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ANISOTROPIC PRESSURE AND FINITE HOT-ELECTRON LARMOR-RADIUS EFFECTS ON RING STABILITY

K. T. Tsang, X. S. Lee, and P. J. Catto

MASTER

Science Applications, Inc.
Plasma Research Institute
934 Pearl Street
Boulder, Colorado 80302

ABSTRACT

The effect of the anisotropic pressure of a hot electron plasma on ballooning-interchange and compressional Alfvén modes are investigated. General eigenmode equations for these modes are derived in the eikonal limit with finite gyro-radius effects retained. A local dispersion relation is obtained in the flute limit for an isotropic Maxwellian background plasma with a bi-Maxwellian hot electron population. Stability is investigated both analytically and numerically.

I. INTRODUCTION

In a recent paper,[1] the stability of a hot electron ring-plasma system in either tandem mirror or Elmo Bumpy Torus (EBT) geometry was investigated. Modes with wave frequency ω below the ion gyrofrequency Ω_i were treated by using the gyrokinetic technique.[2] A set of ballooning-interchange/compressional Alfvén eigenmode equations were derived from the Maxwell equations. This resulting set of equations is general enough to describe low frequency ($\omega < \Omega_i$) modes including interchange, shear and compressional Alfvén, and drift waves. In the limit in which ballooning is negligible, $k_\parallel \approx 0$ (where $k_\parallel$ is the parallel wavenumber), the shear Alfvén and drift waves decouple from the system. One then recovers the dispersion relation, which is the focus of much of the recent ring stability analyses,[1,3,4] and describes the coupling of interchange and compressional Alfvén modes.

However, the calculation in Ref. 1 is based on an isotropic Maxwellian model: the equilibrium distribution functions for the core electrons, ions and hot electrons are assumed to be isotropic Maxwellians. For the hot electrons this assumption is obviously not true because the hot electrons are generated by electron cyclotron heating which heats preferentially in the perpendicular direction. In this work, we will remove the isotropic assumption. In order to obtain the simplest results, we eventually retain only the anisotropy of the hot electrons, while continuing to treat the ions and core electrons as isotropic Maxwellians.

The interest in anisotropic effects in a high beta hot electron plasma is not new. Theoretical investigations[5] in the late 1950's revealed that a high beta plasma having a perpendicular pressure ($p_\perp$) greater than the parallel pressure ($p_\parallel$) could be unstable with respect to the so-called "mirror instability." It is basically a fast (compressional) Alfvén mode which becomes destabilized by pressure anisotropy.

These early results were based on infinite homogenous medium theory. In a hot electron ring-plasma system, it has been shown that for $k_\parallel \approx 0$ a subtle cancellation in the dispersion relation of the compressional Alfvén mode leads instead to a curvature driven compressional Alfvén instability[1,3,4] because of the finite beta effects of the hot electrons. Since it is the perturbed hot electron pressure that is responsible for the subtle cancellation, a careful gyrokinetic calculation including the anisotropic velocity distribution of the hot electrons becomes necessary in order to obtain an accurate picture of the interaction between the mirror driving force and the inhomogeneity.

II. BASIC MODEL

In order to be as general as practical we shall initially assume that all equilibrium distribution functions F_j depend upon the magnetic moment $\mu = v_\perp^2/2B$ as well as energy $E = v^2/2$ and guiding center location $\underset{\sim}{R} \approx \underset{\sim}{r} + \Omega_j^{-1}\underset{\sim}{v}\times\underset{\sim}{\hat{n}}$, where $B = |\underset{\sim}{B}|$, $\underset{\sim}{\hat{n}} = \underset{\sim}{B}/B$, and $\Omega_j = Z_j eB/M_j c$ with $\underset{\sim}{B}$ the unperturbed magnetic field and Z_j and M_j the species charge number and mass. The unperturbed densities N_j and the total unperturbed pressures $p_\perp$ and $p_\parallel$ are assumed to satisfy quasi-neutrality

$$\sum_j Z_j e N_j = 0$$

and pressure balance

$$\nabla(4\pi p_\perp + \tfrac{1}{2}B^2) = \left[1 + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\right]\mathbf{B}\cdot\nabla\mathbf{B} + 4\pi\mathbf{B}\mathbf{B}\cdot\nabla\left(\frac{p_\perp - p_\parallel}{B^2}\right) \quad , \qquad (1)$$

with

$$p_\perp = \sum_j M_j \int d^3v \; \tfrac{1}{2}v_\perp^2 F_j = \sum_j p_{\perp j}$$

and

$$p_\parallel = \sum M_j \int d^3v \; v_\parallel^2 F_j = \sum_j p_{\parallel j} \quad .$$

The species subscript j will be used to denote hot electrons ($j = h$, $Z_h = -1$, $M_h = m$), core electrons ($j = e$, $Z_e = -1$, $M_e = m$), and ions ($j = i$, $Z_i = Z$, $M_i = M$).

Perturbed field quantities will be indicated by a tilda. The perturbed scalar and vector potentials will be denoted by $\tilde{\Phi}$ and $\tilde{\mathbf{A}}$ with $\tilde{A}_\parallel \equiv \hat{\mathbf{n}}\cdot\tilde{\mathbf{A}}$. For any perturbed quantity $\tilde{Q}$, we employ the eikonal ansatz

$$\tilde{Q}(\mathbf{r},t) = \tilde{Q}(\mathbf{r},\omega)\exp[iS(\mathbf{r}) - i\omega t]$$

with $\hat{\mathbf{n}}\cdot\nabla S = 0$, $\mathbf{k}_\perp \equiv \nabla S$, and $\tilde{Q}$ satisfying $k_\perp \gg k_\parallel \equiv |\hat{\mathbf{n}}\cdot\nabla \ln\tilde{Q}|$ and $k_\perp \gg |\nabla_\perp \ln\tilde{Q}(\mathbf{r},\omega)|$.

