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By

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HIGH- n IDEAL AND RESISTIVE SHEAR ALFVÉN WAVES IN TOKAMAKS

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ABSTRACT

Ideal and resistive MHD equations for the shear Alfvén waves are studied in a low- β toroidal model by employing the high- n ballooning formalism. The ion sound effects are neglected. For an infinite shear slab, the ideal MHD model gives rise to a continuous spectrum of real frequencies and discrete eigenmodes (Alfvén-Landau modes) with complex frequencies. With toroidal coupling effects due to nonuniform toroidal magnetic field, the continuum is broken up into small continuum bands and new discrete toroidal eigenmodes can exist inside the continuum gaps. Unstable ballooning eigenmodes are also introduced by the bad curvature when $\beta > \beta_c$. The resistivity (η) can be considered perturbatively for the ideal modes. In addition, four branches of resistive modes are induced by the resistivity: (1) Resistive entropy modes which are stable with frequencies going to zero with resistivity as $\eta^{1/3}$, (2) Tearing modes which are stable ($\Delta' < 0$) with frequencies approaching zero as $\eta^{3/5}$, (3) Resistive periodic shear Alfvén waves which approach the finite frequency end points of the continuum bands as $\eta^{1/2}$, and (4) Resistive ballooning modes which are purely growing with growth rate proportional to $\eta^{1/3}\beta^{2/3}$ as $\eta \rightarrow 0$ and $\beta \rightarrow 0$.

I. INTRODUCTION

The properties of the shear Alfvén waves in a magnetically confined plasma have been extensively studied due to their important implications in plasma heating and instabilities. The properties of the shear Alfvén continuum are crucial to the resonant absorption and mode conversion of the shear Alfvén waves in a nonuniform plasma. The existence of marginally stable global shear Alfvén eigenmodes suggests a new efficient heating scheme. On the other hand, various shear Alfvén instabilities can be excited either by directly tapping the plasma free energy associated with the inhomogeneities in the plasma current, pressure, and magnetic field, or through the combined effect of plasma resistivity and the plasma inhomogeneities. Therefore, a thorough understanding of the shear Alfvén eigenmodes in tokamaks is essential. In this paper, we will be devoted to studying all possible eigenmodes of the high- n (n is the toroidal mode number) shear Alfvén waves in tokamaks by using both the ideal and the resistive MHD models.

Neglecting the ion sound effects, the equation governing the small amplitude shear Alfvén oscillations derived from the linearized incompressible, ideal MHD equations in an infinite plane slab with a sheared magnetic field, $\vec{B} = B_0 \hat{e}_z + B_y(x) \hat{e}_y$, is given by

$$\left(\frac{d}{dx} [\omega^2 - \tilde{\omega}_A^2(x)] \frac{d}{dx} - k_y^2 [\omega^2 - \tilde{\omega}_A^2(x)] \right) \phi(x) = 0, \quad (1)$$

where $\tilde{\omega}_A(x) = k_{\parallel} v_A$, $k_{\parallel} = \vec{k} \cdot \vec{B}/B$, $v_A^2 = B_0^2/\rho$ is the Alfvén speed, and ρ is the mass density. This type of equation has been extensively studied [1-6].

With the shear Alfvén frequency $\tilde{\omega}_A(x)$ replaced by the plasma frequency $\omega_p(x)$, Eq. (1) is essentially the same as the plasma wave equation for describing electrostatic oscillations in a nonuniform cold plasma which has

been studied by Barston [1] and Sedlacek [2]. By a normal mode approach, Barston [1] solved Eq. (1) with the boundary condition that $\phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$, and obtained a continuous spectrum given by the singularities of Eq. (1), i.e., $\omega = \tilde{\omega}_A(x)$. For any frequency ω_0 in the continuum, the eigenfunction $\phi(x)$ has a logarithmic or $1/x$ singularity at the resonant surface x_0 , where $\omega_0^2 = \tilde{\omega}_A^2(x_0)$. The integral superposition of these normal modes leads to damped noncollective modes with amplitude decaying algebraically in time due to phase mixing.

Later, Sedlacek [2] employed the Laplace transform technique to solve the corresponding initial value problem of Eq. (1) by means of the Green's function for a monotonic density profile composed of three linear sections. He showed that the Green's function is a multivalued analytic function of the complex frequency ω with a branch cut at $\omega^2 = \tilde{\omega}_A^2(x)$, which corresponds to the Barston's normal modes. Moreover, he found new simple poles of the Green's function in the lower half of the ω plane as in the case of Landau damping. The poles represent exponentially damped shear Alfvén waves which we call shear Alfvén-Landau eigenmodes. The corresponding eigenfunctions in general exhibit nonanalyticity and can become analytical through nonideal effects such as resistivity or viscosity. These eigenmodes represent surface waves and exist only if there is a region of nonuniformity in ω_A . When the region of nonuniformity vanishes, the dispersion relation yields a volume wave which travels undamped with exactly the Alfvén velocity.

In an axisymmetric toroidal plasma, the magnitude of the toroidal magnetic field is also nonuniform over a magnetic surface. This nonuniformity of the toroidal magnetic field can cause coupling of neighboring poloidal harmonics and results in the breakup of the shear Alfvén continuous spectrum into small bands of continuous spectra [7-9]. In addition, we find in this

paper that stable toroidal shear Alfvén eigenmodes can exist inside the gaps. The gaps between the continuum bands correspond to the local shear Alfvén frequencies located between the corresponding rational surfaces. Dewar et al. [10] and Chance et al. [11] found that such gaps in the continuous spectra of a straight plasma column can also arise from ellipticity of the plasma cross section.

If the magnetic field curvature is taken into account, its combined effect with the plasma pressure gradient can further excite unstable ballooning modes in the bad curvature region [12-15] when β (the ratio of the kinetic pressure to the magnetic pressure) lies between two critical values β_1 and β_2 ($\beta_1 < \beta_2$).

When the plasma resistivity η is considered, several branches of resistive shear Alfvén modes are created. The frequencies of these resistive branches have different η scalings and tend either to zero (local shear Alfvén frequency at rational surface), or to the end points of the ideal continuum bands, or to the ideal toroidal shear Alfvén modes.

In this paper, we will systematically examine the shear Alfvén waves, the toroidal coupling effects on the shear Alfvén spectrum, and the resistivity dependence of the shear Alfvén modes in a low- β , large aspect ratio, axisymmetric toroidal plasma. The WKB-ballooning formalism will be employed to study the high- n (n is the toroidal mode number) shear Alfvén waves. We will show that the nonuniform toroidal field not only breaks up the shear Alfvén continuum, but also creates new discrete marginally stable modes in the continuum gaps. With plasma resistivity, we have classified six branches of discrete eigenmodes in the resistive MHD model: (1) the shear Alfvén-Landau modes which are insensitive to resistivity and toroidicity, (2) the resistive entropy modes [16-18] which are stable with frequencies approaching to zero

with resistivity as $\eta^{1/3}$, (3) the tearing modes [19] which converge to zero as $\eta^{3/5}$ (4) the resistive periodic shear Alfvén modes which approach the finite frequency end points of the continuum gaps as $\eta^{1/2}$ (5) the toroidal shear Alfvén mode which approaches the marginally stable mode in the continuum gap as $\eta \exp(-i\pi/2)$, and (6) the resistive ballooning modes [20-25] which are purely growing with growth rates proportional to $\eta^{1/3} \beta^{2/3}$.

The theoretical model and the corresponding shear Alfvén wave equation are given in Sec. II. In Sec. III, we discuss the shear Alfvén continuum and the shear Alfvén-Landau modes in an infinite shear slab with the ideal MHD model. We then discuss the breakup of the shear Alfvén continuum due to the toroidal coupling effects in Sec. IV. The introduction of the marginally stable ideal toroidal shear Alfvén modes and the ballooning modes are also presented. In Sec. V, we examine the effect of plasma resistivity on the spectrum. The analytical theories and the numerical solutions will be presented for the resistive entropy modes and the tearing modes in Sec. (V-1), the resistive periodic shear Alfvén waves in Sec. (V-2), and for the resistive ballooning modes in Sec. (V-3). Finally, a summary and discussion will be given in Sec. VI.

