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for ZTH Electrical Diagnostics*

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EQUILIBRIUM AND FLUCTUATION ANALYSIS FOR ZTH ELECTRICAL DIAGNOSTICS

by G. Miller and J. C. Ingraham

ABSTRACT

Some of the rationale behind the electrical diagnostics proposed for the Los Alamos Confinement Physics Research Facility, ZTH, is discussed. The axisymmetric equilibrium measurements consist of a poloidal flux array and a toroidally averaged poloidal field array. The equilibrium quantities of interest, for example, the radial magnetic field causing displacement of the outer plasma magnetic surface, are obtained from the measurements by linear combination with constant coefficients. Some possible objectives for the nonaxisymmetric field measurements are discussed.

I. INTRODUCTION

This report discusses some of the rationale behind the electrical diagnostics proposed for the Los Alamos Confinement Physics Research Facility, Reversed Field Pinch experiment, ZTH.

The electrical diagnostics characterize the magnetic field external to the plasma and its time derivative (equivalent to the electric field). The electrical diagnostics can be divided into two groups: 1) those measuring the axisymmetric equilibrium magnetic field and 2) those measuring the small nonaxisymmetric magnetic field components that are also present. Measurements in group 1 need to be quantitatively accurate, as this group of measurements includes many of the most interesting parameters of machine performance, such as toroidal plasma current I_p , toroidal loop voltage V_ϕ , toroidal magnetic flux ψ , external toroidal magnetic field B_ϕ , and Shafranov Λ . These measurements also provide signals for equilibrium feedback control. Measurements in group 2 require less accuracy and are mainly of interest for their qualitative features.

The magnetic field measurements need to be referred to definite coordinate systems. The coordinate systems used here are a cylindrical R, ϕ, Z system and a curved cylindrical r, θ, z system with the z axis along the minor axis of the torus, as shown in Fig. 1.

This choice of coordinate system orientation is, unfortunately, not the universal choice. The following considerations determine the orientation of the coordinate systems shown in Fig. 1. The R, ϕ, Z and r, θ, z systems are right handed. The Z axis is chosen to be upward. By $\hat{Z} \times \hat{R} = \hat{\phi}$, $\hat{\phi}$ is in the direction shown. The $\hat{\phi}$ and $\hat{z}$ directions are chosen to be the same. The angle θ is chosen to increase as $\hat{r}$ is rotated upward. To satisfy $\hat{r} \times \hat{\theta} = \hat{z}$, the r, θ, z coordinate system orientation must be as shown in Fig. 1.

Toroidal curvature is usually ignored in the theory of the nonaxisymmetric magnetic field components. The r, θ, z system then refers to a straight cylinder of length $2\pi R_0$ with identified ends, where R_0 is the major radius of the torus, as shown in Fig. 2.

II. AXISYMMETRIC EQUILIBRIUM QUANTITIES

The magnetic field in the plasma and external to it is approximately axisymmetric, that is, independent of the toroidal angle ϕ . From $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{A}$ is the vector potential and $\mathbf{B}$ the magnetic field,

$$\begin{aligned} B_R &= -\frac{\partial A_\phi}{\partial Z} \\ B_Z &= \frac{1}{R} \frac{\partial}{\partial R} (R A_\phi). \end{aligned} \tag{1}$$

The poloidal flux of magnetic field will be denoted by $\chi^\dagger$ and is given by

$$\chi = \int B_Z R dR d\phi, \quad (2)$$

where the integral is over the disk of radius R at constant Z . In terms of A_ϕ ,

$$\chi = \int R A_\phi d\phi = 2\pi R A_\phi. \quad (3)$$

Going over to the curved cylindrical r, θ, z coordinate system shown in Fig. 1, Eqs. (1) and (3) imply

$$\begin{aligned} B_r &= \frac{1}{2\pi R} \frac{1}{r} \frac{\partial \chi}{\partial \theta} \\ B_\theta &= -\frac{1}{2\pi R} \frac{\partial \chi}{\partial r}. \end{aligned} \quad (4)$$

Equations (4) express the r and θ components of the magnetic field in terms of χ . The z (or ϕ) component of the magnetic field is obtained starting with the relation

$$J = \int j_Z R dR d\phi \quad (5)$$

defining J , the poloidal flux of current in external windings and plasma, where j_Z is the current density in the Z direction. In complete analogy to Eq. (3), using $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, Eq. (5) can be written as

$$\mu_0 J = 2\pi R B_\phi, \quad (6)$$

so that

$$B_z = B_\phi = \frac{\mu_0 J}{2\pi R}. \quad (7)$$

Outside the plasma but inside the external B_ϕ winding, J is the total poloidal current flowing in the external windings, which can be measured directly. Like χ , J is a magnetic surface quantity having the same value everywhere on a magnetic surface.