Defining a displacement $\tilde{\boldsymbol{\xi}}$ via

$$\tilde{\boldsymbol{\xi}} \equiv (c\tilde{\Phi}/\omega B)\mathbf{k}_\perp\times\hat{\mathbf{n}} \quad ,$$

the perturbed species distribution function $\tilde{f}_j$ can be written as [6]

$$\tilde{f}_j = \left\{\left[\frac{Z_j e}{M_j}\tilde{\Phi}\left(\frac{\partial F_j}{\partial E} + \frac{1}{B}\frac{\partial F_j}{\partial \mu}\right) - \frac{Z_j e}{M_j B}\frac{\partial F_j}{\partial \mu}\frac{v_\parallel}{c}\tilde{A}_\parallel\right][1 - J_0\exp(-iL)]\right.$$

$$\left. -\left(J_0\tilde{\boldsymbol{\xi}}\cdot\nabla F_j + J_1\frac{2\Omega_j\mu}{k_\perp v_\perp B}\frac{\partial F_j}{\partial \mu}\tilde{B}_\parallel\right)\exp(-iL) + (\tilde{h}_j + \tilde{g}_j)\exp(-iL)\right. \quad , \qquad (2)$$

where $\tilde{h}_j$ satisfies the gyrokinetic equation

$$v_\parallel \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\tilde{h}_j - i(\omega - \omega_{dj})\tilde{h}_j = \left(\frac{Z_j e}{M_j}\frac{\partial F_j}{\partial E} + \frac{c}{\omega B}\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}F_j\right)\left[i\left(\omega_{dj}\tilde{\Phi}J_o + \frac{\omega v_\perp}{k_\perp c}\tilde{B}_\parallel J_1\right)\right.$$

$$\left. + v_\parallel \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}(J_o\tilde{\Phi}) - v_\parallel \tilde{A}_\parallel \frac{i\omega}{c}J_o\right] . \qquad (3)$$

We have employed the definitions

$$L \equiv \Omega_j^{-1}\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{v}_\perp , \qquad \tilde{B}_\parallel \equiv -i\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\tilde{\underset{\sim}{A}} ,$$

$$\omega_{dj} \equiv (M_j v_\perp^2/2T_{\perp j})\omega_{bj} + 2(M_j v_\parallel^2/2T_{\parallel j})\omega_{\kappa j} , \qquad (4)$$

$$\omega_{bj} \equiv (T_{\perp j}/M_j\Omega_j)\underset{\sim}{k}_\perp\cdot\hat{\underset{\sim}{n}}\times\underset{\sim}{\nabla}\ell n B ,$$

and

$$\omega_{\kappa j} \equiv (T_{\parallel j}/M_j\Omega_j)\underset{\sim}{k}_\perp\cdot\hat{\underset{\sim}{n}}\times(\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\hat{\underset{\sim}{n}}) .$$

The nth order Bessel functions J_n have argument $k_\perp v_\perp/\Omega_j$. In Eq. (2), $\tilde{g}_j$ is the correction to $\tilde{f}_j$ to next order in ω/Ω_j and for isotropic ions it satisfies[2]

$$-\Omega_i \frac{\partial \tilde{g}_i}{\partial\phi} + v_\parallel \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\tilde{g}_i - i(\omega - \omega_{di})\tilde{g}_i = -\frac{i}{c}\underset{\sim}{v}_\perp\cdot\underset{\sim}{A}\left(\frac{Ze}{M}\frac{\partial F_i}{\partial E}\omega + \frac{c}{B}\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}F_i\right) , \qquad (5)$$

with ϕ the gyrophase angle. Only the leading order ion $\tilde{g}_i$,

$$\tilde{g}_i = -\left(\frac{\partial F_i}{\partial E}\omega + \Omega_i^{-1}\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}F_i\right)(\tilde{B}_\parallel \underset{\sim}{k}_\perp\cdot\underset{\sim}{v}_\perp/k_\perp^2 B) , \qquad (6)$$

need be employed to obtain the polarization drift that is necessary to retain in order to recover the compressional Alfvén mode.[1]

III. MAXWELL EQUATIONS

The system of three eigenvalue equations in the three unknowns $\tilde{\Phi}$, $\tilde{B}_\parallel$, and $\tilde{A}_\parallel$ can be derived in a straightforward manner by judiciously employing the appropriate moments of the $\tilde{f}_j$ of Eqs. (2) and (3) for $k_\perp v_\perp/\Omega_j \ll 1$ in the Maxwell equations.