II. SHEAR ALFVÉN WAVE EQUATIONS

Our model is based on the linearized MHD fluid equations in a nonuniform medium. These equations are solved for perturbations subject to certain boundary conditions in $\vec{x}$ space and to the initial conditions of the perturbations. The initial value problem is approached by the Laplace transform for the perturbation $\xi(\vec{x}, t)$ defined by

$$\xi(\vec{x}, \omega) = \int_0^\infty \xi(\vec{x}, t) e^{i\omega t} dt. \quad (2)$$

$\xi^+(\vec{x}, \omega)$ is found by the Green's function technique and is formally given by

$$\xi^+(\vec{x}, \omega) = \int d\vec{x}' G(\vec{x}, \vec{x}', \omega) \xi^+(\vec{x}', t=0), \quad (3)$$

where the Green's function G is inversely proportional to the dispersion function $D(\omega)$, which is determined by the solution of the governing homogeneous equations. In the case of one-dimensional second order differential equations, such as Eq. (11), the Green's function can be expressed in terms of two linearly independent solutions of the governing homogeneous equation, the first ξ_1 satisfying the boundary condition at $x = -\infty$ and the second ξ_2 at $x = \infty$. G can be expressed explicitly as [26]

$$G(x, x', \omega) = D^{-1}(\omega) [\xi_1(x) \xi_2(x') H(x' - x) + \xi_2(x) \xi_1(x') H(x - x')],$$

where $H(x)$ is the unit step-function, and D is related to the Wronskian of ξ_1 and ξ_2 and is independent of x and x' , i.e., $D(\omega) = [\omega^2 - \omega_A^2(x)] \{ [d\xi_2(x)/dx] \xi_1(x) - [d\xi_1(x)/dx] \xi_2(x) \}$. The time behavior of the $\xi(\vec{x}, t)$ can then be obtained by the inverse Laplace transform,

$$\xi(\vec{x}, t) = \frac{1}{2\pi} \int_C \xi(\vec{x}, \omega) e^{-i\omega t} d\omega, \quad (4)$$

where the contour integration path C is parallel to the real axis of the complex ω -plane and is above all singularities of $\xi(\vec{x}, \omega)$ due to the causality condition. The Green's function G has branch cuts and poles. The branch cuts, which are related to singular solutions of the homogeneous equations, give rise to the continuous spectrum. The poles, which are determined by the

zeros of the dispersion function $D(\omega)$, give rise to a discrete spectrum. From the Cauchy residue Theorem, Eq. (4) can be evaluated as

$$\vec{\xi}(\vec{x}, t) = \frac{1}{2\pi} \int_{C_1} \vec{\xi}(\vec{x}, \omega) e^{-i\omega t} d\omega + i \sum_j \text{Res } \vec{\xi}(\vec{x}, \omega_j) e^{-i\omega_j t}, \quad (5)$$

where C_1 represents the contour paths around the branch cuts and the summation runs over all the poles. For logarithmic singularities of the homogeneous solutions, the contour integration along C_1 leads to a well-behaved asymptotic solution decaying algebraically in time due to phase mixing.

We now derive the eigenmode equation for the shear Alfvén waves. Applying the Laplace transform, the linearized MHD fluid equations are given by

$$-i\omega\rho\vec{v} = \vec{j} \times \vec{B} + \vec{J} \times \delta\vec{B} - \nabla\delta p, \quad (6)$$

and

$$\vec{E} + \vec{v} \times \vec{B} = \eta\vec{j}, \quad (7)$$

where ρ , $\vec{B}$, $\vec{J}$, and η are the equilibrium density, magnetic field, current density, and resistivity, respectively. Note that we have neglected the initial conditions in Eqs. (6) and (7), since we will be devoted to studying the eigenmode problem. The perturbed quantities are the electric field $\vec{E}$, the magnetic field $\delta\vec{B}$, the current density $\vec{j}$, the velocity $\vec{v}$, and the pressure δp . In terms of the electrostatic potential ϕ and the vector potential $\vec{A}$, $\vec{E}$ and $\delta\vec{B}$ can be expressed as

$$\vec{E} = -(\nabla\phi - i\omega\vec{A}/c), \quad (8)$$

and

$$\delta \vec{B} = \nabla \times \vec{A} , \quad (9)$$

Equations (6) and (7) are to be closed by the quasineutrality condition

$$\nabla \cdot \vec{j} = 0 , \quad (10)$$

the adiabatic pressure law

$$-i\omega\delta p + \vec{v} \cdot \nabla P + \gamma_s P \vec{v} \cdot \vec{v} = 0 , \quad (11)$$

where γ_s is the ratio of specific heat, the Ampere's law

$$\nabla^2 \vec{A} = -\vec{j} , \quad (12)$$

and the equilibrium force balance equation

$$\vec{j} \times \vec{B} = \nabla P , \quad (13)$$

where P is the equilibrium pressure. From Eqs. (6) and (7) we obtain solutions for $\vec{v}$ and $\vec{j}$ given by

$$\begin{aligned} \vec{v} = & [\vec{E} \times \vec{B} + \eta(\vec{I} - \hat{e}_b \hat{e}_b) \cdot (\vec{j} \times \delta \vec{B} - \nabla \delta p)] / (B^2 - i\omega\eta) \\ & - i(\delta \vec{B} \cdot \nabla P + \vec{B} \cdot \nabla \delta p) \vec{B} / \omega \rho B^2 , \end{aligned} \quad (14)$$

$$\vec{j} = [(\vec{j} \times \delta\vec{B} - \nabla\delta p) \times \vec{B} - i\omega\rho (\vec{I} - \hat{e}_b \hat{e}_b) \cdot \vec{E}] / (B^2 - i\omega\rho\eta) + (\vec{E} \cdot \vec{B}) \vec{B} / \eta B^2, \quad (15)$$

where $\vec{I}$ is the unit dyadic tensor and $\hat{e}_b = \vec{B}/|\vec{B}|$.

Now in a toroidal plasma we consider perturbations with short wavelengths perpendicular to $\vec{B}$ in comparison with parallel wavelengths, $k_{\parallel}/k_{\perp} \ll 1$. The resistivity is ordered such that $S^{-1} = \tau_A/\tau_R \ll 1$, where the resistive diffusion time is $\tau_R = a^2/\eta$, the Alfvén time is $\tau_A = qR/V_A$, a is the minor radius, R is the major radius, $V_A^2 = B^2/\rho$, and S defines the magnetic Reynold's number. By ignoring effects associated with the fast magnetosonic waves and the slow sound waves, we will also neglect the $\nabla \cdot \vec{v}$ term in the adiabatic pressure law. Note that for a low- β , large aspect ratio tokamak, the $\nabla \cdot \vec{v}$ term is small when $\omega^2 \gg (c_s/qR)^2$, where c_s is the ion sound speed, and q the safety factor and R the major radius. With these assumptions, we have $\omega\rho\eta/B^2 \ll 1$, and

$$\nabla_{\perp}(\delta p + \delta\vec{B} \cdot \vec{B}) \approx 0. \quad (16)$$

The quasineutrality condition then reduces to [20]

$$\vec{B} \cdot \nabla \left[\frac{1}{\eta B^2} (\vec{B} \cdot \nabla\phi - i\omega \vec{A} \cdot \vec{B}) \right] - \frac{i\omega\rho}{B^2} \nabla_{\perp}^2 \phi + 2 \frac{\vec{k} \times \vec{B}}{B^2} \cdot \nabla_{\perp} \delta p = 0 \quad (17)$$

where the magnetic field curvature $\vec{k} = \hat{e}_b \cdot \nabla \hat{e}_b$. The parallel Ampere's law becomes

$$\nabla^2(\vec{A} \cdot \vec{B}) = \frac{1}{n} (\vec{B} \cdot \nabla \phi - i\omega \vec{A} \cdot \vec{B}), \quad (18)$$

and the equation of state reduces to

$$-i\omega \delta p + \frac{\partial P}{\partial \psi} \frac{\partial \phi}{\partial \alpha} = 0, \quad (19)$$

where α and ψ are related to the equilibrium magnetic field by $\vec{B} = \nabla \alpha \times \nabla \psi$. Eqs. (17)-(19) form the linearized resistive ballooning equations for the shear Alfvén waves.