The axisymmetric equilibrium measurements may be thought of as measurements to determine $\chi(r, \theta)$ as a function of r and θ outside the plasma edge at $r = r_p$. The quantity $\chi(r, \theta)$ is expanded as a Fourier series in θ :

$$\chi(r, \theta) = \sum_m [\chi_{mc}(r) \cos(m\theta) + \chi_{ms}(r) \sin(m\theta)]. \quad (8)$$

Note that for $m = 0$, only the $\cos(m\theta)$ ($= 1$) term is present in Eq. (8). As shown in Appendix A, the Fourier coefficients in Eq. (8) satisfy coupled, linear second-order ordinary differential equations in r , obtained from Maxwell's equations $\nabla \cdot \mathbf{B} = 0$, and $\nabla \times \mathbf{B} = 0$ in the vacuum region outside the plasma. Thus, $\chi_m(r)$ (χ_m is either χ_{mc} or χ_{ms}) is defined by two parameters, which here we assume to be $\chi_m(r_p)$ and $\chi'_m(r_p)$. If terms up to $m = m_{max}$ are included in Eq. (8), then $4m_{max}$ (for $m = 1$ to m_{max}) + 2 (for $m = 0$) parameters are required to specify $\chi(r, \theta)$. For example, if $m_{max} = 2$, then 10 parameters are required. The basic idea of the method we will use to determine these parameters by measurements is to use an analytic solution of Maxwell's equations with 10 free parameters to represent $\chi(r, \theta)$, and to vary the parameters in a least-squares fitting procedure in order to best fit the measured quantities, which are χ and B_θ ($\sim \partial\chi/\partial r$) at different poloidal locations. The number of measured quantities is planned to be 13 for ZTH, with at least 10 required.

[†] Another definition of poloidal flux is sometimes used: poloidal flux is defined as the flux in the poloidal direction between the point in question and the magnetic axis. This flux is $\chi - \chi_{axis}$ in terms of our definition of χ .

The shape of the magnetic surface at $r = r_p$ can be determined from χ . The relation defining a magnetic surface is

$$\chi(r, \theta) = \text{const.} \quad (9)$$

We represent the shape of the magnetic surface using the equation

$$r = r_p + \sum_{m=1}^{m=m_{\max}} [\xi_{mc} \cos(m\theta) + \xi_{ms} \sin(m\theta)], \quad (10)$$

and expand Eq. (9) around $r = r_p$ (neglecting χ'_m compared to χ'_0) to obtain

$$\xi_m = -\frac{\chi_m(r_p)}{\chi'_0(r_p)}. \quad (11)$$

Another quantity of interest is Shafranov Λ , defined by^{1,2}

$$\Lambda = -\frac{B_{\theta 1c} + B_{r 1c}}{B_{\theta 0}} \frac{R}{r_p}, \quad (12)$$

where $B_{\theta 1c}$ and $B_{r 1c}$ are the $\cos \theta$ Fourier components of B_θ and B_r , and $B_{\theta 0}$ is the symmetrical component of B_θ , at $r = r_p$. The quantity $B_{r 1c}$ will normally be very small, giving a well-centered equilibrium. Using Eq. (4), Λ , in the centered case, is given by

$$\Lambda = -\frac{\chi'_{1c}(r_p)}{\chi'_0(r_p)} \frac{R}{r_p}. \quad (13)$$

Equation (13) is used to experimentally determine Λ as will be discussed subsequently.