We first form quasi-neutrality, $\sum Z_j e \int d^3v\ \tilde{f}_j = 0$, by using $\sum Z_j e N_j = 0$ and noting $\int d^3v\ \tilde{g}_j \exp(-iL) = 0$ for the $\tilde{g}_j$ of Eq. (6). The resulting expression may then be written as

$$\sum Z_j e \int d^3v\ J_0 \tilde{h}_j = (\tilde{B}_\parallel + \tilde{\xi}\cdot\nabla B)\sum Z_j e \int d^3v\ \frac{\mu}{B}\frac{\partial F_j}{\partial \mu}$$

$$+\left(\frac{k_\perp c}{B}\right)^2\left\{\tilde{\Phi}\left[\sum M_j N_j - \frac{1}{\omega}\sum \frac{1}{\Omega_j} k_\perp\times\hat{n}\cdot\nabla(M_j \int d^3v\ \tfrac{1}{2}v_\perp^2 F_j)\right]\right.$$

$$\left. -(\tfrac{3}{4}\tilde{B}_\parallel + \tilde{\xi}\cdot\nabla B)\sum \frac{M_j^2}{Z_j e}\int d^3v\ \mu^2 \frac{\partial F_j}{\partial \mu}\right\} \quad . \tag{7}$$

Similarly, the $k_\perp\times\hat{n}$ component of Ampere's law, $(ik_\perp^2 c/4\pi)\tilde{B}_\parallel = \sum Z_j e \int d^3v\ k_\perp\times\hat{n}\cdot v_\perp \tilde{f}_j$, may be evaluated to obtain

$$[(1 - C)\tilde{B}_\parallel - (4\pi/B)\tilde{\xi}\cdot\nabla p_\perp]$$

$$= -\sum \frac{4\pi M_j}{B}\int d^3v\ \tfrac{1}{2}v_\perp^2 \frac{2\Omega_j}{k_\perp v_\perp} J_1 \tilde{h}_j + \frac{4\pi k_\perp^2 c}{B^2}\left\{\frac{3}{2}\tilde{\Phi}\left[\sum(p_{\perp j}/\Omega_j) - \tfrac{1}{2}\sum \frac{M_j}{\omega\Omega_j^2} k_\perp\times\hat{n}\cdot\nabla \int d^3v\ (\tfrac{1}{2}v_\perp^2)^2 F_j\right]\right\}$$

$$+ \frac{4\pi}{B^2}(\tilde{B}_\parallel + \tilde{\xi}\cdot\nabla B)\sum M_j \int d^3v\ (\tfrac{1}{2}v_\perp^2)^2 \frac{\partial F_j}{\partial \mu} - \frac{2\pi}{B^3}k_\perp^2(\tilde{B}_\parallel + \tfrac{3}{2}\tilde{\xi}\cdot\nabla B)\sum \frac{M_j}{\Omega_j^2}\int d^3v\ (\tfrac{1}{2}v_\perp^2)^3 \frac{\partial F_j}{\partial \mu}\ , \tag{8}$$

with

$$p_{\perp j} \equiv M_j \int d^3v\ \tfrac{1}{2}v_\perp^2 F_j$$

and where C is defined via

$$\tilde{B}_{\parallel}C \equiv \frac{4\pi}{ik_{\perp}^2c} \sum Z_j e \int d^3v \; \underset{\sim}{k}_{\perp}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{v}_{\perp}\tilde{g}_j \; \exp(-iL) \; . \tag{9}$$

To evaluate C , we follow Ref. 1 and multiply Eq. (5) by $\underset{\sim}{k}_{\perp}\cdot\underset{\sim}{v}_{\perp}$, integrate over all $\underset{\sim}{v}$, and employ the lowest order $\tilde{g}_i$ of Eq. (6) to obtain

$$Ze\int d^3v \; \underset{\sim}{k}_{\perp}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{v}_{\perp}\tilde{g}_i \approx (icM/B)\int d^3v \; (\omega - \omega_{di})\underset{\sim}{k}_{\perp}\cdot\underset{\sim}{v}_{\perp}g_i \quad .$$

As a result, we may write

$$\tilde{B}_{\parallel}C \approx \frac{4\pi M}{k_{\perp}^2 B}\int d^3v \; (\omega - \omega_{di})\underset{\sim}{k}_{\perp}\cdot\underset{\sim}{v}_{\perp}\tilde{g}_i \quad , \tag{10}$$

with $\tilde{g}_i$ given by Eq. (6). Carrying out the velocity space integral yields

$$C \approx \frac{1}{k_{\perp}^2 v_A^2}\left\{\omega^2 - 2\omega\omega_{bi} - \omega\omega_{\kappa i} - \omega\frac{c}{ZeBN_i}\underset{\sim}{k}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}p_i + \omega_{bi}\frac{MN_i}{p_i\Omega_i}\underset{\sim}{k}_{\perp}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\int d^3v \; F_i\left(\frac{v_{\perp}^2}{2}\right)^2\right.$$
$$\left. + \; \omega_{\kappa i}\frac{MN_i}{p_i\Omega_i}\underset{\sim}{k}_{\perp}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\int d^3\underset{\sim}{v} \; F_i\frac{v_{\perp}^2}{2}v_{\parallel}^2\right\} \tag{11}$$

where $v_A = (B^2/4\pi MN_i)^{1/2}$ is the Alfvén speed.

Finally, we employ the parallel Ampere's law

$$(k_{\perp}^2c/4\pi)\tilde{A}_{\parallel} = \tilde{J}_{\parallel} = \sum Z_j e\int d^3v \; v_{\parallel}\tilde{f}_j \quad , \tag{12}$$

along with Eq. (2) to obtain

$$B\cdot\nabla\left\{\frac{k_{\perp}^2\tilde{A}_{\parallel}}{B}\left[1 + \frac{4\pi}{B^2}(p_{\perp} - p_{\parallel})\right]\right\} = \frac{4\pi}{c}\sum Z_j e\int d^3v \; v_{\parallel}\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}(\tilde{h}_j J_o) \tag{13}$$