Next, we consider perturbations with high toroidal mode number n and use $\delta \equiv 1/n \approx O(k_{\parallel}/k_{\perp})$ as an expansion parameter to develop an asymptotic solution of Eqs. (17)-(19). Employing the high- n ballooning mode representation [13], the perturbed quantities can be expressed as, e.g.,

$$\phi(\psi, \theta, \zeta) = \sum_{\ell=-\infty}^{\infty} \hat{\phi}(\psi, \theta - 2\pi\ell, \zeta), \quad (20)$$

where a (ψ, θ, ζ) coordinate system is adopted, and the domain of $\hat{\phi}$ extends from $-\infty$ to ∞ . If we let $\alpha = \zeta - q(\psi)\theta$ in a toroidal system, then $2\pi\psi$ is the poloidal flux between the magnetic axis and a constant ψ surface. θ and ζ are the generalized poloidal and toroidal angles with a period of 2π , and q is the safety factor and is a monotonic function of ψ only. In this coordinate system, $\vec{B} \cdot \nabla = J^{-1}(\partial/\partial\theta)_{\alpha, \psi}$, where the Jacobian $J = (\nabla\psi \times \nabla\theta \cdot \nabla\zeta)^{-1}$. It is clear that Eqs. (17)-(19) are satisfied by the perturbed quantities over an infinite range in θ with no periodicity constraint. Then we express $\hat{\phi}$ by the WKB representation,

$$\hat{\phi} = \phi(\psi, \theta, \delta) \exp[i\chi(\alpha, \psi)], \quad (21)$$

where χ describes the rapid cross field variations and ϕ the slow variations along the field lines on the equilibrium scale so that $\vec{B} \cdot \nabla \chi = 0$. For an axisymmetric toroidal system, χ is separable and can be expressed as

$$\chi = n(\alpha + \int \theta_K(\psi) dq), \quad (22)$$

where θ_K is to be determined by a higher order radial nonlocal analysis. In the lowest order in δ , Eqs. (17) - (19) then reduce to a single second order differential equation in θ for every ψ and θ_K . The final resistive ballooning equation is given by [20]

$$\begin{aligned} & \left(-1 \frac{\partial}{\partial \theta} \frac{|V\alpha|^2}{B^2 (1 + i\eta |V\chi|^2 / \omega)} \frac{\partial}{\partial \theta} \phi + \frac{\rho \omega^2}{B^2} |V\alpha|^2 \phi \right. \\ & \left. + 2 \frac{\partial P}{\partial \psi} \frac{\vec{k} \cdot \hat{e}_b \times \nabla \alpha}{B} \phi \right) = 0. \end{aligned} \quad (23)$$

For the purpose of analytical studies we consider an axisymmetric, large aspect ratio, low- β toroidal plasma with concentric, circular magnetic surfaces. Employing a (r, θ, ζ) coordinate system, where r is the minor radius, θ the poloidal angle and ζ is the toroidal angle, the equilibrium magnetic field is given by $\vec{B} = B_0 (1 - \epsilon \cos \theta) [\hat{e}_\zeta + (\epsilon/q) \hat{e}_\theta]$, $\epsilon = r/R$. The high- n resistive ballooning equation, Eq. (22), reduces to

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left[\frac{(1 + s^2 \theta^2)}{1 + \frac{i\nu}{\Omega} (1 + s^2 \theta^2)} \frac{\partial}{\partial \theta} \phi \right] + \Omega^2 (1 + 2\epsilon \cos \theta) (1 + s^2 \theta^2) \phi \\ & + \alpha_p (\cos \theta + s \theta \sin \theta) \phi = 0, \end{aligned} \quad (24)$$

where $s = r q' / q$, $v = \eta (n q / r)^2 / \omega_A$, $\Omega = \omega / \omega_A$, $\omega_A = V_A / q R$, $V_A^2 = B_0^2 / \rho$, $\alpha_p = -2 q^2 R (d p / d r) / B_0^2$, and we have set $\theta_K = 0$ for simplicity. The ϵ term is obtained from B^2 and $\mathcal{J}$ in Eq. (23) in the large aspect ratio limit. If we neglect the toroidal effect by setting $\epsilon = \alpha_p = 0$ and let $s \theta = k_x / k_y$, Eq. (24) becomes the resistive shear Alfvén wave equation in an infinite plane slab with a sheared magnetic field in the Fourier transformed space. The solutions of Eq. (24) will be presented in the following sections.

III. SHEAR ALFVÉN CONTINUUM AND THE SHEAR ALFVÉN

LANDAU MODES IN AN INFINITE SHEAR SLAB

We first study Eq. (24) in the absence of resistivity and toroidal effects ($v = \epsilon = \alpha_p = 0$). In this limit, Eq. (24) deals with shear Alfvén waves in an infinite plane slab with sheared magnetic field; its x space representation is given by Eq. (1) with $\tilde{\omega}_A^2(x) = (\omega_A k_y x)^2$ and can be expressed in the form

$$\left[\frac{d}{dr} (\Omega^2 - r^2) \frac{d}{dr} - (\Omega^2 - r^2) \right] \phi = 0 \quad (25)$$

where $r = k_y x$, x is measured from the mode rational surface. It is clear that Eq. (25) will give rise to singular solutions for $\Omega^2 = r_0^2$, which defines the continuous spectrum. The singular solutions are singular at every point r_0 where $\Omega^2 = r_0^2$. For $\Omega^2 = 0$, the solution of Eq. (25) for $r \ll 1$ exhibits a singular behavior as r^{-1} and can be written as

$$\phi = C_1 + C_2 / r \quad (26)$$

where C_1 and C_2 are two constants to be determined by boundary conditions. For $\Omega^2 = r_0^2 \neq 0$, the solution of Eq. (25) has a logarithmic singularity at r_0 and can be written as

$$\phi(z) = C_1 + C_2 \ln|r - r_0| \quad (27)$$

It is also worthwhile to know the solutions of the continuous spectrum in the Fourier transformed space, in which the shear Alfvén wave equation is given by

$$\left[\frac{d}{d\theta} (1 + s^2 \theta^2) \frac{d}{d\theta} + \Omega^2 (1 + s^2 \theta^2) \right] \phi = 0. \quad (28)$$

For large $|\theta|$, the asymptotic solution for finite Ω can be written as a superposition of right and left going waves, i.e.,

$$\phi \sim [C_1 \exp(i\Omega\theta) + C_2 \exp(-i\Omega\theta)]/\theta. \quad (29)$$

For $|r \pm \Omega| \ll 1$ the corresponding real space solution, which is defined by the inverse Fourier transform of $\phi(\theta)$, is determined by the large θ solution and is given by Eq. (27). For $\Omega = 0$, the solution of Eq. (28) is given by

$$\phi(\theta) = C_1 + C_2 \tan^{-1}(s\theta) \quad (30)$$

The corresponding real space solution for $|r| \ll 1$ is determined by the large θ behavior of Eq. (30) and is given by Eq. (26)

As pointed out by previous authors [2-6], Eq. (25) will admit discrete complex frequency eigenmodes (shear Alfvén-Landau modes) by analytic

continuation of the Green's function into the lower half of the ω plane. In our case, it is easier to consider the solution of the shear Alfvén-Landau modes in the Fourier transformed space, described by Eq. (28). First, we transform Eq. (28) by setting $\phi = (1 + s^2 \theta^2)^{1/2} \psi$ and we have

$$\psi'' + Q\psi = 0, \quad (31)$$

where $Q = \Omega^2 - s^2/(1 + s^2 \theta^2)^2$. For large θ , $Q \approx \Omega^2$ and $\psi \sim \exp(\pm i\Omega\theta)$. Due to the causality condition, ψ must be spatially decaying for unstable eigenmodes ($\text{Im}\Omega > 0$) and we have $\psi \sim \exp(i\Omega\theta)$ as $\theta \rightarrow \infty$. By analytic continuation of θ into the complex plane, we can solve the shear Alfvén-Landau modes by the WKB method with the knowledge of the turning points θ_T defined by $Q(\theta_T) = 0$. The corresponding WKB eigenvalue conditions are then determined by

$$\int_{-\theta_T}^{\theta_T} Q^{1/2} d\theta = (N + 1/2)\pi, \quad N = 0, 1, \dots \quad (32)$$

For the shear Alfvén-Landau modes, Q can be approximated by

$$Q \approx \Omega^2 - s^2(1 - 2s^2\theta^2) \quad (33)$$

for $|s\theta| < 1$. Then Eq. (31) becomes a Weber-type equation and the eigenvalue is given by

$$\Omega^2 = s^2[1 - i \hat{\alpha} \sqrt{2} (2N + 1)], \quad (34)$$

where $\hat{\alpha} = \text{Sign}(\text{Re}\Omega)$. In order to better understand the solutions of the shear Alfvén-Landau modes, it is helpful to examine the corresponding Stokes'

diagram of the function $Q(\theta)$, which is shown in Fig. 1, for the case with $s = 1$ and the corresponding eigenvalue is $\Omega = 0.681 - 0.616i$. In Fig. 1 there are two pairs of turning points ($\pm\theta_T$ and $\pm\theta_T^*$), one pair of poles ($\pm\theta_S$), and the associated Anti-Stokes lines are also shown. The boundary condition, $\phi \sim \exp(i\Omega\theta)$ as $\theta \rightarrow \infty$, dictates that regions (I) and (I') be dominant, and hence, regions (II) and (II') be subdominant. It is clear that the eigenfunctions of the shear Alfvén-Landau mode is bounded between $\pm\theta_T$ and is unbounded along real θ . The eigenvalue Ω is obtained by a numerical shooting code along the shooting axis as shown in Fig. 1. We note that the analytical eigenvalue, given by Eq. (34), is roughly 50% larger than the numerical shooting solution and is roughly equal to the numerical solution of the WKB condition, Eq. (32), $\Omega = 0.973 - 0.7178i$.

