta t$ and the complete dynamic state of the PDC Bit-Bit Sub System is established at time $t + \Delta t$. Once the state at time $t + \Delta t$ has been computed, the complete process may be repeated and the solution computed for each succeeding time increment. The discussion which follows reviews the mathematical development of the iterative algorithm used in GEODYN to solve Equation 6.11.

In order to achieve our objective, it becomes useful for us to define a vector function $\{f(\Psi)\}$ given by

$$\begin{aligned} \{f(\Psi)\} \equiv & [\tilde{K}_{\eta}]\{\Psi(t+\Delta t)\} - \{\hat{F}_{FR\eta}(t+\Delta t)\} \\ & - \{\tilde{F}_{\eta}(t+\Delta t)\} \end{aligned} \quad (6.13)$$

If $\{\Psi^*\}$ represents the solution to Equation 6.11 for the current time step Δt , it is clear from comparing Equations 6.11 and 6.13 that we must have

$$\{f(\Psi^*)\} = \{0\} \quad (6.14)$$

i.e. any vector $\{\Psi(t+\Delta t)\}$ satisfying (see Equation 6.13)

$$\{f(\Psi)\} = \{0\} \quad (6.15)$$

represents a solution to Equation 6.11.

Consider the system of nonlinear equations represented by Equation 6.15 and suppose that the solution vector at the end of iteration $(i-1)$, designated $\{\Psi_{i-1}\}$, differs from the actual solution $\{\Psi^*\}$ to Equation 6.11 by a small vector $\{\Delta\Psi_i^*\}$, i.e.

$$\{\Delta\Psi_i^*\} = \{\Psi^*\} - \{\Psi_{i-1}\} \quad (6.16)$$

If we assume $\{f(\Psi)\}$ to be sufficiently differentiable at $\{\Psi_{i-1}\}$, we may employ the Taylor Expansion (see also Equations 6.14 and 6.16) to obtain

$$\begin{aligned}
\{f(\Psi^*)\} &= \{f(\Psi_{i-1} + \Delta\Psi_i^*)\} = \{0\} \\
&= \{f(\Psi_{i-1})\} + \left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)} \{\Delta\Psi_i^*\} \\
&\quad + \text{Higher Order Terms} = \{0\}
\end{aligned} \tag{6.17}$$

where $\left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)}$ is the Jacobian Matrix given by

$$\left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)} \equiv \left[\frac{\partial \{ f \}}{\partial \{ \Psi \}} \right] \Big|_{\{\Psi_{i-1}\}} \tag{6.18}$$

Defining

$$\{\Delta\Psi_i\} \equiv \{\Psi_i\} - \{\Psi_{i-1}\} \tag{6.19}$$

where $\{\Psi_i\}$ is a solution for iteration i which is assumed closer to $\{\Psi^*\}$ than $\{\Psi_{i-1}\}$, we can truncate to only linear terms of Equation 6.17 (i.e. we retain only the first two terms on the right hand side of Equation 6.17) to obtain the standard Newton-Raphson approximation

$$\{f(\Psi_{i-1})\} + \left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)} \{\Delta\Psi_i\} \approx \{0\} \tag{6.20}$$

which can be solved for $\{\Delta\Psi_i\}$ to obtain

$$\{\Delta\Psi_i\} = - \left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)}^{-1} \{f(\Psi_{i-1})\} \quad (6.21)$$

Following the solution for $\{\Delta\Psi_i\}$ derived in Equation 6.21 above, we can utilize Equation 6.19 to obtain

$$\{\Psi_i\} = \{\Psi_{i-1}\} + \{\Delta\Psi_i\} \quad (6.22)$$

The value of $\{\Psi_i\}$ (Equation 6.22) obtained at the end of iteration i , is a better approximate solution to Equation 6.11 and, furthermore, serves as a starting value in another step in the iteration process for iteration $i+1$. Equations 6.21 and 6.22 define a recursive relationship built into GEODYN which proceeds until terminated at some pre-defined level of accuracy.