Λ is the asymmetry in the strength of the poloidal magnetic field, normalized by R/r , around the outer magnetic surface. In the Reversed Field Pinch this asymmetry can have either sign; that is, the poloidal field, in going around the outer magnetic surface, can be stronger on the inside (in major radius) or stronger on the outside, depending on the equilibrium profiles.

An interesting relation involving Λ can be derived using equilibrium relations within the plasma,^{1,2} namely,

$$\Lambda = \frac{1}{2} L_\theta + \beta_\theta - 1, \quad (14)$$

where

$$\begin{aligned} L_\theta &= \frac{2}{r_p^2 B_\theta^2(r_p)} \int_0^{r_p} B_\theta^2 r dr \\ \beta_\theta &= \frac{4}{r_p^2 B_\theta^2(r_p)} \int_0^{r_p} p r dr. \end{aligned} \quad (15)$$

This relation constrains the choice of equilibrium profiles used to model the plasma equilibrium.

The measurements to determine the parameters defining $\chi(r, \theta)$ are 1) a poloidal flux array, measuring $\chi(r_{\chi i}, \theta_{\chi i})$, at N_χ poloidal locations $i = 1, N_\chi$, and 2) a poloidal magnetic field array, measuring toroidally averaged $B_\theta(r_{Bi}, \theta_{Bi})$ at N_B poloidal locations, $i = 1, N_B$, as shown in Fig. 3. As long as the number of measurements $N_\chi + N_B$ is greater than or equal to the number of parameters, a least-squares fit may be used to determine the parameters from the measurements. Since this is a linear fitting problem, the fit parameters will be given by some constant matrix times the measured quantities (considered as a column vector). As discussed in Appendix B, we plan to have $N_\chi = 8$ and $N_B = 8$.

To economize on the number of data channels, the B_θ array signals are preprocessed into Fourier modes as follows:

$$B_{\theta 0} = \frac{1}{N_B} \sum_{i=1}^{N_B} B_{\theta i},$$

for $m = 0$ and

$$\begin{aligned} B_{\theta mc} &= \frac{2}{N_B} \sum_{i=1}^{N_B} B_{\theta i} \cos(m\theta_{Bi}), \\ B_{\theta ms} &= \frac{2}{N_B} \sum_{i=1}^{N_B} B_{\theta i} \sin(m\theta_{Bi}), \end{aligned} \quad (16)$$

for $m \neq 0$. Thus the vector representing measured quantities has B_θ mode amplitudes truncated at a low value (5, corresponding to $m_{max} = 2$) instead of a larger number of discrete B_θ values (8). The B_θ values represent toroidal averages of a number of measurements at different toroidal locations. As discussed in Appendix B, four different toroidal locations are planned.

As an example and a test case, consider the case of N_χ evenly spaced poloidal flux loops and a B_θ mode array both at radius $r = r_p$. The aspect ratio R_0/r_p is assumed very large. The flux parameters are then given by

$$\begin{aligned} \chi_0 &= \frac{1}{N_\chi} \sum_{i=1}^{N_\chi} \chi_{\chi i}, \\ \chi_{mc} &= \frac{2}{N_\chi} \sum_{i=1}^{N_\chi} \chi_{\chi i} \cos(m\theta_{\chi i}), \\ \chi_{ms} &= \frac{2}{N_\chi} \sum_{i=1}^{N_\chi} \chi_{\chi i} \sin(m\theta_{\chi i}), \\ \chi'_m &= -2\pi R_0 B_{\theta m}. \end{aligned} \quad (17)$$

In the general case of finite aspect ratio, $r_\chi \neq r_B \neq r_p$, and with the measuring array circles possibly shifted with respect to the plasma, the sparse matrix defined by Eqs. (17) tends to fill out, but the general nonsparse matrix is still recognizably related (an example is given later on).

An additional complication is the metal liner surrounding the plasma. The axisymmetric current j_z in the liner is given by Ohm's law:

$$j_z = \frac{E_z}{\eta} = -\frac{\dot{\chi}}{2\pi R \eta}, \quad (18)$$

where η is the resistivity of the liner material and the dot denotes the time derivative. From $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$,

$$\mu_0 j_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta). \quad (19)$$

Equation (19) is valid for the toroidally curved r, θ, z coordinate system.

