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Abstract

We have computed the scale breaking Λ parameters of the Euclidean and Hamiltonian formulations of the lattice regulated $O(N)$ and $SU(N) \times SU(N)$ spin systems in 1 + 1 dimensions in terms of the Λ_{PV} parameter of the Pauli-Villars regulated continuum models. Using lattice perturbation theory, the renormalized mass gap has been determined in terms of Λ_{PV} for each model. These results are compared to analogous calculations in $SU(N)$ gauge theories.

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Several correspondences have been suggested over the past five years between two dimensional spin systems and four dimensional gauge theories. The earliest correspondence is due to A. A. Migdal¹⁾ who presented an approximate renormalization group transformation which predicted that the coupling constant renormalization problem in $SU(N) \times SU(N)$ spin systems is essentially identical to that in the $SU(N)$ gauge theories. Within this approximation scheme both models were predicted to have but one phase at all coupling. This result constituted the first bit of numerical evidence for quark confinement in $SU(N)$ gauge theories.

Since that time strong coupling expansions²⁾ and Monte Carlo computer simulations³⁾ have been used to study these theories more systematically. We wish to continue such studies and compare these theories in both their weak coupling regimes using ordinary perturbation theory and in their strong and intermediate coupling regimes using lattice perturbation theory.

One of our main objectives here is to set the scale of non-perturbative calculations of the spin theorys' mass gaps (inverse correlation lengths) in terms of their Λ parameters which control their weak coupling deviations from free field behavior. These are the spin theory analogues of recent calculations of the Λ parameters in $SU(N)$ continuum and lattice gauge theories⁴⁾. These calculations then allow us to study the N dependence of physical quantities in the renormalized, Lorentz invariant continuum limits of the models. With the lattice scaffolding thus removed, we find that the $O(N)$, $SU(N) \times SU(N)$ and $SU(N)$ models have rather different N dependence. If we let M be the mass gap of the spin system and let T be the string tension of the gauge model, our calculations yield relations of the form

$$M = C_N \Lambda_{PV} \quad (1)$$

where Λ_{PV} is the Pauli-Villars' Λ parameter of the continuum limit of a spin system and

$$\sqrt{T} = G_N \Lambda_{PV} \quad (2)$$

for the gauge theories. The constants C_N and G_N for the various models are listed in Table 1. We note that the C_N decrease as N increases for the $O(N)$ models but increase for the $SU(N) \times SU(N)$ spin theories and the $SU(N)$ gauge theories. And the absolute magnitudes of the G_N are substantially larger for the $SU(N)$ gauge theories than the $SU(N) \times SU(N)$ spin systems.

These numerical results invite explanation. The vector and matrix models clearly have very different $N \rightarrow \infty$ dynamical features. The $N \rightarrow \infty$ limit of the vector models is the (mean) spherical model for which $\lim_{N \rightarrow \infty} C_N = 1$ is known to be exact. It is interesting that C_3 is the largest constant in the $O(N)$ sequence of models. Is this because only the $O(3)$ model has instantons? Our numerical value for $C_3 = 3.40 \pm .30$ can be compared against a recent numerical renormalization group analysis of that model⁵⁾. That analysis gave the correlation length ξ in units of the Euclidean lattice Λ_L parameter, $\xi^{-1} = (100 \pm 30) \Lambda_L$. From the calculations to be discussed below, we have $\Lambda_L / \Lambda_{PV} = 27.212$ and observing that ξ^{-1} is precisely the mass gap, the result of ref. 5 can be written as $M = (3.7 \pm 1.1) \Lambda_{PV}$ in good agreement with our calculations. The $SU(N) \times SU(N)$ models possess C_N which increase substantially with increasing N . This is characteristic of lattice models which disorder at smaller coupling increasingly more abruptly as N increases. It is plausible that this trend is a manifestation of the Gross-Witten third order phase transition⁶⁾ which a single site matrix model possesses in the $N \rightarrow \infty$ limit. Similar remarks apply to the $SU(N)$ gauge theories except we also note that their values of G_N are even larger than in the $SU(N) \times SU(N)$ spin systems. It is intriguing to suggest that this difference is due to

instantons in the gauge theories which are absent in all the $SU(N) \times SU(N)$ spin models. Of course, it could also be due to other effects--the higher dimensionality of space-time certainly plays some role. Nonetheless, we emphasize that the results listed in Table 1 may be used as a testing ground for various hypotheses concerning the natural disordering mechanisms (vortices, vortex loops, etc.) in asymptotically free models.