Rewriting Eq. (13) by employing Eq. (3), and then using quasi-neutrality as given by Eq. (7) and the perpendicular Ampere's law of Eq. (8) to eliminate $\sum Z_j e\int d^3v \; \tilde{h}_j J_o$ and $\sum Z_j e\int d^3v \; \omega_{bj}\tilde{h}_j J_o$, respectively, yields after a considerable amount of algebra

$$
\frac{v_A^2}{k_\perp^2}\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\left\{\frac{k_\perp^2}{B}\left[1+\frac{4\pi}{B^2}(p_\perp - p_\parallel)\right]\frac{i\omega}{c}\tilde{A}_\parallel\right\} + \tilde{\Phi}\Big\{\omega^2 - \omega(\omega_p + \omega_b - \omega_\kappa) + (\omega_\kappa\omega_\parallel/b)
$$

$$
- (2\omega_b/b\hat{\beta})(\omega_\kappa - C\omega_b) + \omega_b\omega_g - \omega_\kappa\Big[\sum(p_{\perp j}/\Omega_j)\Big]^{-1}\sum(M_j/\Omega_j^2)\Big[\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}(\tfrac{1}{2}\int d^3v\ v_\perp^2 v_\parallel^2 F_j
$$

$$
- (\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\ln B)\int d^3v\ v_\perp^2(v_\parallel^2 - \tfrac{1}{4}v_\perp^2)F_j\Big]\Big\} + (\tilde{B}_\parallel + \underset{\sim}{\tilde{\xi}}\cdot\underset{\sim}{\nabla}B)(2\omega/\rho c b\hat{\beta})\Big[\sum(p_{\perp j}/\Omega_j)\Big]\Big\{\omega_b(1 - C)
$$

$$
+ \tfrac{1}{2}\hat{\beta}\omega_\perp + \frac{3}{4}b\hat{\beta}\Big[\omega - \omega_g - \omega_b(\rho B^2/G)\Big(\sum p_{\perp j}/\Omega_j\Big)^{-2}\sum(M_j/\Omega_j^2)\int d^3v\ \mu^3\partial F_j/\partial\mu\Big]\Big\}
$$

$$
- \frac{4\pi i\omega}{c^2}\frac{v_A^2}{k_\perp^2}\sum Z_j e\int d^3v\ v_\parallel\tilde{h}_j\hat{n}\cdot\underset{\sim}{\nabla}J_o - (\omega B/k_\perp^2 c\rho)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\Big\{(\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\underset{\sim}{\hat{n}})\sum M_j\int d^3v\ v_\parallel^2\tilde{h}_j J_o
$$

$$
+ (\underset{\sim}{\nabla}\ln B)\sum M_j\int d^3v\ \frac{1}{2}v_\perp^2\tilde{h}_j\Big[J_o - \frac{2\Omega_j}{k_\perp v_\perp}J_1\Big]\Big\} = 0\ . \qquad (14)
$$

In Eq. (14) we have defined the following quantities:

$$
\begin{aligned}
\rho &\equiv \sum M_j N_j\ ,\\
\hat{\beta} &\equiv (8\pi/B^2)\Big[\sum(p_{\perp j}/\Omega_j)\Big]/\Big(\sum\Omega_j^{-1}\Big)\\
b &\equiv \rho^{\ 1}k_\perp^2\Big(\sum\Omega_j^{-1}\Big)\sum(p_{\perp j}/\Omega_j)\ ,\\
\omega_b &\equiv \rho^{-1}\Big(\sum p_{\perp j}/\Omega_j\Big)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\ln B\ ,\\
\omega_\kappa &\equiv \rho^{-1}\Big(\sum p_{\perp j}/\Omega_j\Big)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot(\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\underset{\sim}{\hat{n}})\ ,\qquad (15)\\
\omega_\perp &\equiv \rho^{-1}\Big(\sum\Omega_j^{-1}\Big)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}p_\perp\ ,\\
\omega_\parallel &\equiv \rho^{-1}\Big(\sum\Omega_j^{-1}\Big)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}p_\parallel\ ,\\
\omega_p &\equiv \rho^{-1}\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\sum(\underset{\sim}{\nabla}p_{\perp j})/\Omega_j\ ,
\end{aligned}
$$

and

$$
\omega_g \equiv \Big[\sum(p_{\perp j}/\Omega_j)\Big]^{-1}\sum(M_j/8\Omega_j^2)\underset{\sim}{k}_\perp\times\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\Big(\int d^3v\ v_\perp^4 F_j\Big)\ .
$$

Rather than employ Eqs. (7), (8), and (14) as the system of three eigenvalue equations for $\tilde{\Phi}$, $\tilde{B}_{\parallel}$, and $\tilde{A}_{\parallel}$, it is convenient to replace Eq. (7) by the $\Sigma Z_j e \int d^3v\, v_{\parallel} J_0$ moment of Eq. (3) with the parallel Ampere's law of Eq. (12) employed to eliminate $\Sigma Z_j e \int d^3v\, v_{\parallel} \tilde{h}_j$ and thereby obtain to lowest order

$$(4\pi/k_{\perp}^2c^2)\sum Z_j e \int d^3v\, J_0 [v_{\parallel}^2 \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \tilde{h}_j + i v_{\parallel} \omega_{dj} \tilde{h}_j]$$

$$= \frac{i\omega}{c}\tilde{A}_{\parallel}\left[1 + \frac{4\pi}{B^2}(p_{\perp} - p_{\parallel})\right] - (\hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \tilde{\Phi} - \frac{i\omega}{c}\tilde{A}_{\parallel})\Big\{(4\pi/k_{\perp}^2c^2)\sum(Z_j^2e^2N_j/M_j)$$

$$- (4\pi/k_{\perp}^2B^2\omega)\Big[\sum \Omega_j \underset{\sim}{k}_{\perp} \times \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} p_{\parallel j} + (\underset{\sim}{k}_{\perp} \times \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \ln B)\sum \Omega_j (p_{\perp j} - p_{\parallel j})\Big]\Big\}. \qquad (16)$$

Equations (8), (14), and (16) along with Eqs. (2) and (3) are the convenient system of eigenvalue equations for an anisotropic plasma.