Observing that $\{f(\Psi_{i-1})\}$ can also be interpreted as representing $\{E_{i-1}\}$, the error remaining after evaluating Equation 6.13 at the end of iteration $(i-1)$, i.e.

$$\{E_{i-1}\} = \{f(\Psi_{i-1})\} \quad (6.23)$$

We can substitute Equation 6.23 into the right hand side of Equation 6.21 to obtain the following alternative (but equivalent) form of viewing the iterative process.

$$\{\Delta\Psi_i\} = - \left[\frac{\partial f}{\partial \Psi} \right]_{(i-1)}^{-1} \{E_{i-1}\} \quad (6.24)$$

A key to the execution of the iterative solution process described above, is the proper definition of the general Jacobian form $\left[\frac{\partial f}{\partial \Psi} \right]$ (Equations 6.17, 6.18, 6.21 and 6.24). In view of the relationship expressed by Equation 6.13, it is clear we can expand $\left[\frac{\partial f}{\partial \Psi} \right]$ to obtain

$$\left[\frac{\partial f}{\partial \Psi} \right] = [\tilde{K}_\eta] - \left[\frac{\partial \hat{F}_{FR\eta}}{\partial \Psi} \right] - \left[\frac{\partial \tilde{F}_\eta}{\partial \Psi} \right] \quad (6.25)$$

where

$$\left[\frac{\partial \hat{F}_{FR\eta}}{\partial \Psi} \right] = \left[\frac{\partial \{\hat{F}_{FR\eta}\}}{\partial \begin{bmatrix} \Psi \end{bmatrix}} \right] \quad \text{a)} \quad (6.26)$$

$$\left[\frac{\partial \tilde{F}_\eta}{\partial \Psi} \right] = \left[\frac{\partial \{\tilde{F}_\eta\}}{\partial \begin{bmatrix} \Psi \end{bmatrix}} \right] \quad \text{b)}$$

In view of the fact that, in previous discussion, we have assumed the Effective Force Vector $\{\tilde{F}_\eta\}$ to remain constant over the time step Δt , it follows that

$$\left[\frac{\partial \tilde{F}_\eta}{\partial \Psi} \right] = [0] \quad (6.27)$$

Now the substitution of Equation 6.27 into the right hand side of Equation 6.25 yields

$$\left[\frac{\partial f}{\partial \Psi} \right] = [\hat{K}_\eta] - \left[\frac{\partial \hat{F}_{FR\eta}}{\partial \Psi} \right] \quad (6.28)$$

Equation 6.25 above illustrates that $\left[\frac{\partial f}{\partial \Psi} \right]$ is an equivalent stiffness matrix in which a few selected components are augmented by the bit-formation interaction forces. Since this matrix is used solely for the selection of the increment $\{\Psi\}$ during the iterative process, it behoves us to introduce a computationally efficient approximation which requires only one computation of $\left[\frac{\partial f}{\partial \Psi} \right]$ per time step. Simply, we assume that our time step Δt is small enough such that we can neglect the contributions of $\left[\frac{\partial \hat{F}_{FR\eta}}{\partial \Psi} \right]$ in the construction of $\left[\frac{\partial f}{\partial \Psi} \right]$, i.e. we assume

$$\left[\frac{\partial \hat{F}_{FR\eta}}{\partial \Psi} \right] \approx [0] \quad (6.29)$$

Whereupon the substitution of Equation 6.29 into Equation 6.28 yields the following approximation of $\left[\frac{\partial f}{\partial \Psi} \right]$

$$\left[\frac{\partial f}{\partial \Psi} \right] \approx [\tilde{K}_\eta] \quad (6.30)$$

Now substituting Equation 6.30 into Equation 6.24 we obtain

$$\{\Delta \Psi_i\} = -[\tilde{K}_\eta]_{(i-1)}^{-1} \{E_{i-1}\} \quad (6.31)$$

We should note from Equation 6.24 that, as $\left[\frac{\partial f}{\partial \Psi} \right]$ is defined in Equation 6.28, we have to address the inversion (or solution triangularization) of a $\left[\frac{\partial f}{\partial \Psi} \right]$ matrix of order $(\hat{N} \times \hat{N})$ (see Equation 5.41) during each iteration of the time step Δt . By implementing the approximation of Equations 6.30 and 6.31, we need only address the inversion of $\left[\frac{\partial f}{\partial \Psi} \right]$ (or $[\tilde{K}_\eta]$) at the start of an analysis, a simplification which both eases the difficulties and reduces the cost of the computational effort. Computation of $\{\Delta \Psi_i\}$ through the process implied by Equation 6.31 reduces our algorithm to a form of Modified Newton-Raphson technique.