By integrating Eq. (19) across the small thickness δ of the liner, we obtain

$$[B_\theta] = \mu_0 j_z \delta = -\frac{\mu_0 \dot{\chi} \delta}{2\pi R \eta}, \quad (20)$$

where the square bracket denotes the jump across the liner of the enclosed quantity. Using Eq. (4), Eq. (20) implies that

$$\left[\frac{\partial \chi}{\partial r} \right] = \frac{\mu_0 \delta}{\eta} \dot{\chi}. \quad (21)$$

The quantities $\partial \chi / \partial r$ and $\dot{\chi}$ appearing in Eq. (21) are functions of θ . When η is independent of θ , Eq. (21) also applies to Fourier components.

Up to this point, the liner has been imagined to be a uniform annulus with thickness δ and resistivity η . The liner is actually a bellows. If δ is taken as the thickness of the liner material, the resistivity must be increased by a bellows factor f that is the ratio of the stretched-out bellows length to the actual compressed length. The experimentally measurable toroidal resistance of the liner is given by

$$R_\phi = -\frac{\dot{\chi}_0}{I_l},$$

where I_l is the total liner current resulting from driving voltage $-\chi_0$. Using Eq. (18),

$$R_\phi = \frac{f R_0 \eta}{r_l \delta}, \quad (22)$$

where R_0 is the major radius of the torus and r_l is the mean radius of the liner. For nominal ZTH parameters, $R_\phi = 12 \times 10^{-3} \text{ Ohm}$ ($f = 2.4$, $R_0 = 2.4 \text{ m}$, $\eta = 1.3 \times 10^{-3} \text{ Ohm-m}$, $r_l = 0.409 \text{ m}$, $\delta = 0.06 \text{ inch}$). Equation (22) may be used to experimentally determine the bellows factor using known values of the other quantities.

Because the liner is a bellows made by compressing and bending a straight cylinder into a torus, the bellows factor f is not constant but varies like R_0/R . This is because the actual compressed length of the bellows ($\sim 2\pi R$) is shorter on the inside in major radius than on the outside. Thus $f\eta$ that should appear in Eq. (21) is a function of θ and the Fourier analysis of Eq. (21) becomes more complicated. This effect is ignored here, but is required elsewhere (see Appendix C—this effect must be included when the result of interest is, to leading order, a toroidal effect, as is the case for the vertical field produced by poloidally uniform liner current).

Equation (21), applied to Fourier components, gives the jump of χ' across the liner. The quantity χ itself is continuous across the liner. This follows from Eq. (3) and continuity of A_ϕ , which follows from Eq. (1). Thus, using Eq. (21) we can transform the vector of equilibrium parameters obtained outside the liner to the inside surface of the liner.

Appendix B describes the proposed diagnostic package for ZTH. Appendix C discusses vacuum shots that might be useful for test and calibration purposes. In what follows we describe in more detail the equilibrium analysis.

As a numerical example, we consider a possible ZTH case (see Appendix B) with

$$\begin{aligned} r_p &= 0.380 \text{ m} \\ r_l &= 0.409 \text{ m} \\ r_\chi &= 0.431 \text{ m} \\ r_B &= 0.433 \text{ m} \\ R_0 &= 2.400 \text{ m}. \end{aligned} \quad (23)$$

It should be noted that thermal expansion can change some of these dimensions slightly, when the liner is operated at greatly different temperatures. The angles θ_χ and θ_B are equal and evenly spaced about the center of the liner and plasma with

$$\theta_i = \frac{2\pi}{8} \left(i - \frac{1}{2} \right) \quad \text{for } i = 1, 8.$$

The B_θ modes and the representation of χ are both truncated at $m = 2$.