Equation (16) is best viewed as the kinetic generalization of the ideal MHD constraint relating $i\omega\tilde{A}_{\parallel}/c$ and $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \tilde{\Phi}$. It has previously[1] been referred to as the "parallel Ohm's law;" but it is more accurate to note that it is really just the parallel Ampere's law rewritten by using the appropriate moment of the equation for $\tilde{h}_j$. The misnomer for Eq. (16) arises because Eq. (14) is normally referred to as the parallel Ampere's law; however, it is easily seen to be a linear combination of quasi-neutrality and the two components of Ampere's law. As a result, Eq. (14) is best viewed as a more convenient form of quasi-neutrality than Eq. (7), since Eqs. (8) and (16) are really the parallel and perpendicular components of Ampere's law, respectively.

III. BI-MAXWELLIAN MODEL

We assume that the equilibrium distribution functions for the ions ($j = i$) and the core electrons ($j = e$) are isotropic Maxwellians, while that of the hot electrons ($j = h$) is a bi-Maxwellian. We therefore take

$$F_j = N_j \pi^{-3/2} \alpha_{\parallel j}^{-1/2} \alpha_{\perp j}^{-1} \exp[-(v_\parallel/\alpha_{\parallel j})^2 - (v_\perp/\alpha_{\perp j})^2] \quad , \tag{17}$$

where $v_\parallel$ and $v_\perp$ are the parallel and perpendicular velocity components at the midplane of a mirror cell. For the ions and background electrons $\alpha_{\perp j} \equiv \alpha_j = \alpha_{\parallel j} = 2T_j/M_j$ ($j = i,e$); while for the hot electrons $\alpha_{\perp h}^2 = 2T_\perp/m$ and $\alpha_{\parallel h}^2 = 2T_\parallel/m$, with $T_\perp$ and $T_\parallel$ the perpendicular and parallel temperatures.

In order to obtain a tractable system of equations from Eqs. (2), (3), (8), (14), and (16) we consider the flute limit $|\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}| \sim k_\parallel \to 0$. In this limit Eq. (16) is not required since the ballooning $\tilde{A}_\parallel$ term in Eq. (14) may be neglected. In addition, only moments of $\tilde{h}_j$ even in $v_\parallel$ enter in Eqs. (8) and (16) so that only the lowest order solution of Eq. (3) is required, namely,

$$\tilde{h}_j = -(\omega - \omega_{dj})^{-1}\left(\frac{Z_j e}{M_j}\frac{\partial F_j}{\partial E} + \frac{c}{\omega B}\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}F_j\right)\left(\omega_{dj}\tilde{\Phi}J_0 + \frac{\omega v_\perp}{k_\perp c}\tilde{B}_\parallel J_1\right) \quad . \tag{18}$$

Defining

$$A_j \equiv \alpha_{\perp j}^2/\alpha_{\parallel j}^2 \quad ,$$

$$\omega_{*j}^T = (\alpha_{\perp j}^2/2\Omega_j)\underset{\sim}{k}_\perp\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\ln F_j \quad ,$$

and the integrals

$$(I_{1j},I_{2j},I_{3j},I_{4j},I_{5j},I_{6j}) \equiv \int \frac{d^3v\,F_j}{N_j}\frac{(A_j\omega - \omega_{*j}^T)}{(\omega - \omega_{dj})}\left[\left(\frac{v_\parallel}{\alpha_{\parallel j}}\right)^4, \left(\frac{v_\parallel v_\perp}{\alpha_{\parallel j}\alpha_{\perp j}}\right)^2, \left(\frac{v_\perp}{\alpha_{\perp j}}\right)^4,\right.$$

$$\left.\left(\frac{v_\parallel}{\alpha_{\perp j}}\right)^2\left(\frac{v_\perp}{\alpha_{\perp j}}\right)^4, \left(\frac{v_\perp}{\alpha_{\perp j}}\right)^6, \left(\frac{v_\parallel}{\alpha_{\parallel j}}\right)^4\left(\frac{v_\perp}{\alpha_{\perp j}}\right)^2\right] \quad , \tag{19}$$

Eqs. (8) and (14) may be written as

$$D_1\tilde{\Phi}\rho c\left[\sum(p_{\perp j}/\Omega_j)\right]^{-1} + (2\omega D_2 B/b\hat{\beta})\tilde{\Psi} = 0 \tag{20}$$

and

$$D_3\tilde{\Psi} = D_2\tilde{\Phi}\rho c\left[B\omega\sum(p_{\perp j}/\Omega_j)\right]^{-1} \quad , \tag{21}$$

with

$$\tilde{\Psi} \equiv B^{-1}(\tilde{B}_{\parallel} + \tilde{\underset{\sim}{\xi}}\cdot\underset{\sim}{\nabla}B) \quad .$$