It should be noted that, though the generalized formation interaction forces $\{\hat{F}_{FR\eta}\}$ are neglected in the computation of $\left[\frac{\partial f}{\partial \Psi} \right]$ in the scheme provided by Equation 6.30, the full effect of these forces is included in the computation of the error

vector $\{E\}$ (refer to Equations 6.13 and 6.23) used in the computation (Equation 6.31) of the refined estimate of $\{\Delta\Psi\}$ for the next iteration. The computation procedure, outlined above, can be summarized as follows.

Solution Initiation Phase

The first step in the process involves the execution of the modal analysis described in Chapter V and the subsequent computation of the reduced Generalized Mass $[\hat{M}_\eta]$, Damping $[\hat{C}_\eta]$ and Stiffness $[\hat{K}_\eta]$ Matrices via Equations 5.46 and 5.48. Given values of γ and β , of the time integration operator of Equations 6.3, and definition of the time step Δt , we implement Equations 6.5 in calculating the constants a_0 through a_5 characterizing the modified integration operator of Equations 6.4.

Initializing A Time Step

In executing the current time step from time t to time $t + \Delta t$, we have available (from the just completed time step from time $t - \Delta t$ to time t) the generalized displacement, velocity and acceleration vectors $\{\hat{\eta}(t)\}$, $\{\dot{\hat{\eta}}(t)\}$ and $\{\ddot{\hat{\eta}}(t)\}$. At this point we specify $\{\hat{F}_{x\eta}(t+\Delta t)\}$ and compute (from $\{\hat{\eta}(t)\}$) $\{\hat{F}_{BD\eta}(t+\Delta t)\}$; the former representing the direct forces to be applied over the current time step and the latter representing the corresponding body forces.

Finally, in terminating the time step initialization process, we compute the Effective Force Vector $\{\tilde{F}_\eta(t+\Delta t)\}$ from Equation 6.8.

The Iteration Process

We assume that the iterative process is proceeding such that we are transitioning from an iteration $i-1$ to the present iteration i . Given a value of $\{\Delta\Psi_i\}$ computed during iteration $(i-1)$ from Equation 6.31, we compute $\{\Psi_i(t+\Delta t)\}$ from Equation 6.22. We should note that $\{\Psi_i(t+\Delta t)\}$ is meant to indicate the particular value (or estimate) of $\{\Psi(t+\Delta t)\}$ operative during iteration i . In the special case where $i=1$ (i.e. we are in the first iteration), we initialize the value of $\{\Delta\Psi_1\}$ to the null vector $\{0\}$, i.e.

$$\{\Delta\Psi_1\} = \{0\} \quad (6.32)$$

Next we compute the reduced generalized displacement, velocity and acceleration vectors $\{\hat{\eta}_i(t+\Delta t)\}$, $\{\dot{\hat{\eta}}_i(t+\Delta t)\}$ and $\{\ddot{\hat{\eta}}_i(t+\Delta t)\}$ (for time $t + \Delta t$, iteration i) through Equations 6.4 and 6.10 and implement these quantities in the computation of the reduced generalized contact $\{\hat{F}_{f\eta i}(t+\Delta t)\}$ and Friction $\{\hat{F}_{\mu\eta i}(t+\Delta t)\}$ force vectors of Equation 5.45. Equation 6.2

is now used to construct the reduced Generalized Formation Interaction Force Vector $\{\hat{F}_{FRni}(t+\Delta t)\}$ for time $t + \Delta t$, iteration i .