Now we shall summarize our calculations in the context of the $SU(N) \times SU(N)$ spin systems. Consider the Euclidean formulation on a square two dimensional lattice with the Action,

$$S = -\frac{1}{g^2} \sum_{x, \mu} \text{tr} [(U(x)U^\dagger(x + \mu) + \text{H.c.})] \quad (3)$$

where $U(x)$ is a $SU(N)$ matrix at the site x and μ is a unit lattice vector. This Action has the global invariance group $SU(N) \times SU(N)$. It is asymptotically free with a two-loop β function⁷⁾,

$$-\beta(\lambda) = b_0 \lambda^2 + b_1 \lambda^3 + \dots \quad (4)$$

where $\lambda = g^2 N$ is the coupling constant appropriate for discussing the $N \rightarrow \infty$ limit and $b_0 = 1/8\pi$, $b_1 = 1/2(8\pi)^2$. Eq. (4) governs the behavior of the bare lattice coupling as the lattice spacing "a" is varied,

$$-\beta(\lambda) = a \partial \lambda / \partial a \quad (5)$$

A Λ_L parameter (L for "lattice" regulated) is introduced to set the scale in the variation of λ with a. It follows from Eq. (4) and (5) that,

$$\Lambda_L = \frac{1}{a} (b_0 \lambda)^{-b_1/b_0} e^{-1/b_0 \lambda} [1 + O(\lambda)] \quad (6)$$

Of course, the scale in Eq. (6) is not determined by the renormalization group considerations. The choice in Eq. (6) has the virtue that with it the coupling constant satisfies,

$$\lambda = \frac{1}{b_0 \ln(1/a\Lambda_L) + \frac{b_1}{b_0} \ln \ln(1/a\Lambda_L)} \quad (7)$$

and the two loop contribution to the denominator does not include a constant term which would alter the scale in a one-loop calculation of λ . In this sense, the choice of scales in Eq (6) is optimal. For each regularization scheme there is a different version of Eq. (6). For a Hamiltonian lattice "a" would be the spatial lattice separation and in a continuum Pauli-Villars regulated theory "a" would be the reciprocal of the regulator mass. To find the relations between these different Λ -parameters, one must calculate the one-loop β function of the $SU(N) \times SU(N)$ model and retain the finite pieces of the renormalizations to set the scales in the logarithms. We have done this for the spin systems by generalizing the background field technique to the Action of Eq. (3), to an anisotropic formulation of the model so that the Hamiltonian theory could be considered, and to the continuum model regulated in the Pauli-Villars manner. The background field technique will be discussed at length elsewhere. We found,

$$\Lambda_{PV} / \Lambda_L = \sqrt{32} \exp [\pi(N^2 - 2)/2N^2] \quad (8)$$

and for the Hamiltonian model,

$$\Lambda_{PV} / \Lambda_H = 8 \exp [(N^2 - 2)/N^2] \quad (9)$$

The intermediate and strong coupling physics in these models is obtained by computing the Hamiltonian strong coupling expansions for the mass gaps of this family of spin systems. The Hamiltonian following from Eq (3) is,

$$H = \frac{g^2 \sqrt{n}}{2a} \sum_{\ell} \left\{ \tilde{E}_{\ell}^2 - \frac{2}{4} \text{tr} [U^+(\ell)U(\ell + 1) + \text{H.c.}] \right\} \quad (10)$$

where "a" is a spatial distance, the sum $\sum_{\ell}$ runs over the sites of a one dimensional lattice and $\tilde{E}_{\ell}^2$ is the quadratic Casimir operator for $SU(N)$. Eq. (10)

contains two coupling constants, $\sqrt{\eta}$ and g . At strong coupling ($g^2 \gg 1$) the lowest lying $2N$ -fold degenerate family of excitations are described by the states,

$$|\alpha\beta\rangle = \frac{1}{\sqrt{V}} \sum_{\ell} \frac{1}{\sqrt{N}} U_{\alpha\beta}(\ell) |0\rangle \quad (11)$$

where V is the number of spatial sites, and $|0\rangle$ is the strong coupling vacuum, $E_{\ell}^2 |0\rangle = 0$. The mass gap is computed in ordinary perturbation theory in $x = 1/\lambda^2$,

$$M = \frac{\lambda\sqrt{\eta}}{2a} \left(\frac{N^2-1}{2N^2} \right) \sum_{m=0}^{\infty} c_m x^m \quad (12)$$