IV. BREAKUP OF THE CONTINUOUS SPECTRUM AND THE TOROIDAL SHEAR ALFVEN WAVES IN A TOROIDAL SYSTEM

In this section we study Eq. (24) in the ideal limit ($\nu = 0$). It has been extensively studied [12-15] that unstable ballooning modes are introduced by the local bad curvature effects when $\alpha_p > \alpha_{pc}$. Our numerical results also show that the shear Alfvén-Landau modes are insensitive to the toroidal effects. Therefore, we will only consider the toroidal coupling effects due to finite ϵ on the continuous spectrum in this section. We find that the coupling of neighboring poloidal harmonics results in the breakup of the continuous spectrum into small continuum bands with the gaps located at the intersections of the shear Alfvén frequencies of different poloidal harmonics. In addition, we will show that marginally stable toroidal shear Alfvén eigenmodes can exist inside the gaps. Since the curvature (finite α_p) has little effect on the discrete Alfvén waves and the continuum gaps, for simplicity we will set $\alpha_p = 0$ in the following discussion.

By setting $\alpha_p = v = 0$ and $\phi = (1 + s^2 \theta^2)^{1/2}$, Eq. (24) reduces to

$$\left[\frac{d^2}{d\theta^2} + \Omega^2 (1 + 2\epsilon \cos\theta) - F \right] \phi = 0, \quad (35)$$

where $F = s^2 / (1 + s^2 \theta^2)^2$. At first, we note that for large θ , F ($\sim \theta^{-4}$) is negligibly small and Eq. (37) is essentially the Mathieu Equation [27] which admits infinite pairs of real characteristic values Ω . These pairs of characteristic values define the gaps between the continuum bands. The lowest characteristic values are approximately given by $\Omega^2 \approx 1/[4(1 \pm \epsilon)]$. For Ω inside the continuum gaps, ϕ is divergent at large θ . By performing a two scale analysis on Eq. (35), we will then show the existence of the stable toroidal shear Alfvén wave inside the gap defined by Eq. (35) due to the contribution from F .

Consider $\Omega^2 = 1/4$ and let

$$\phi = \psi_c(\theta) \cos(\theta/2) + \psi_s(\theta) \sin(\theta/2), \quad (36)$$

where ψ_c and ψ_s have slow θ variations compared with the fast θ variations in $\cos(\theta/2)$ and $\sin(\theta/2)$. After averaging over fast θ , Eq. (35) becomes

$$\psi_c'' + Y_+ \psi_c + \psi_s' = 0, \quad (37)$$

and

$$\psi_s'' + Y_- \psi_s - \psi_c' = 0, \quad (38)$$

where $Y_{\pm} = \Gamma_{\pm} - F$, $\Gamma_{\pm} = \Omega^2(1 \pm \varepsilon) - 1/4$, $\Psi'' = d^2\Psi/d\theta^2$. If we let $x = s\theta$, then Eqs. (37) and (38) can be combined to give

$$\left[\left(\frac{d}{dx} \frac{s^2}{Y_-} \frac{d}{dx} + 1 \right) \left(s^2 \frac{d^2}{dx^2} + Y_+ \right) + \frac{d}{dx} \frac{s^2}{Y_-} \frac{d}{dx} \right] \Psi_c = 0. \quad (39)$$

For $\varepsilon \ll 1$, we will solve Eq. (39) by the method of asymptotic matching by considering four regions of x : (i) $x \sim 1$, (ii) $x \sim \varepsilon^{-1/4}$, (iii) $x \sim \varepsilon^{-1/2}$, and (iv) $x \sim \varepsilon^{-1}$. Taking $\Gamma_{\pm} \sim O(\varepsilon)$, then in these four regions, Eq. (39) reduces to

$$\left(s^2 \left[\frac{d}{dx} (1 + x^2)^2 \frac{d}{dx} - 1 \right] \left[\frac{d^2}{dx^2} - \frac{1}{(1 + x^2)^2} \right] + \frac{d}{dx} (1 + x^2)^2 \frac{d}{dx} \right) \Psi_1 = 0 \quad (40)$$

in the region (i), to

$$\frac{d}{d\theta_2} \left(\frac{d\Psi_2/d\theta_2}{\Gamma_- - (s^2\varepsilon/\theta_2^4)} \right) = 0 \quad (41)$$

in the region (ii) with $\theta_2 = \varepsilon^{1/4}x$, and to

$$\frac{d^2\Psi_3}{d\theta_3^2} = 0 \quad (42)$$

in the region (iii) with $\theta_3 = \varepsilon^{1/2}x$, and to

$$\left[\frac{d^2}{d\theta_4^2} + \left(\frac{\Gamma_+ \Gamma_-}{\varepsilon^2 s^2} \right) \right] \Psi_4 = 0 \quad (43)$$

in the region (iv) with $\theta_4 = \varepsilon x$.

The solutions of Eqs. (41) - (43) in regions (ii) - (iv) are given by

$$\psi_2 = b_0 (\Gamma_- \epsilon^{1/4} x + s^2 \epsilon^{1/4} / 3x^3) + b_1, \quad (44)$$

$$\psi_3 = c_1 + c_2 \epsilon^{1/2} x, \quad (45)$$

and

$$\psi_4 = d \exp[-(-\Gamma_+ \Gamma_-)^{1/2} x/s], \quad (46)$$

where b_0 , b_1 , c_1 , c_2 , and d are integration constants.

It is difficult to obtain an exact solution of Eq. (40) in region (i). Instead, we will obtain solutions of Eq. (40) in the limits $s \ll 1$ and $s \gg 1$. For $s \ll 1$, the solution of Eq. (40) is given by

$$\psi_1 = a_1 \psi_o + a_2 \psi_e, \quad (47)$$

where

$$\psi_o = \psi_o(0) + s^2 \psi_o(1) + O(s^4),$$

$$\psi_e = \psi_e(0) + s^2 \psi_e(1) + O(s^4),$$

$$\psi_o(0) = \frac{x}{(1+x^2)} + \tan^{-1} x,$$

$$\begin{aligned} \psi_o(1) = & \frac{x}{(1+x^2)} \left[\frac{5}{8} + \frac{5}{12} (1+x^2)^{-1} + \frac{28}{3} (1+x^2)^{-2} \right] \\ & + \left[\frac{5}{8} + (1+x^2)^{-2} \right] \tan^{-1} x - \frac{1}{24} \left(\frac{x}{1+x^2} + \tan^{-1} x \right)^3, \end{aligned}$$

$$\psi_e^{(0)} = 1,$$

$$\begin{aligned} \psi_e^{(1)} = & -(1+x^2)^{-1} \left[\frac{1}{8} - (1+x^2)^{-1} - \frac{1}{12} (1+x^2)^{-2} \right] \\ & - \frac{1}{4} \tan^{-1} x [x(1+x^2)^{-1} + \frac{1}{2} \tan^{-1} x], \end{aligned}$$

and a_1 and a_2 are integration constants.

For $s \gg 1$, Eq. (40) can be solved by the method of asymptotic matching by considering $x \sim 1$ region and $x \sim s$ region. For $x \sim 1$, Eq. (40) reduces to

$$\left[\frac{d}{dx} (1+x^2)^2 \frac{d}{dx} - 1 \right] \left[\frac{d^2}{dx^2} - \frac{1}{(1+x^2)^2} \right] \hat{\psi}_1 = 0, \quad (48)$$

and the solution is given by

$$\hat{\psi}_1 = (a_1 + a_2 x \tan^{-1} x + a_3 x + a_4 \tan^{-1} x) (1+x^2)^{1/2}, \quad (49)$$

where a_1, a_2, a_3, a_4 are integration constants. For $x \sim s$, Eq. (40) reduces to

$$\frac{d}{dx} x^4 \frac{d}{dx} \left(\frac{d^2}{dx^2} + \frac{1}{s^2} \right) \hat{\psi}_1 = 0, \quad (50)$$

with the solution given by

$$\hat{\psi}_1 = \hat{a}_1 + \hat{a}_2 \left[\frac{s}{x} + I(x) \right] + \hat{a}_3 \cos\left(\frac{x}{s}\right) + \hat{a}_4 \sin\left(\frac{x}{s}\right), \quad (51)$$

where a_1, a_2, a_3, a_4 are integration constants, and $I(x)$ is related to the sine integral and is defined by

$$I(x) = \left(\int_0^\beta \frac{\sin(\beta)}{\beta + x/s} d\beta \right)_{\beta=0}. \quad (52)$$

For $x/s \ll 1$, $I(x) \sim O(x/s)$, and for $s/x \gg 1$, $I(x) \sim 2(s/x)^3 + O(s/x)^5$.