Equations 6.13 and 6.23 now provide us an estimate of $\{E_i\}$, the residual error remaining after iteration i has been completed. This permits us to compute a value of $\{\Delta\psi_{i+1}\}$, via Equation 6.31, for the next iteration in the sequence. At this point we compare $\{\Delta\psi_i\}$ and $\{\Delta\psi_{i+1}\}$ to determine whether or not our convergence criterion has been satisfied. If the convergence criterion has been satisfied, we exit the inner iterative loop (Figure 2.1) and move back into the outer incremental loop to undertake computation of the next time step (which takes us from time $t + \Delta t$ to time $t + 2\Delta t$). On the other hand, if the convergence criterion remains unsatisfied after iteration i , we implement $\{\Delta\psi_{i+1}\}$ in completing calculation of the next iteration (i.e. iteration $(i + 1)$).

The Role of the Transformations in the Integration Process

The numerical scheme outlined above assumes that the integration of the Equations of Motion takes place entirely in terms of the reduced generalized degrees of freedom $\{\hat{n}\}$. In order to properly compute the force vectors acting on the Bit-Bit Sub Finite Element Model at varying stages of the integration process, we require penetration displacements,

absolute velocities and absolute accelerations (see Chapter IV) which correspond to the model's physical degrees of freedom and which are related to the Inertial, Non-Rotating, Global X_g, Y_g, Z_g Coordinate System of Figure 4.1-a. Thus, in executing the above scheme, we are in a position where we must continually implement (in both directions) the two aforementioned classes of transformations represented by:

- a. the transformation from the reduced, generalized, degrees of freedom $\{\hat{n}\}$ to the physical degrees of freedom $\{u\}$ expressed in terms of the Rotating X, Y, Z Coordinate System of Figure 4.1-b (see Equation 5.44) and
- b. the transformation from the Rotating, Non-Inertial X, Y, Z coordinates to the Global Fixed X_g, Y_g, Z_g coordinates (Figures 4.1).

VII. REFERENCES

- 1.1 Apostol, M.C., Baird, J.A., Jordan, S., A State-Of-The Art Feasibility Study To Quantify And Define A Phased Series Of Tasks Which When Implemented Will Characterize The Interaction Between Arbitrary, Dynamically Loaded, Stratapax Bit Configurations And The Downhole Assembly With The Bottom Hole Formation; Jordan, Apostol, Ritter Associates, Inc., Drilling Resources Development Corporation, Technical Report Number JAR-TR-81-11, August 1981.
- 1.2 GEODYN User Instruction Manual, SAND84-7102, to be published in July 1984.
- 2.1 Clough, R.W.; "Numerical Integration of the Equations of Motion", from Lectures on Finite Element Methods in Continuum Mechanics; J.T. Oden and E.R.A. Oliveira, Editors; University of Alabama in Huntsville Press; 1973.
- 2.2 Bathe, K.J. and Wilson, E.L.; Numerical Methods in Finite Element Analysis; Prentice Hall Inc.; 1976.
- 3.1 Zienkiewicz, O.C.; THE FINITE ELEMENT METHOD in Engineering Science; McGraw Hill, 3rd Edition, 1977.
- 3.2 Clough, R.W.; "Numerical Integration of the Equations of Motion", from Lectures on Finite Element Methods in Continuum Mechanics; J.T. Oden and E.R.A. Oliveira, Editors; University of Alabama in Huntsville Press; 1973.
- 3.3 FEMGEN/FEMVIEW, Interactive Graphics Finite Element Pre- and Post-Processors, Users Manuals, 1983.
- 3.4 Przemieniecki, J.S.; THEORY OF MATRIX STRUCTURAL ANALYSIS, McGraw Hill Book Company, 1968.
- 5.1 Bathe, K.J. and Wilson, E.L.; Numerical Methods in Finite Element Analysis; Prentice Hall Inc.; 1976.
- 5.2 Smith, B.T., Boyle, J.M., Garbow, B.S., Ideke, Y., Klema, V.C. and Moler, C.B.; Matrix Eigensystem Routines - EISPACK Guide; from Lecture Notes in Computer Science, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

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