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LINEAR HELICAL STELLARATOR

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Stability Limit of a $l = 3$ Linear Helical Stellarator

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ABSTRACT

The stability limit of a linear helical system with small magnetic axis expansion was calculated.

Estimation of the stability limit of a toroidal stellarator is frustrated by the number of expansion parameters. Thus it is difficult to arrive at a physical insight of stability limit. Even numerical results are subject to question because the three-dimensional nature makes the stability estimate rather difficult. In what follows, we use the Suydam stability condition¹ with the assumption that the average magnetic field strength yields the field curvature necessary to evaluate the pressure gradient from the Suydam criterion.

Shafranov has expressions for ψ , ϕ , and $u = 1/2$ in convenient forms on pages 42-43 of Ref. 2. We shall adopt his notation and estimate the critical β for the straight stellarator. Here ψ is the magnetic surface and ϕ is the rotational transform.

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The magnetic potential is given by

$$\phi = B_0 \left(s + \varepsilon \frac{u'}{m-2} \rho^m \sin mu \right), \quad (1)$$

where s is the distance of a straight stellarator and $I_m[(2\pi n/L)\rho]$ was approximated by the first term, and $\vec{B} \cdot \vec{\nabla} \psi = 0$ leads to

$$\psi = \rho^2 + \frac{2\varepsilon}{m\rho^{m-2}} \rho^m \cos mu. \quad (2)$$

Here $B_0 = \text{constant}$ is the longitudinal field and

$$mu = m\omega + \frac{2\pi}{L} ns, \quad u' = \frac{2\pi}{L} \frac{n}{m}. \quad (3)$$

The cylindrical coordinates (ρ, ω, s) are used. L is the length of the stellarator ($=2\pi R$). It is also assumed that $\rho(2\pi n/L) = \rho n/R \ll 1$, and ρ_s is the radius where a separatrix exists (i.e., $\partial\psi/\partial\rho = 0$). In what follows, we shall look at $m \geq 3$. So we can let $\varepsilon = 1$. Near the axis

$$\frac{1}{2\pi} = \frac{(m-1)n}{m^2} \left(\frac{\rho^2}{\rho_s^2} \right)^{m-2}. \quad (4)$$

At the separatrix, if the above equation holds,

$$\frac{1}{2\pi} = \frac{m-1}{3m^2} n. \quad (5)$$

Now we calculate $\bar{B}^2/B_0^2$ in the area bounded by $\psi = \text{constant}$.

That is

$$\frac{B^2}{B_0^2} = \int d\omega \left[1 + 2 \frac{u'^2}{m^2 \rho_s^{m-2}} \cos^2 \mu u + \frac{u'^2}{\rho_s^2} \right. \\ \left. \times \left(\frac{\rho}{\rho_s} \right)^2 \left(\frac{2\pi n}{L} \right)^2 \cos^2 \mu u + \frac{u'^2 \rho^2}{m^2 \rho_s^2 (m-2)} \rho^{2(m-1)} \right] \quad (6)$$

and

$$\frac{\bar{B}^2}{B_0^2} = \frac{I}{\tau_0 B_0^2} \quad (7)$$

Near the axis $\psi = \text{constant}$ surface is approximated by $\rho_0 + \rho_1$

where

$$\rho_1 = - \frac{1}{c} \frac{\rho_0^m}{m \rho_s^{m-2}} \cos \mu u \quad (8)$$

Hence to the lowest order

$$\frac{I}{B_0^2} = \int d\omega \left(\frac{1}{2} \rho_0^2 + 2 \left(\frac{2\pi n}{L} \right)^2 \frac{1}{m^2} \frac{\rho_0^{m+1}}{\rho_s^{m-2}} \cos(\mu u) \cdot \rho_1 \right. \\ \left. + \frac{1}{2m^3} \left(\frac{2\pi n}{L} \right)^2 \frac{\rho_0^{2m}}{\rho_s^{2m-2}} \right) \\ = - \frac{2}{c} - 2 \left(\frac{2\pi n}{L} \right)^2 \frac{1}{m^3} \frac{\rho_0^{2m}}{\rho_s^{2m-4}} + \frac{1}{m^3} \left(\frac{2\pi n}{L} \right)^2 \frac{\rho_0^{2m}}{\rho_s^{2m-2}}$$

$$= \pi \rho_o^2 \left[1 - \left(\frac{2\pi n o s}{Lm} \right)^2 \frac{1}{m} \left(\frac{\rho_o}{\rho_s} \right)^{2m-2} \right] \quad (9)$$

or

$$\frac{r}{B_o^2 \pi \rho_o^2} = \frac{\bar{B}^2}{B_o^2} = 1 - u^2 \frac{1}{m} x_o^{2m-2} \quad (10)$$

where

$$u = \frac{2\pi n o s}{Lm} \ll 1 \quad (11)$$

and $x_o \equiv \rho_o / \rho_s$.