By asymptotic matchings of the solutions in Eqs. (44)-(51) we obtain the dispersion relation for the discrete toroidal shear Alfvén eigenmode. We first consider even parity solutions. For $s \ll 1$ the dispersion relation is given by

$$\Omega = \pm \frac{1}{2} \left(1 + \frac{\epsilon \left[\left(1 - \frac{s^2 \pi^2}{32} \right)^2 - \frac{s^2 \pi^2}{16} \right]}{\left[\left(1 - \frac{s^2 \pi^2}{32} \right) + \frac{s^2 \pi^2}{16} \right]} \right)^{-1/2}. \quad (53)$$

For $s \gg 1$, we have

$$\Omega = \pm \frac{1}{2} \left(1 - \epsilon \frac{(1 - \pi^2/64s^2)}{(1 + \pi^2/64s^2)} \right)^{-1/2}. \quad (54)$$

For odd parity we find that the discrete toroidal shear Alfvén eigenmode does not exist. To show this, we first obtain the dispersion relation from the asymptotic matching for $s \ll 1$,

$$\pi s (-\Gamma_+ \Gamma_-)^{1/2} = 4\Gamma_- \quad (55)$$

Since $\Gamma_+ > 0$ for $\Omega^2 > [4(1 + \epsilon)]$ and $\Gamma_- < 0$ for $\Omega^2 < [4(1 - \epsilon)]^{-1}$, Eq. (55) does not have solutions inside the continuum gap $[4(1+\epsilon)]^{-1} < \Omega^2 < [4(1 - \epsilon)]^{-1}$. Similar conclusion can be drawn for the $s \gg 1$ case.

Figure 2 shows the eigenvalues, Ω , versus shear, s , for the parameters $\alpha_p = \nu = 0$, and $\varepsilon = 0.2$. Numerical eigenfrequencies from solving Eq. (23) are consistent with the analytical solutions of Eqs. (53) and (54) for the toroidal shear Alfvén waves with even parity in θ . The numerical value of Ω approaches the lower bound of the gap, $[2(1+\varepsilon)^{1/2}]^{-1}$, as $s \rightarrow \infty$. A typical eigenfunction of the toroidal shear Alfvén wave is shown in Fig. 3 for the parameters $s = 1$, $\varepsilon = 0.2$, $\alpha_p = \nu = 0$, and $\Omega = 0.54$. The envelope of the eigenfunctions is highly peaked with $\theta \sim 1$, but has a slowly varying tail as $\exp[-(\Gamma_+ \Gamma_-)^{1/2} \theta]$, where $\Gamma_{\pm} \sim O(\varepsilon)$. The eigenfunction has a period of 4π , consistent with our two-scale representation of the eigenfunction.

V. RESISTIVE MHD EIGENMODES

When the plasma resistivity ν is introduced, its effects on the discrete ideal modes (the shear Alfvén-Landau modes, the toroidal shear Alfvén modes, and the ideal ballooning modes) can be considered as perturbations. For the toroidal shear Alfvén waves, the eigenfrequencies approach the ideal limit frequencies linearly with ν along the -90° line in the complex frequency plane. A plot of the eigenfrequencies versus ν is given in Fig. 4 with the parameters $\varepsilon = 0.2$, $\alpha_p = 0$, $s = 1$. A regular perturbation analysis has confirmed this $(-i\nu)$ scaling of the perturbed frequencies and will not be presented here.

In addition to these ideal modes, the plasma resistivity introduces four branches of resistive modes, the resistive entropy modes, the tearing modes, the resistive periodic shear Alfvén modes, and the resistive ballooning modes. In Sec. V-1, we will discuss the resistive entropy modes and the tearing modes, and in Sec. V-2 the resistive periodic shear Alfvén modes. The resistive ballooning modes will be presented in Sec. V-3, and its relations

with the ideal ballooning mode will also be discussed there. As $\nu \rightarrow 0$, these resistive modes will converge either to the finite frequency endpoints of the continuum gaps or to $\omega = 0$, the local shear Alfvén frequency at the rational surface.

V-1. RESISTIVE ENTROPY MODES AND TEARING MODES

The resistive entropy modes were first studied by Boris [16] in a finite slab geometry by employing a boundary layer analysis in x space. He found that the eigenfrequencies lie on loci at $\pm 30^\circ$ below the real ω -axis. As the resistivity ν goes to zero, the eigenfrequencies scale as $\nu^{1/3}$ and approach the extremum of the square of the local shear Alfvén frequencies. He gave an explanation of the 30° characteristic that when matching the ideal MHD solution to the resistive layer solution, the ordering of $\nu^{1/3}$ of the resistive layer makes the purely real ideal shear Alfvén frequency pick up a complex part by $\arg(i^{1/3}) = 30^\circ$. Recently, Ryu et al. [17] had confirmed Boris' results by numerical studies in a cylindrical geometry. Connor et al. [18] had studied this eigenmode branch in an infinite sheared slab in the Fourier transformed space. Their equation is essentially the same as Eq. (24) with $\epsilon = \alpha_p = 0$ and is solved by a power series analysis in the small ν limit with the eigenfrequencies given by

$$\omega = \nu^{1/3} (2N + 5)^{2/3} (3^{1/2} - 1)/2, \quad N = 0, 2, 4 \quad (56)$$

Here we will present the solutions of Eq. (24) for the resistive entropy modes by the method of asymptotic matching. For simplicity, we will set $\alpha_p = \epsilon = 0$ in Eq. (24) for this mode because these two parameters have only minor effects on its solutions. By setting $x = s\theta$, Eq. (24) becomes

$$\left\{ \frac{d}{dx} \frac{1+x^2}{\left[1 + \frac{iv}{\Omega} (1+x^2)\right]} \frac{d}{dx} + \left(\frac{\Omega}{s}\right)^2 (1+x^2) \right\} \phi = 0 \quad (57)$$

For $v \ll 1$, we will order $v/\Omega \ll 1$, $\Omega^2 \ll 1$, $s \sim 1$, and consider x in two regions, the ideal region ($x \sim 1$) and the resistive region ($x \gg 1$). For $x \sim 1$, Eq. (57) reduces to

$$\frac{d}{dx} (1+x^2) \frac{d}{dx} \phi_I = 0, \quad (58)$$

which has the solution,

$$\phi_I = c_1 + c_2 \tan^{-1} x, \quad (59)$$

where c_1 and c_2 are integration constants. In the resistive region ($x \gg 1$), Eq. (57) reduces to

$$\left[\frac{d}{dy} - \frac{y^2}{1+y^2} \frac{d}{dy} - Q^2 y^2 \right] \phi_R = 0, \quad (60)$$

where $y^2 = (iv/\Omega)x^2$, $Q^2 = i\Omega^3/s^2v$. With the substitution

$$\phi_R = \frac{1}{y^2} \frac{d}{dy} (y^{1/2} \psi) \quad (61)$$

Eq. (60) reduces to the Whittaker's equation

$$\frac{d^2 \psi}{dz^2} - \left(\frac{1}{4} + \frac{Q}{4z} + \frac{5}{16z^2} \right) \psi = 0 \quad (62)$$

where $z = Qy^2$. The asymptotic decaying solution of Eq. (62) for $\text{Re}(z) \rightarrow \infty$ is given by

$$\Psi = \exp\left(-\frac{z}{2}\right) z^{5/2} U\left(\frac{Q+5}{4}, \frac{5}{2}, z\right), \quad (63)$$

where $U(a,b,z)$ is the Kummer's function which is related to the confluent hypergeometric functions [27] $M(a,b,z)$ by

$$U(a,b,z) = \frac{\pi}{\sin \pi b} \left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right), \quad (64)$$

where Γ is the Gamma function. For $|z| \ll 1$, we have

$$\phi_R \approx 3\pi Q \left(\frac{1}{\Gamma((Q-1)/4)\Gamma(5/2)} + \frac{Q^{1/2}}{3y\Gamma((Q+5)/4)\Gamma(-1/2)} \right). \quad (65)$$

Now considering the even parity solution in the ideal region, we set $c_2 = 0$ in Eq. (59). Matching the resistive solution, Eq. (65), to the ideal solution, Eq. (59), we have the eigenvalue condition

$$Q = -(4N + 5), \quad N = 0, 1, 2, \dots, \quad (66)$$

and the eigenfrequencies of the resistive entropy mode with even parity is given by

$$\Omega = \pm v^{1/3} [s(4N + 5)]^{2/3} \exp\left(\mp \frac{i\pi}{6}\right). \quad (67)$$

Note that Eq. (67) is identical with Eq. (56) obtained by Connor et al. [18].