with $c_0 = 1$. By direct calculation one finds that the coefficients c_m have finite $N \rightarrow \infty$ limits. One can renormalize the disordered lattice theory by holding M fixed as " a " is varied. Then λ varies in a precise way and a familiar argument shows that⁸⁾,

$$-\frac{\beta(\lambda)}{\lambda} = \frac{1}{1 - 2xW'/W} \quad (13)$$

where $W = \sum c_m x^m$ of Eq. (12). By using Eq. (4) for weak coupling and Eq. (13) for strong and intermediate coupling, one can patch together a β -function for all λ . The results of these calculations which used strong coupling series to $O(\lambda^{-16})$ and extrapolated the series to weak coupling with Padé approximates are shown in Fig. 1a. We note that as N increases the crossover between weak and strong coupling becomes more abrupt and shifts to weaker coupling. And finally, the constants C_N can be found by fitting the mass gap series with the weak coupling scaling law Eq. (6) at the matching point between the strong and weak coupling expansions for the β functions. Using Eq. (9) this relation can be written in the form Eq. (1) completely free of lattice scaffolding. These results are collected in Table 1a. The coupling constant $\sqrt{\eta}$ of Eq. (10) must be calculated in the weak coupling region and at the matching point to determine the constants C_N with precision. The same calculations which yielded

Λ_{PV}/Λ_H give $\eta = 1 + \frac{\lambda}{\pi} \left(\frac{N^2-1}{2N} \right) + O(\lambda^2)$. Physically, η represents a finite renormalization of the speed of light. Choosing η according to our formula guarantees that the matching procedure produces a theory which is Lorentz invariant in the continuum limit. This renormalization effect has been ignored in the past and will be discussed in detail elsewhere.

Precisely the same calculations were done for the $O(N)$ models.

The Λ -parameters were related,

$$\Lambda_{PV} / \Lambda_L = \sqrt{32} \exp [\pi/2(N-2)] \quad (14)$$

and

$$\Lambda_{PV} / \Lambda_H = 8 \exp [1/(N-2)] \quad (15)$$

Eq. (14) has been obtained previously by G. Parisi⁹). Note that since $O(4) \approx SU(2) \times SU(2)$ Eq. (14) and (15) at $N = 4$ must agree with Eq. (8) and (9) at $N = 2$. In addition, we know from the solution of the mean spherical model that $\Lambda_{PV} / \Lambda_L \rightarrow \sqrt{32}$ ¹⁰⁾ and $\Lambda_{PV} / \Lambda_H \rightarrow 8$ ¹¹⁾ as $N \rightarrow \infty$. Our general equations satisfy all these checks. The strong coupling calculations employed the Hamiltonian,

$$H = \frac{g\sqrt{\eta}}{2a} \sum_m [J^2(m) - x \underline{n}(m) \cdot \underline{n}(m+1)], \quad x = 2/g^2 \quad (16)$$

where J^2 is the angular momentum-squared in N dimensions and $\underline{n}(m)$ is an N -component unit vector. If one introduces the scaled coupling constant,

$$\bar{g} \equiv (N-2)g, \quad y \equiv 2/\bar{g}^2 = x/(N-2)^2 \quad (17)$$

then the mass gap series has the form

$$M = \frac{\bar{g}\sqrt{\eta}}{2a} \left(\frac{N-1}{N-2} \right) \sum_m a_m y^m \quad (18)$$

and the coefficients have finite $N \rightarrow \infty$ limits. The parameter η has a weak coupling expansion $\eta = 1 + \bar{g}/(N-2)\pi + O(\bar{g}^2)$. We have calculated the first five terms in Eq. (18) for all N and have obtained strong coupling expansions for the theory's β functions. The weak coupling β function is,

$$\beta(\bar{g}) = \frac{1}{2\pi} \bar{g}^{-2} + \frac{1}{(2\pi)^2 (N-2)} \bar{g}^{-3} + \dots \quad (19)$$

which reduces to the one-loop term in the $N \rightarrow \infty$ limit. The β functions and constants C_N are shown in Fig. 1b and Table 1, respectively.