The D_j's of Eqs. (20) and (21) are defined as follows:

$$\begin{aligned} D_1 \equiv\ & \omega^2 - \omega(\omega_p + \omega_b - \omega_\kappa) + (\omega_\kappa\omega_{\parallel}/b) + \omega_b\omega_g - \omega_\kappa\omega_q \\ & - 4(\omega_\kappa^2/b\hat{\beta})\sum(I_{1j} - b_j I_{6j})\beta_{\parallel j}A_j^{-1} \\ & + 2(\omega_b/b\hat{\beta})\left\{C\omega_b - \omega_\kappa\left[1 + \sum\beta_{\perp j}b_j(1 - A_j^{-1}) - (\tfrac{1}{2})\sum\beta_{\parallel j}b_j I_{4j}\right]\right\} \ , \end{aligned} \tag{22}$$

$$\begin{aligned} D_2 \equiv\ & \omega_\kappa\left\{1 + (\tfrac{1}{2})(\beta_{\parallel} - \beta_{\perp}) - \sum\beta_{\parallel j}\left[I_{2j} - (3/4)b_j I_{4j}\right]\right\} \\ & - \omega_b\left[C + (3/4)\sum\beta_{\perp j}b_j(A_j - 1) - (1/8)\sum\beta_{\perp j}b_j I_{5j}\right] + (3/4)b\hat{\beta}(\omega - \omega_g) \ , \end{aligned} \tag{23}$$

and

$$D_3 \equiv 1 - C - \sum\beta_{\perp j}(A_j - 1)\left[1 - (3/2)b_j\right] + (\tfrac{1}{2})\sum\beta_{\perp j}\left[I_{3j} - (\tfrac{1}{2})b_j I_{5j}\right] \ , \tag{24}$$

with

$$\omega_q \equiv \left(\sum p_{\perp j}/\Omega_j\right)^{-1}\sum(M_j/\Omega_j^2)\,\underset{\sim}{k}_{\perp}\times\hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\int d^3v\ (v_{\perp}^2/2)v_{\parallel}^2 F_j \quad ,$$

$$b_j = k_{\perp}^2 p_{\perp j}/M_j N_j \Omega_j^2 \quad ,$$

$$\beta_{\parallel} = \sum\beta_{\parallel j} = \sum 8\pi p_{\parallel j}/B^2 \quad ,$$

and

$$\beta_{\perp} = \sum\beta_{\perp j} = \sum 8\pi p_{\perp j}/B^2 \quad .$$

Equations (20) and (21) may then be combined to obtain the local dispersion relation for interchange and compressional Alfvén modes:

$$D = D_1 D_3 + 2D_2^2(b_i\beta_i)^{-1} = 0 \quad . \tag{25}$$

In a weakly inhomogeneous plasma in the absence of the hot electron ring and finite ion gyration radius corrections $D_2 \approx 0$ and Eq. (25) yields the branches $D_1 = 0$ (interchange) and $D_3 = 0$, which gives rise to the compressional Alfvén mode for isotropic pressure plasmas. If the contribution of nonadiabatic perturbed pressures to D_3 are ignored ($I_{3j} \approx 0$), then the so-called "mirror instability"[5] is possible when $1 - \Sigma\beta_{\perp j}(A_j - 1) < 0$. In the next section, the effects of the ring and inhomogeneity are systematically investigated.

IV. LOCAL STABILITY ANALYSIS

In the limit when background ions and electrons are much colder than the hot electrons, we can assume $|\omega| \ll |\omega_{*i}|$, $|\omega_{*e}|$, $|\omega_{di}|$, $|\omega_{de}|$ with $\omega_{*j} = (\alpha_{\perp j}^2/2\Omega_j)\underset{\sim}{k}_\perp\times\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\ln N_j$. We also assume that the finite gyroradius terms may be ignored compared with the hot electron gyroradius terms. The integrals in D for the core ions and electrons can be simplified to

$$I_{2j} \approx \tfrac{1}{2} , \quad I_{3j} \approx 2 , \quad I_{4j} \approx 1 \quad \text{and} \quad I_{5j} \approx 6 \quad .$$

Another simplification usually adopted is the long-thin approximation, which means $\omega_{\kappa j} \ll \omega_{bj}$. Therefore I_{1j} and I_{6j} do not appear in the dispersion relation. Consequently, we define the small parameter $\varepsilon \equiv \omega_{\kappa i}/\omega_{bi}$ and assume $|\varepsilon| \ll 1$.

To evaluate I_{2h}, I_{4h}, and T_{5h}, we first note that

$$\omega_{*h}^{T} = \omega_{*h} - (v_{\perp h}/\alpha_{\perp h})^2(1 - A)\omega_{bh} \quad ,$$

where the ω_{bh} term is obtained by keeping E,μ constant in the spatial gradient of F_h, and $A = A_h$. In the limit $|\omega| \ll |\omega_{*h}|, |\omega_{bh}|$, we have

$$I_{2h} \approx \tfrac{1}{2}\left(\frac{\omega_{*h}}{\omega_{bh}} + A - 1\right) ,$$

$$I_{4h} \approx \tfrac{1}{2}\left[\frac{\omega_{*h}}{\omega_{bh}} + 2(A - 1)\right] \quad ,$$

and

$$I_{5h} \approx 2\left[\frac{\omega_{*h}}{\omega_{bh}} + 3(A - 1)\right] \quad .$$

The integral I_{3h} has to be evaluated to higher order in ε and ω/ω_{bh} because the lowest order terms cancel in D_3 due to the equilibrium perpendicular pressure balance. Evaluating I_{3h} to higher order yields

$$I_{3h} = \frac{\omega_{*h}}{\omega_{bh}}(1 - \frac{\varepsilon}{A}) + \frac{\omega}{\omega_{bh}}\left(\frac{\omega_{*h}}{\omega_{bh}} - 1 + \varepsilon\right) + (A - 1)(2 - \frac{\varepsilon}{A}) \quad .$$