For the odd parity solution in the ideal region, we set $c_1 = 0$ in Eq. (59) and $\phi_I = c_2[(\pi/2) - (1/x)]$ for $x \gg 1$. Then Δ' , which is defined by the ratio of the coefficient of the asymptotically small solution to that of the large solution, is given by $\Delta' = -2/\pi$. Again, matching the resistive solution Eq. (65) to the ideal solution, we obtain the eigenvalue condition

$$\Delta' = \left(\frac{-i\Omega^5}{2v^3}\right)^{1/4} \frac{\Gamma[(Q-1)/4]\Gamma(5/2)}{3\Gamma[(Q+5)/4]\Gamma(-1/2)} \quad (68)$$

Note that for the resistive entropy modes, $(\Omega^5/v^3) \sim v^{-1/3} \gg 1$, and for Eq. (68) to have solutions we demand $\Gamma[(Q+5)/4] \gg 1$. If we write $\Omega = \Omega_0 + \delta$, where $\delta \ll \Omega_0$ and Ω_0 is given by Eq. (67), then Eq. (68) can be solved perturbatively and we have the frequency corrections of the odd parity resistive entropy modes

$$\frac{\delta}{\Omega_0} = \frac{(2is^2v^3/\Omega_0^5)^{1/4}\Gamma(-1/2)}{\pi(4N+5)\Gamma(-N-3/2)\Gamma(5/2)\{d/dx[\Gamma(x)^{-1}]\}_{x=-N}}, \quad N = 0, 1, 2, \dots \quad (69)$$

Note that δ/Ω_0 is of the order of $v^{1/3}$. Also note that the odd parity resistive entropy modes were not discussed by Connor et al. [18] because they did not match the resistive solution to the ideal solution.

From the numerical shooting solutions of the resistive entropy modes, the inclusion of ϵ and α_p only modifies the eigenvalues slightly. The numerical eigenfrequencies Ω versus resistivity are presented in Fig. 5 for the $N = 0$ even parity mode with the set of parameters $s = 1$, $\epsilon = \alpha_p = 0$. We see that for $v \ll 1$ we have $\Omega \sim v^{1/3}$, but for $v > 1$, $\text{Im}(\Omega) \sim v$. A typical even parity eigenfunction with $\Omega = (0.253, -0.1424)$ is plotted in Fig. 6 and the corresponding odd parity eigenfunction with $\Omega = (0.257 - 0.1516i)$ in Fig. 7 for

the parameters $s = 1$, $v = 10^{-3}$, $\epsilon = \alpha_p = 0$. Note that for large θ , both the even and odd parity solutions have almost identical behavior.

We should also note that Eq. (68) also admits the tearing modes with the frequency scaling $\Omega \sim v^{3/5}$. To obtain the tearing mode eigenvalues, we first note that $Q^2 = i\Omega^3/s^2v \sim v^{4/5} \ll 1$. By neglecting Q contribution in the Gamma functions in Eq. (68), the dispersion relation of the tearing modes is given by

$$\Delta' = \frac{2\Gamma(3/4)}{\Gamma(1/4)} \left(\frac{-i\Omega^5}{s^2v} \right)^{1/4} \quad (70)$$

or

$$\Omega = i v^{3/5} s^{2/5} \left(\frac{\Delta' \Gamma(1/4)}{2\Gamma(3/4)} \right)^{4/5} \quad (71)$$

Note that if $\Delta' > 0$, we will have purely growing tearing mode. In the present case, $\Delta' = -2/\pi$, and Ω is complex and proportional to $\exp(-3i\pi/10)$ and $\exp(13i\pi/10)$. Figure 8 shows the eigenfrequencies of the tearing mode versus v for the parameters $s = 1$, $\epsilon = \alpha_p = 0$. For $v < 1$, Ω has a $v^{3/5}$ dependence and for $v > 1$, Ω approaches $(1 - iv)$.

To understand the v scaling of Ω for $v \gg 1$, we let $\Omega = -iv + \delta$, $\delta \sim O(1)$, and we have $(\Omega/iv + 1) \sim O(v^{-1})$. Consider $x \sim v^{-1/2}$ and Eq. (57) reduces to

$$\left[\frac{d}{dx} \left(\frac{\delta}{iv} + x^2 \right)^{-1} \frac{d}{dx} + v^2 \right] \phi = 0 \quad (72)$$

Now let $\phi = dW/dx$ and Eq. (72) further reduces to a Weber-type equation for w

$$\left(\frac{d^2}{dx^2} - i \frac{v\delta}{s^2} + \frac{v^2 x^2}{s^2}\right)W = 0 \quad , \quad (73)$$

and the eigenvalues are given by

$$\Omega = -iv \pm (2M + 1)S, \quad M = 0, 1, 2, \dots \quad (74)$$

Note that $M = 0, 2, 4, \dots$ correspond to the odd parity modes of ϕ and $M = 1, 3, 5, \dots$ the even parity modes. From Fig. 5, we see that the even parity $N = 0$ eigenstate for $v \ll 1$ converges to the even parity $M = 1$ eigenstate as v becomes larger than one. We also find that the odd parity $N = 0$ eigenstates for $v \ll 1$ correspond to the odd parity $M = 2$ eigenstate for $v \gg 1$. On the other hand, the $M = 0$ eigenstate for $v \gg 1$ will transform into the stable odd parity tearing mode [19] with $v^{3/5}$ frequency scaling as $v \rightarrow 0$.

V-2. RESISTIVE PERIODIC SHEAR ALFVÉN WAVES

We have shown that in Sec. IV that the periodic nature of the toroidal magnetic field can break up the ideal slab continuous spectrum into bands of continuous spectra separated by the gaps. The size of the continuum gaps in Ω is of the order of ϵ . When the resistivity is present, we find the resistive periodic shear Alfvén waves with the eigenfrequencies lying on loci at $\pm 45^\circ$ below the real ω -axis, as $v \rightarrow 0$. These eigenfrequencies scale as $v^{1/2}$ and approach the endpoints of the continuum gaps. To study the resistive periodic shear Alfvén waves we will set $\alpha_p = 0$ in Eq. (24), because α_p has very little effect on the mode.

Letting

$$\Phi = \left(\frac{1 + s^2 \theta^2}{1 + (iv/\Omega)(1 + s^2 \theta^2)} \right)^{1/2} \phi \quad ,$$

Eq. (24) reduces to

$$\phi'' + Q\phi = 0, \quad (75)$$

where

$$Q = \Omega^2 \left[1 + \frac{iv}{\Omega} (1 + s^2 \theta^2) \right] (1 + 2\epsilon \cos \theta) - G,$$

and

$$G = s^2 \frac{[1 + (iv/\Omega)(1 - 3s^2 \theta^2)](1 + s^2 \theta^2)}{(1 + s^2 \theta^2)^2 [1 + (iv/\Omega)(1 + s^2 \theta^2)]^2}.$$

We now perform a two-scale analysis and write

$$\phi = \psi_c \cos(\hat{\Omega}_0 \theta) + \psi_s \sin(\hat{\Omega}_0 \theta), \quad (76)$$

where ψ_c and ψ_s have slow θ variations due to small resistivity and toroidicity ϵ , and $\hat{\Omega}_0 = \pm m/2$ ($m = 1, 2, \dots$) is where the continuum gaps are located. In the following we will show only the case with $\hat{\Omega}_0 = 1/2$. After averaging over fast θ variations, Eq. (75) becomes

$$\psi_c'' + X_+ \psi_c + \psi_s' = 0, \quad (77)$$

and

$$\psi_s'' + X_- \psi_s - \psi_c' = 0, \quad (78)$$

where $x_{\pm} = \Omega^2 [1 + (1 \pm \epsilon) (1 + s^2 \theta^2)] (1 \pm \epsilon) - 1/4 - G$. For $v \ll 1$, $vs^2 \theta^2 / \Omega < 1$, $vs^4 \theta^4 \sim 1$, and also assuming $\Psi'' < \Psi$, Ψ' , then Eqs. (77) and (78) can be combined into a Weber-type equation,

$$\Psi'' + \frac{[\Omega^2 (1 \pm \epsilon) (1 + vs^2 \theta^2 / \Omega) - 1/4]}{[1 + (\Omega^2 (1 \mp \epsilon) - 1/4)^{-1}]} \Psi = 0 \quad (79)$$