The parameters defining χ at the outside surface of the liner at $r = r_l$ are obtained from measured quantities using

$$X(r_l) = AY + BZ, \quad (24a)$$

where

$$\begin{aligned}
 X(r) &\equiv \begin{pmatrix} \lambda_0 \\ \lambda_{1c} \\ \lambda_{1s} \\ \lambda_{2c} \\ \lambda_{2s} \\ r\lambda'_0 \\ r\lambda'_{1c} \\ r\lambda'_{1s} \\ r\lambda'_{2c} \\ r\lambda'_{2s} \end{pmatrix} \\
 Y &\equiv \begin{pmatrix} \lambda_{\chi_1} \\ \vdots \\ \lambda_{\chi_8} \end{pmatrix} \\
 Z &\equiv r_l^2 \begin{pmatrix} B_{\theta 0} \\ B_{\theta 1c} \\ B_{\theta 1s} \\ B_{\theta 2c} \\ B_{\theta 2s} \end{pmatrix}
 \end{aligned} \tag{24b}$$

and

$$\begin{aligned}
 A &= \begin{pmatrix} 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\ 0.231 & 0.096 & -0.096 & -0.231 & -0.231 & -0.096 & 0.096 & 0.231 \\ 0.096 & 0.231 & 0.231 & 0.096 & -0.096 & -0.231 & -0.231 & -0.096 \\ 0.178 & -0.178 & -0.178 & 0.178 & 0.178 & -0.178 & -0.178 & 0.178 \\ 0.178 & 0.178 & -0.178 & -0.178 & 0.178 & 0.178 & -0.178 & -0.178 \\ 0.002 & 0.002 & 0.001 & -0.001 & -0.002 & -0.002 & -0.001 & 0.001 \\ 0.013 & -0.005 & 0.005 & 0.013 & 0.013 & 0.005 & -0.006 & -0.013 \\ 0.005 & -0.013 & -0.013 & -0.005 & 0.005 & 0.013 & 0.013 & 0.005 \\ 0.042 & 0.040 & 0.041 & -0.039 & -0.039 & 0.041 & 0.040 & -0.042 \\ 0.041 & -0.042 & 0.039 & 0.040 & -0.040 & -0.039 & 0.042 & 0.041 \end{pmatrix} \\
 B &= \begin{pmatrix} 2.038 & -0.179 & 0.005 & 0 & 0 \\ -0.358 & 2.045 & 0.001 & -0.185 & 0 \\ 0 & 0 & 2.046 & 0 & -0.185 \\ 0 & -0.179 & 0 & 2.032 & 0 \\ 0 & 0 & -0.179 & 0 & 2.031 \\ -39.030 & 3.332 & -0.192 & 0.017 & 0.017 \\ 6.660 & -39.070 & -0.018 & 3.528 & 0.002 \\ 0 & 0 & -39.090 & 0 & 3.526 \\ 0.001 & 3.342 & -0.002 & -38.910 & 0 \\ 0 & 0 & 3.340 & 0 & -38.900 \end{pmatrix}.
 \end{aligned} \tag{24c}$$

The quantities X , Y , and Z are defined to have units of flux (Webers). Thus the matrices A and B are dimensionless. The large terms in A and B correspond closely to the values from the cylindrical approximation given by Eq. (17).

The final step in the axisymmetric analysis is to transform the equilibrium parameters inside the liner at $r = r_l$ [after using Eq. (21) to go through the liner], to the plasma edge at $r = r_p$. There is a linear matrix, similar to that given by Eq. (24a), relating the two sets of quantities. For the numerical case under consideration, given by Eq. (23), this transformation matrix is

$$X(r_p) = CX(r_l), \tag{25a}$$

where

$$C = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & -0.073 & 0 & 0 & 0 & 0 \\ 0 & 1.003 & 0 & 0 & 0 & 0 & -0.074 & 0 & 0 & 0 \\ 0 & 0 & 1.003 & 0 & 0 & 0 & 0 & -0.074 & 0 & 0 \\ 0 & 0 & 0 & 1.011 & 0 & 0 & 0 & 0 & -0.074 & 0 \\ 0 & 0 & 0 & 0 & 1.011 & 0 & 0 & 0 & 0 & -0.074 \\ 0 & 0.006 & 0.006 & 0 & 0 & 1.001 & 0.006 & 0.006 & 0 & 0 \\ 0 & -0.073 & 0.001 & 0 & 0 & 0.012 & 1.003 & 0.001 & 0 & 0 \\ 0 & 0 & -0.074 & 0 & 0 & 0 & 0 & 1.003 & 0 & 0 \\ 0 & -0.006 & 0 & -0.295 & 0 & 0.001 & 0.006 & 0 & 1.011 & 0 \\ 0 & 0 & -0.006 & 0 & -0.295 & 0 & 0 & 0.006 & 0 & 1.011 \end{pmatrix}. \quad (25b)$$

In the above matrices, quantities less in absolute value than 0.0005 are truncated to zero.