Using the preceding approximations, D_1, D_2, and D_3 can be rewritten as

$$D_1 = \omega^2 + \frac{2\omega_{bi}^2}{b_i\beta_i}\left\{C - \varepsilon_1 - b_h\left[1 + \frac{\varepsilon}{2A} + \frac{\varepsilon}{2A}(A - 1)(2 + \beta_{\perp h})\right]\right\}$$

$$D_2 = \omega_{bi}\left\{\varepsilon_1 - C - \frac{\varepsilon}{2}\beta_c + b_h + \frac{3}{4A}\varepsilon b_h\left[1 - \beta_{\perp h}(A - 1)\right]\right\} \quad , \tag{26}$$

and

$$D_3 = \varepsilon_1 - C + b_h + \beta_c(1 + \frac{1}{\beta_{\perp h}}) - \frac{\omega}{\omega_{bh}}(1 + \frac{\beta_{\perp h}}{2}) \quad ,$$

where $\varepsilon_1 = (1 + A^{-1})\varepsilon$, $\beta_c = \beta_i + \beta_e$, $\varepsilon_o = \omega_{\kappa i}/\omega_{*i}$ and $C = (\omega/k_\perp V_A)^2$. It is interesting to note that, in D_3 of Eq. (26), the mirror instability driving term, $\beta_{\perp h}(1 - 3b_h/2)(1 - A)$, is cancelled by similar terms in I_{3h} and I_{5h} . As a result, the hot electron pressure anisotropy cannot drive the mirror instability in the high phase velocity limit.

Ignoring the products of small parameters in Eq. (26) and normalizing ω by $k_\perp V_A$, we obtain the local dispersion relation

$$d_1 d_3 + w d_2^2 = 0 \tag{27}$$

where

$$w = k_\perp^{-2}(\partial \ln B/\partial r)^2 ,$$
$$d_1 = z^2 + w(z^2 - \varepsilon_2) ,$$
$$d_2 = \varepsilon_2 - z^2 ,$$
$$d_3 = \varepsilon_2 - z^2 + \beta_c(1 + 1/\beta_{\perp h}) - qz ,$$
$$z = \omega/k_\perp V_A ,$$
$$q = k_\perp V_A(1 + \beta_{\perp h}/2)/\omega_{bh} ,$$

and

$$\varepsilon_2 = \varepsilon_1 + b_h$$

Equation (27) reduces to Eq. (28) of Ref. 1 if A - 1 . Therefore an analytic technique similar to that used in Ref. 1 can be applied to Eq. (27). The generalization of the Nelson-Van Dam-Lee (NVL) stability boundary[7] can be estimated from the condition:

$$\frac{\varepsilon_2^2}{\varepsilon_2 + \beta_c(1 + 1/\beta_{\perp h})} - \varepsilon_2 < 0 ,$$

which gives

$$\beta_c < \frac{2\varepsilon_o(1 + A^{-1}) - \beta_{\perp h} b_h}{1 + \beta_{\perp h}} \tag{28}$$

if $\varepsilon_2 < 0$. From Eq. (28), we conclude that the NVL stability boundary is lowered by the anisotropy and the hot electron gyroradius effects. From the expression of d_3 , it is also easy to conclude that anisotropy slightly improves the stability of the curvature driven compressional Alfvén mode because of the multiplier $1+A^{-1}$ in front of the curvature term ε . In addition to NVL and the compressional Alfvén stability boundaries, there is a new stable region provided by the hot electron gyroradius effect as pointed out in Ref. 1. This stable region, given by $\varepsilon_2 > 0$, becomes[8]

$$\beta_{\perp h} > 2\varepsilon_0 b_h^{-1}(1 + A^{-1}) \tag{29}$$

The analytic estimate given above is confirmed by a numerical solution of Eq. (27), which is a quartic equation in z . By fixing the following set of parameters: $r = 10$ cm, $\delta = N_h/N_i = 0.1$, $\varepsilon_0 = 0.05$, $L_n = |\partial \ell n N/\partial r|^{-1} = 2$ cm, and $\rho_h = 0.3$ cm, we are able to solve Eq. (27) and map out the stability region in β_c,$\beta_{\perp h}$ space for different values of poloidal mode number m , and anisotropy A . In Figures (1), the stability boundaries for $m = 5$, 10, and 20 are represented by dotted, solid and broken lines, respectively. Figure (1a) shows the stability boundaries for $A = 2$, while Fig. (1b) shows those for $A = 1$. When $m = 5$, we observe that $k_\perp L_n = (m/r)L_n \sim 1$ and, strictly speaking, the eikonal ansatz is not applicable. Therefore the stability region for $m = 5$ shown in Figs. 1 is not to be taken too seriously. Nonetheless, it shows the stable region is shrinking for low m , which agrees with previous work.[4] The behavior of the NVL stability boundaries in Figs. (1) with respect to m (for $m \geq 10$) and A agrees with the prediction of Eq. (28). The compressional Alfvén stability boundaries also behave as expected. The new stable region for $\beta_{\perp h} \gtrsim 0.5$ and $m = 20$ is roughly the same as predicted by Eq. (29).