The eigenvalues are given by

$$\Omega^2 (1 \pm \epsilon) - 1/4 = (2N + 1) \left(\frac{-1 \pm v \Omega^2 (1 \pm \epsilon) [\Omega^2 (1 \mp \epsilon) + 3/4]}{\Omega^2 (1 \mp \epsilon) - 1/4} \right)^{1/2} \quad (80)$$

where $N = 0, 1, 2, \dots$. If we write $\Omega = \Omega_0 + \delta$ with $\delta \ll \Omega_0$, then we have

$$\Omega_{0\pm}^2 = 1/4(1 \pm \epsilon), \quad (81)$$

and

$$\delta_{\pm} = \mp (N + 1/2)s \left(\frac{-1 \pm v [\Omega_{0\pm}^2 (1 \mp \epsilon) + 3/4]}{\Omega_{0\pm}^2 (1 \pm \epsilon) [\Omega_{0\pm}^2 (1 \mp \epsilon) - 1/4]} \right)^{1/2} \quad (82)$$

Note that the $v^{1/2}$ scaling of $\delta_{\pm}$ is related to the $v^{1/2}$ scaling of the turning points of the potential Eq. (79).

Numerical solutions of Eq. (24) are shown in Fig. 9 for the resistive periodic shear Alfvén eigenfrequencies in the complex Ω - plane for the set of parameters $v = 10^{-6}$, $\epsilon = 0.2$, $\alpha_p = 0$, $s = 1$. Figure 9 clearly shows the 45° characteristics of the eigenfrequencies. The dependence of the eigenfrequencies in v is shown in Fig. 10 for the parameters $\alpha_p = 0$, $s = 1$, $\epsilon = 0.2$. In Fig. 10 the $v^{0.47}$ scaling of the damping rates is close to

the $v^{1/2}$ analytical prediction given in Eq. (76). A typical eigenfunction of the resistive periodic shear Alfvén mode is shown in Fig. 11 for the parameters $\alpha_p = 0$, $s = 1$, $\epsilon = 0.2$, $v = 10^{-5}$, and $\Omega = 0.44 - 0.0145i$. The period of the oscillation is roughly 4π , consistent with our analytical solution.

V-3 RESISTIVE BALLOONING MODES

In the presence of the magnetic field curvature ($\alpha_p \neq 0$), the ideal ballooning modes are excited from the ideal continuum when $\alpha_p > \alpha_{pc}$. When the plasma resistivity is introduced, the purely growing resistive ballooning modes²⁰⁻²⁴ are excited for all $\alpha_p > 0$. For small v and α_p , the growth rates scale as $v^{1/3} \alpha_p^{2/3}$ and accumulate at $\omega = 0$. For simplicity, we set $\epsilon = 0$ in Eq. (24).

For small v Eq. (24) can be simplified by a two-scale representation of ϕ with

$$\phi = \phi(\theta) + \phi_c(\theta) \cos\theta + \phi_s(\theta) \sin\theta, \quad (83)$$

where ϕ , ϕ_c , and ϕ_s are slowly varying quantities in comparison with the trigonometric functions. Substituting Eq. (83) into Eq. (24) and averaging over the fast varying scale, we obtain an envelope equation for ϕ ,

$$\left(\frac{d}{dx} \frac{(1+x^2)}{1+(v/\gamma)(1+x^2)} \frac{d}{dx} - \left(\frac{\gamma}{s} \right)^2 (1+x^2) + D_s \frac{[1+(v/\gamma)(1+x^2 x^2)]}{\{1+\gamma^2 [1+(v/\gamma)(1+x^2)]\}} \right) \phi = 0, \quad (84)$$

where $x = s\theta$, $\Omega = i\gamma$, and $D_s = \alpha_p^2 / 2s^2$.

Equation (84) may be solved by the method of asymptotic matching

when $v < 1$ and $D_s < 1$. We will take the ordering $\gamma \sim (vD_s)^{1/3}$, and introduce a small parameter $\lambda = (v^2/D_s)^{1/6} < 1$ so that $(v/\gamma) \sim \lambda^2$. Then, x can be divided into three regions: (1) $x \sim 1$, (2) $x \sim \lambda^{-1}$, and (3) $x \sim \lambda^{-1}D_s^{-1/2}$. For $x \sim 1$, Eq. (84) reduces to

$$\frac{d}{dx} (1 + x^2) \frac{d}{dx} \phi_I = 0. \quad (85)$$

For $x \sim \lambda^{-1}$, we have

$$\frac{d}{dy} \frac{y^2}{1+vy^2/\gamma\lambda^2} \frac{d}{dy} \phi_{II} = 0, \quad (86)$$

where $y = \lambda x$. Now defining $B = D_s v/\gamma - \gamma^2/s^2$ and assuming $B \sim D_s^2 \lambda^2$, Eq. (84) reduces to

$$\left[\frac{d^2}{dy^2} + \frac{v}{\gamma\lambda^2} \left(\frac{D_s + By^2}{\lambda^2} \right) \right] \phi_{III} = 0. \quad (87)$$

in the region $x \sim \lambda^{-1}D_s^{-1/2}$. The solution of Eqs. (85)-(87) are

$$\phi_I = a_1 + a_2 \tan^{-1} x, \quad (88)$$

$$\phi_{II} = b_1 + b_2 \left(\frac{vy}{\gamma\lambda^2} - y^{-1} \right), \quad (89)$$

and

$$\phi_{III} = U(-A, z), \quad (90)$$

where

$$A = (-v D_s^2 / 4 \gamma B)^{1/2} > 0, \quad (91)$$

a_1, a_2, b_1, b_2 are integration constants, $z^2 = (-4vB/\gamma\lambda^4)^{1/2} y^2$ and $U(-A, z)$ is the parabolic cylinder function which decays as $z \rightarrow \infty$, and for $z \rightarrow 0$

$$U(-A, z) \approx \left(1 - \frac{\sqrt{2} \tan(\pi/4 - \pi A/2) \Gamma(3/4 + A/2) z}{\Gamma(1/4 + A/2)}\right) \quad (92)$$

By asymptotic matchings of the solutions in Eqs. (89)-(91), the dispersion relation for the resistive ballooning modes with even parity in x is given by

$$(-v D_s^2 / 4 \gamma B)^{1/2} = [2m + 1/2], \quad m = 0, 1, 2, \dots, \quad (93)$$

or

$$\gamma^3 = (s^2 D_s v) \{1 + D_s / 4 (2m + 1/2)^2\}, \quad (94)$$

where m corresponds to the number of the nodes in the eigenfunctions. Note also from Eq. (93) that our assumption, $B \sim D_s^2 \lambda^2$, is consistent. For the odd parity solutions, we have the dispersion relation

$$\left(\frac{-v D_s^2}{4 \gamma B}\right)^{1/2} = \left(2m + \frac{1}{2}\right) + \frac{2}{\pi} \tan^{-1} \left[\frac{\sqrt{2} \Gamma(1/4 + A/2)}{\pi \Gamma(3/4 + A/2)} \left(\frac{-v^3}{4 B \gamma}\right)^{1/4} \right], \quad (95)$$

where A is given by Eq. (91), and $m = 0, 1, 2, \dots$. For $m \gg 1$, $(-v^3/4 B \gamma)^{1/4} \sim (v/D_s^2)^{1/2} \ll 1$, the eigenfrequencies are approximately given by

$$\gamma^3 = s^2 D_s v \left\{1 + \frac{D_s}{4 \left[2m + 1/2 + \left(2\sqrt{2}/\pi^2\right) (v/s D_s^2)^{1/3} (2m + 1/2)^{1/2}\right]^2}\right\} \quad (96)$$

Figure 12 shows the numerical growth rates, obtained from integrating Eq. (24), versus α_p for both even and odd parity modes with the parameters $\nu = 10^{-3}$, $s = 1$, $\varepsilon = 0$. The m 's correspond to different eigenstates defined in Eqs. (93) and (95). The growth rates of the ideal ballooning modes ($\nu = 0$) are also plotted with critical $\alpha_{pc} = 0.625$. We see that these numerical results are consistent with our analytical solutions Eqs. (94) and (96), except for the $m = 0$ even parity eigenmode. When $\alpha_p > \alpha_{pc}$, the $m = 0$ even parity resistive ballooning mode transforms into the ideal ballooning mode, which is destabilized by, and localized in, the local bad curvature region. The resistivity scaling of the $m = 0$ even parity mode changes from $\nu^{1/3}$ for $\alpha_p < \alpha_{pc}$ to ν for $\alpha_p > \alpha_{pc}$. The two scale representation of the eigenfunction, Eq. (83), is inadequate for the unstable ideal ballooning mode, which explains the failure of our analytical results for the $m = 0$ even parity eigenstate when $\alpha_p > \alpha_{pc}$.