Finally consider the $SU(N)$ gauge theories in $3 + 1$ dimensions. The relations between the Λ parameters have been calculated¹²⁾,

$$\Lambda_{PV} / \Lambda_L = 42.2297 \exp(-3\pi^2/11N^2) \quad (20)$$

and¹³⁾

$$\Lambda_{PV} / \Lambda_H = 69.1045 \exp\left[-\frac{2\pi^2}{11} \left(\frac{N^2 + 1/2}{N^2}\right)\right] \approx \quad (21)$$

We have discussed the strong coupling analyses of the string tension elsewhere¹⁴⁾ and have collected the β functions in Fig. 1c. and the constants of Eq. (2) in Table 1. We observe that the trends in these results are similar to the $SU(N) \times SU(N)$ models but they are even more dramatic. For example, the $N = \infty$ β function goes from the weak coupling matching point to 90 percent of its strong coupling value in about 2.5 units of $\lambda = g^2 N$ while the $N = \infty$ curve of the $SU(N) \times SU(N)$ spin system requires 8.5 units to do the same. The constants G_N also show a more pronounced growth with N . However, although detailed analyses of the spin systems suggests that our calculations of the C_N are quite reliable, short strong coupling series for gauge theories in $3 + 1$ dimensions are not so successful. Recall that for $SU(3)$ where higher order calculations have been done a detailed fitting procedure was necessary to estimate C_3 ¹⁵⁾. We have applied the same techniques to the short series for general N to produce estimates for $N = 5, 6$ and infinity. The reader should treat these results for large N as educated guesses only. The functions of Fig. 1c, however, are certainly more reliable. The G_N are difficult to estimate simply because they are exponentially dependent on the coupling at the strong-weak matching point. The results for $SU(2)$ and $SU(3)$ are, in fact, in good agreement with computer simulations³⁾.

In summary, we have calculated the Λ parameters for two families of asymptotically free spin systems in $1 + 1$ dimensions and have thereby set the scale of the continuum limits of non-perturbative lattice calculations and have observed interesting trends in our results. Explanations of these and the analogous gauge theory calculations should shed light on the disordering mechanisms and the $N \rightarrow \infty$ limits of these theories.

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References

1. A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, (1975) 810.
2. J. B. Kogut, R. B. Pearson, and J. Shigemitsu, Phys. Rev. Lett. 43, (1975) 484.
G. Münster, DESY preprint 80/44, May, 1980.
F. Green and S. Samuel, IAS preprint, Dec., 1980.
3. M. Creutz, Phys. Rev. D21 (1980) 2308.
4. A. Hasenfratz and P. Hasenfratz, Phys. Lett. 93B (1980) 165.
5. S. H. Shenker and J. Tobochnik, to be published.
6. D. Gross and E. Witten, Phys. Rev. D21 (1980) 446; S. Wadia, to be published.
7. A. McKane and M. Stone, Nucl. Phys. B163 (1980) 169.
8. For discussion and additional illustrations see J. B. Kogut, invited lectures at the 1980 Les Houches Winter School, to be published.
9. G. Parisi, to appear in Phys. Lett.
10. D. J. Thouless and M. E. Elzain, J. Phys C11 (1978) 3525.
11. M. Srednicki, Phys. Rev. B20 (1979) 3783. J. Banks, Phys. Lett. 93B, (1980) 161.
12. R. Dashen and D. J. Gross, Princeton University preprint, 1980.
13. D. J. Gross, invited talk at the Workshop on Lattice Gauge Theory, July 28 - August 1, 1980. To be published.
14. J. B. Kogut and J. Shigemitsu, Phys. Rev. Lett. 45 (1980) 410.
15. J. B. Kogut, R. B. Pearson and J. Shigemitsu, to appear in Phys. Lett.

Table 1

The constants C_N for $O(N)$ and $SU(N) \times SU(N)$ spin systems and the constants G_N for $SU(N)$ gauge theories for various N . C_N and G_N are defined in Eq. (1) and (2), respectively.

O(N)		SU(N) $\times$ SU(N)		SU(N) Gauge	
N	C_N	N	C_N	N	G_N
3	$3.40 \pm .30$	2	$1.83 \pm .20$	2	3.71 ± 0.93
4	$1.83 \pm .20$	6	$4.92 \pm .10$	3	6.63 ± 1.44
6	1.34	8	$5.47 \pm .10$	5	12-13
10	1.25	∞	$6.05 \pm .10$	6	13-15
∞	1.00			30	16-18

Figure Caption

1. β functions for (a.) $SU(N) \times SU(N)$ spin, (b.) $O(N)$ spin, and (c.) $SU(N)$ gauge theories. The scales of (a.) and (b.) have been chosen so that the $SU(2) \times SU(2)$ curve and the $O(4)$ curves are identical. Note that λ labelling (c.) differs from the definition in ref. (14) by a square root.

Fig. 1a

Fig. 1b

Fig. 1c

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