V. CONCLUSION

The effect of an anisotropic hot electron pressure in an electron-ring plasma is carefully retained in the gyrokinetic derivation of the ballooning-interchange/compressional Alfvén eigenmode equation. We assume an equilibrium bi-Maxwellian distribution function for the hot electrons. In this set of eigen-equations, terms that drive the fluid fire-hose and mirror instabilities are readily identifiable. The fire-hose instability is uninteresting because we always have $p_\perp > p_\parallel$ for the hot electrons and also because the perturbed parallel vector potential, $\tilde{A}_\parallel$, decouples from $\tilde{\phi}$ and $\tilde{B}_\parallel$ in the flute limit. In addition, the fluid driving terms for the mirror instability cancel with the anisotropic contribution to the hot electron compressibility in the widely assumed high phase velocity $(\omega - \omega_{dh} \gg k_\parallel \alpha_{\parallel h})$ limit. Therefore, the mirror instability does not exist in this limit and only the curvature driven compressional Alfvén and interchange modes are possible.

The local dispersion relation for the interchange/compressional Alfvén modes is obtained explicitly including the effect of anisotropic hot electron pressure and finite hot electron gyroradius. All species are assumed to be in the high phase velocity regime. We conclude from this dispersion relation that: (i) the Nelson-Van Dam-Lee stability boundary is degraded in the presence of anisotropic hot electron pressure and the finite hot electron gyroradius effect; (ii) the stability of the curvature driven compressional Alfvén mode is improved by the anisotropy, and (iii) the finite hot electron gyroradius effect opens up a new stable region in high $\beta_{\perp h}$ for high poloidal mode number. However, for low poloidal mode number, the stable region shrinks and the eikonal ansatz employed in our study is no longer valid.

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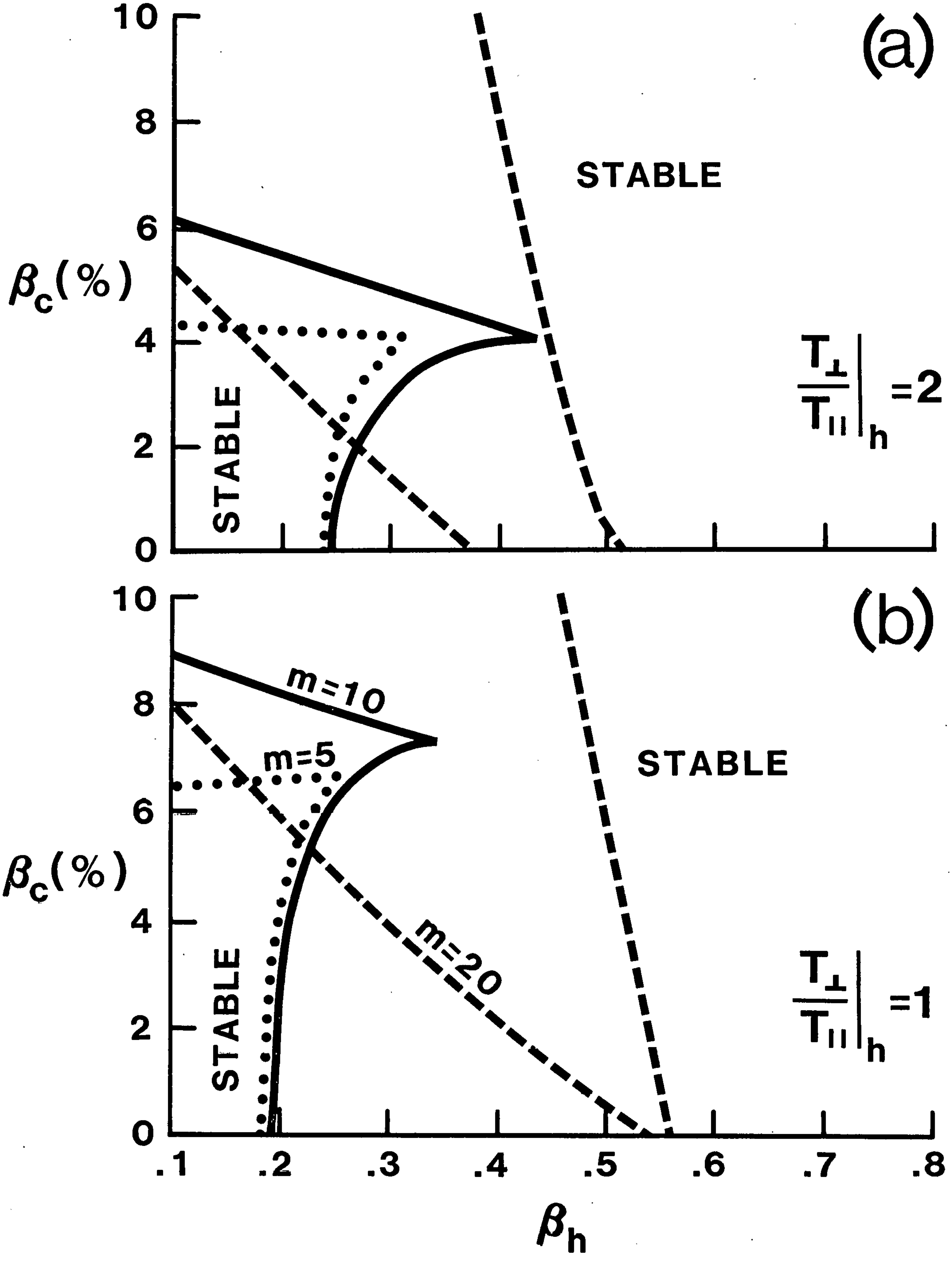

Fig. 1 Plot of marginal stability boundaries for m = 5 (dotted), m = 10 (solid), and m = 20 (broken) lines. Other parameters are r = 10 cm, $\delta = 0.1$, $\varepsilon_o = 0.05$, $L_n = 2$ cm, and $\rho_h = 0.3$ cm.