Since $\bar{B}^2 / B_o^2$ is the average in the domain bordered at $\psi = \rho_o^2$, the average magnetic field at ρ_o , $\bar{B}^2(\rho_o)$, is given by

$$\frac{\bar{B}^2(\rho_o)}{B_o^2} = \frac{1}{2\pi \rho_o} \frac{d}{d\rho_o} \pi \rho_o^2 \frac{\bar{B}^2}{B_o^2} = \frac{1}{2\rho_o} \frac{d}{d\rho_o} \rho_o^2 \frac{\bar{B}^2}{B_o^2} = 1 - u^2 x_o^{2m-2}. \quad (11a)$$

Thus, the average curvature $1/R_o$ is given by

$$\frac{1}{R_c} = \left| \frac{d}{d\rho_o} \left(\frac{\bar{B}^2(\rho_o)}{B_o^2} \right)^{1/2} \right| = \frac{m-1}{\rho_s} u^2 x_o^{2m-3}. \quad (11b)$$

Now we estimate $1/L_s$ as

$$\frac{1}{L_s} = \frac{c}{2-R} \frac{d}{dc} . \quad (12)$$

From Eq. (4)

$$\frac{1}{L_s} = \frac{2}{R} \frac{(m-1)n(m-2)}{m^2} \left(\frac{c}{2} \right)^{m-2} . \quad (13)$$

The Suydam criterion then yields [see Eq. (20) of Ref. 1]

$$\beta \leq \frac{[(\ln p)' (\ln B_0)']^{-1}}{4L_s^2} \quad (= \beta_c) , \quad (14)$$

where the prime indicates differentiation in c . Usually $[(\ln p)'] \gg [(\ln B_0)']$, thus instead of Eq. (14), we write

$$\beta \leq \frac{[(\ln \beta)' (\ln \bar{B})']^{-1}}{4L_s^2} . \quad (15)$$

Using the expression obtained in Eqs. (11b) and (13) [Note

$$[(\ln B_0)'] = 1/R_c]$$

$$\frac{d\beta}{dc} \geq - \frac{R_c}{4L_s^2} = - \frac{1}{c_s} \frac{(m-1)^2 (m-2)^2}{m^2 (m-1)} x_0^{2m-5} . \quad (16)$$

We obtain for $m \geq 3$

$$-\beta(x_0) \geq - \frac{(m-1)(m-2)}{m^2} \frac{1}{2} (x_0^{2m-4} - x_0^{2m-4}) . \quad (17)$$

Here $\beta(x'_0) = 0$. The average β can be computed to yield

$$\langle \beta \rangle = \frac{\int_0^{x'_0} \beta(x) 2\pi x dx}{\pi x'^2_0} \leq \frac{(m-1)(m-2)^2}{4m^2} x'^{2m-4}_0. \quad (18)$$

If we let $x'_0 = 1/2$, we get

$$\langle \beta \rangle (m=3) \leq 1.39\% \text{ and } \langle \beta \rangle (m=4) \leq 0.176\%. \quad (19)$$

We now investigate what will happen if $l = 2$ helical winding is added. In the limit of $2\pi\rho_0 n/mL \rightarrow 0$, the contribution of $l = m = 2$ helical winding is only for the bad curvature and not for shear.

The expression corresponding to Eq. (6) is

$$\begin{aligned} \frac{I}{B^2_0} = \int d\omega \int \rho d\rho \left\{ 1 + \epsilon u'^2 \rho^2 \frac{2\pi n}{L} \cos mu + \epsilon^2 u'^2 \rho^2 \right. \\ \left. + 0 \left[\left(\frac{2\pi n}{Lm} \rho \right)^4 \right] \right\}. \end{aligned} \quad (20)$$

The average field $\bar{B}^2/B^2_0$ for small ϵ is

$$\frac{\bar{B}^2}{B^2_0} = \frac{I}{\pi \rho^2_0 B^2_0} = 1 - \frac{1}{2} u'^2 \epsilon^2 \rho^2_0, \quad (21)$$

$$\frac{\bar{B}^2(\rho_0)}{B^2_0} = \frac{1}{2\pi\rho_0} \frac{d}{d\rho_0} \pi \rho^2_0 \frac{\bar{B}^2}{B^2_0} = 1 - u'^2 \epsilon^2 \rho^2_0. \quad (22)$$

The curvature $1/R_c$ is

$$\frac{1}{R_c} = \frac{d}{ds} \left(\frac{B^2(z_0)}{B_0^2} \right)^{1/2} = u'^2 \varepsilon^2 z_0. \quad (23)$$

The curvature is additive, so for the $l = 2$ and $l = 3$ system,

$$\frac{1}{R_c} = \frac{u^2}{s} (2x_0^3 + \varepsilon^2 x_0). \quad (24)$$

Like Eq. (16), we obtain

$$\frac{ds}{dz} \geq - \frac{1}{s} \frac{4}{9} \frac{x_0^3}{2x_0^2 + \varepsilon^2}, \quad (25)$$

$$\frac{ds}{dx_0} \geq - \frac{2}{9} x_0 + \frac{2}{9} \frac{\varepsilon^2 x_0}{2x_0^2 + \varepsilon^2}, \quad (26)$$

or

$$\varepsilon(x_0) \leq \frac{1}{9} (x_0'^2 - x_0^2) - \frac{\varepsilon^2}{18} \ln \left(\frac{2x_0'^2 + \varepsilon^2}{2x_0^2 + \varepsilon^2} \right). \quad (27)$$

If we impose the condition that ε due to $l = 2$ at $x_0 = 0$ equals $\varepsilon(z_s/2)$ by $l = 3$, we get

$$\frac{\varepsilon^2}{4} = \frac{1}{18} \quad \text{or} \quad \varepsilon = \frac{2^{1/2}}{3}. \quad (28)$$

The maximum ε at $x_0 = 0$ for $x_0' = 1/2$ is reduced from 2.78% to 1.32% and average ε is reduced from 1.39% to 0.82%.

The results seem to be rather pessimistic, but deviation from the small x expansion will increase the β limit. This will be treated in a following paper.

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