We should mention that the resistive ballooning modes have been studied by many authors [20-25]. Ignoring ion sound effects, the previous studies [24,25] obtained only the leading order contribution in α_p for the even parity modes. Including plasma compressibility [21-25] (finite ion sound speed), the resistive ballooning can be stable for either ν or α_p less than some critical values, given approximately by $(C_s/qR)^2 > \gamma^2$. However, we should also note that when the plasma compressibility is important, other kinetic effects such as FLR, ω_* must be included and the condition for neglecting ion sound effect is $\omega(\omega - \omega_{*e}) > (C_s/qR)^2$.

VI. SUMMARY AND DISCUSSIONS

In this paper, we have extensively studied shear Alfvén waves for a low- β , large aspect ratio toroidal plasma using both the ideal and the

resistive MHD models, which can be described by a single second order differential equation, Eq. (24), in terms of the extended poloidal coordinate θ by employing the high- n WKB-ballooning mode formalism. In the absence of resistivity and toroidal effects ($\nu = \epsilon = \alpha_p = 0$), Eq. (24) describes the ideal MHD shear Alfvén waves in a Fourier transformed space for an infinite plane slab with sheared magnetic field. Besides the shear Alfvén continuum $\omega = \omega(x_0)$, which corresponds to singular solution at every spatial point x_0 , we have found the discrete damped shear Alfvén-Landau modes with complex frequencies by analytical continuation of θ into the complex plane and imposing the outgoing wave boundary condition at large θ . With toroidal effects, the purely growing ideal ballooning modes are excited by the local bad curvature when $\alpha_p > \alpha_{pc}$. We have also found that the toroidal couplings between neighboring poloidal harmonics due to the nonuniform toroidal magnetic field can break up the shear Alfvén continuum into small bands of continuous spectra with the gaps located at the local coinciding shear Alfvén frequencies of the corresponding poloidal harmonics. The widths of the gaps are of the order of ϵ . In addition, we have found that discrete toroidal shear Alfvén eigenmode may exist inside the continuum gaps and the modes are localized in between the corresponding mode rational surfaces. By employing the method of asymptotic matching, we have shown that only the even parity eigenmodes exist inside the lowest continuum gaps and the corresponding eigenfrequencies vary from the lower endpoints of the gaps when $s = 0$ to the upper limits of the gaps when $s \rightarrow \infty$.

When plasma resistivity is introduced, four branches of resistive shear Alfvén modes are induced. The frequencies of the resistive entropy modes approach zero as $\nu^{1/3}$ along loci at $\pm 30^\circ$ below the real ω -axis as the resistivity ν vanishes. For large resistivity $\nu > 1$, the eigenfrequencies are

given by $\Omega = -iv \pm (2N+1)s$, where $N = 1, 2, 3, \dots$. For the tearing modes, the frequencies scale as $v^{3/5}$ as $v \rightarrow 0$ and approach $\Omega = -iv \pm s$ for $v > 1$. For the resistive periodic shear Alfvén waves, the eigenfrequencies lie on loci at $\pm 45^\circ$ below the real ω -axis and approach the finite frequency end points of the continuum gaps as $v^{1/2}$ when v vanishes. For the purely growing resistive ballooning modes, the growth rates are proportional to $v^{1/3} \alpha_p^{2/3}$ for small v and α_p with the exception that when $\alpha_p > \alpha_{pc}$, the $m = 0$ eigenstate of the resistive ballooning modes approach linearly with v to the unstable ideal ballooning modes as $v \rightarrow 0$. The effects of plasma resistivity on the ideal MHD modes can be treated perturbatively. For the toroidal shear Alfvén waves the eigenfrequency approaches the ideal limit frequency linearly with v along the -90° line in the complex ω -plane.

Our results may explain many features of the numerical results of Glasser et al. [22], who integrated the high- n resistive MHD ballooning mode equation (a fourth order differential equation) by using numerically generated axisymmetric toroidal equilibria. However, the stabilization of the resistive ballooning modes, when either v or α_p is less than some critical value, can not be accounted by the present model in an equilibrium with circular, concentric magnetic flux surfaces. Especially when $\gamma^2 \ll (c_s/qR)^2$, the effects of compressibility have been shown to stabilize the resistive ballooning modes [21-25]. Other kinetic effects such as FLR, ω_* may play a more important role in this limit.

Finally, we should mention that the toroidal shear Alfvén eigenmodes have been numerically verified for the low- n modes by using the PEST code. For example, the eigenfunction of the $n = 1$ mode is dominated by $m = 1$ and $m = 2$ components which are localized around $q = 1.5$ surface. Unlike the mechanism of Alfvén resonance heating and mode conversion which crucially depend on the

location of the singular surface, the low- n toroidal shear Alfvén mode may be more effectively applied to wave heating and current drive due to its global eigenmode structure. Another application of this mode is in plasmas with energetic particles. When the magnetic drift frequency of the energetic particles roughly equals the eigenfrequency of the toroidal shear Alfvén wave, it can be destabilized by the magnetic drift resonances. Recent neutral beam injection experiments in PLT and PDX have found high frequency ($f \approx 100\text{kHz}$) oscillations in Mirnov coils and soft X-ray signals. These oscillations are dominantly $n = m = 5$ and occurred near the $q \approx 1$ surface where beam particles have high concentration. Our numerical calculations have found that the toroidal shear Alfvén mode may account for these oscillations. These results will be presented in a future article.

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Figure Captions

- Fig. 1 The Stokes' diagram of the potential function $Q(\theta)$ in the complex θ plane for $s = 1$.
- Fig. 2 The eigenfrequencies $\Omega (= \omega/\omega_A)$ versus shear s for the even parity toroidal shear Alfvén eigenmodes for three values of $\epsilon = 0.05, 0.1, 0.2$, and the fixed parameters are $\alpha_p = \nu = 0$.
- Fig. 3 The eigenfunction of the toroidal shear Alfvén modes for the parameters $s = 1$, $\epsilon = 0.2$, and $\alpha_p = \nu = 0$. The eigenfrequency is $\boxed{\Omega = 0.54}$.
- Fig. 4 The eigenfrequencies Ω versus resistivity ν for the toroidal shear Alfvén modes with the parameters $\epsilon = 0.2$, $\alpha_p = 0$, $s = 1$.
- Fig. 5 The eigenfrequencies Ω versus resistivity ν for the even parity resistive entropy modes with the parameters $s = 1$, $\epsilon = \alpha_p = 0$.
- Fig. 6 The eigenfunction of the even parity resistive entropy mode versus θ for the parameters $s = 1$, $\nu = 10^{-3}$, $\epsilon = \alpha_p = 0$, and $\Omega = (0.253, -0.142)$.
- Fig. 7 The eigenfunction of the odd parity resistive entropy mode versus θ for the same parameters as in Fig. 6, but $\Omega = (0.257, -0.151)$.
- Fig. 8 The eigenfrequencies Ω versus ν for the tearing modes with the parameters $s = 1$, $\epsilon = \alpha_p = 0$.
- Fig. 9 The eigenfrequencies for different resistive periodic shear Alfvén eigenstates in the complex ω plane with the parameters $\nu = 10^{-6}$, $\epsilon = 0.2$, $\alpha_p = 0$, $s = 1$.
- Fig. 10 The eigenfrequencies of the even parity resistive periodic shear Alfvén waves versus ν for the parameters $\epsilon = 0.2$, $\alpha_p = 0$, $s = 1$.

Fig. 11 The eigenfunction of the resistive periodic shear Alfvén mode versus θ for the parameters $\varepsilon = 0.2$, $\nu = 10^{-5}$, $\alpha_p = 0$, $s = 1$, and $\Omega = (0.44, -0.0145)$.

Fig. 12 The growth rates of the resistive ballooning modes versus α_p for both the even (E) and odd (O) parity modes with the parameters $\nu = 10^{-3}$, $s = 1$, $\varepsilon = 0$. The m's correspond to different eigenstates defined in Eqs.(93) and (95).

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Fig. 1

Fig. 2

Fig. 3

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Fig. 5

Fig. 6

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Fig. 12

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