As an example of the use of these concepts we discuss the problem of control of the equilibrium and the determination of I_p , V_ϕ , ψ , B_ϕ , and Λ , as mentioned in the introduction. The diagnostics proposed for ZTH are discussed in Appendix B.

Equations (24a) and (25a) express the vector $X(r_p)$ at the plasma radius as a linear combination of the measured quantities: the vectors Y and Z , and their time derivatives [time derivatives enter in through the liner jump condition given by Eq. (21)]. The equilibrium control problem consists of insuring that the various Fourier components of the plasma displacement at $r = r_p$, given by Eq. (11), are zero. Equivalently, Fourier amplitudes χ_m in Eq. (8) should all be zero for $m \geq 1$. These Fourier amplitudes are the second through fifth components of the vector $X(r_p)$, and thus the equilibrium control system should attempt to keep these four signals zero.

As mentioned in the introduction, some of the most interesting parameters of machine performance are the quantities I_p , V_ϕ , ψ , B_ϕ , and Λ . Appendix D gives explicit formulae for these quantities. The basic idea of their determination is as follows. The plasma plus liner current, $I_p + I_l$, will be measured by a Rogowskii loop around the liner. The liner current, I_l can then be subtracted to obtain the plasma current. Alternatively, Eq. (4) for $B_\theta(r_p)$ can be substituted in the expression

$$I_p = \int B_\theta r_p d\theta,$$

giving,

$$I_p \approx \frac{r_p}{R_0} \chi'_0(r_p),$$

neglecting second-order effects. The loop voltage at the plasma surface, $V_\phi(r_p)$, is simply $-\dot{\chi}_0(r_p)$, with $\chi_0(r_p)$ given by the first component of $X(r_p)$. Toroidal flux ψ in the plasma is measured by a voltage loop the short way around the torus, using the relation

$$\dot{\psi} = -V_\phi.$$

The measured flux must be corrected for the flux between the loop and the plasma surface. The toroidal field B_ϕ in this region is given by Eq. (7); however, inside the liner, J includes a contribution J_l from poloidal current in the liner. Shafranov $\Lambda(r_p)$ is given by Eq. (13).

Another equation involving Λ , in addition to Eq. (14), is the approximate relation expressing balance of $\hat{R}$ -directed force per unit length acting on the plasma,^{1,2}

$$\frac{\mu_0 I_p^2}{4\pi R} \left(\Lambda(r_p) + \log\left(\frac{8R_0}{r_p}\right) - \frac{1}{2} \right) + I_p B_Z = 0, \quad (26)$$

where B_Z is the applied vertical field. The applied vertical field B_Z has three sources: 1) the poloidal field windings, consisting of current loops in the $\hat{\phi}$ direction, each of which contributes linearly to the vertical field (see Appendix E); 2) shell currents (the shell gap prevents a net shell current but $m \neq 0$ Fourier components of the shell current are largely unaffected by the gap); and 3) liner currents.

The vertical field produced by shell or liner currents can be calculated by treating the shell or liner currents as a large number N of discrete current loops (see Appendix E), each discrete current loop replacing the current carried in an angular segment $\Delta\theta$ of the shell or liner. From Eq. (18), the discretized current loop current is

$$I_i = \int j_z r d\theta dr = \frac{\chi r_i \Delta\theta \delta}{2\pi R_i f \eta}, \quad (27)$$

where r_i and θ_i describe the location of the current element with angular and radial extents $\Delta\theta$ and δ , R_i is the major radius, η is the shell or liner resistivity, and f is the bellows factor ($f = 1$ for the shell). As already discussed, $f \sim 1/R_i$. This effect is of importance and is not neglected here. Because of the gap in the shell, the net toroidal shell current is zero, and the shell current distribution calculated from Eq. (27), which assumes no gap, must be modified by subtracting the net toroidal current obtained from Eq. (27). Neglecting the effect of the shell gap on current distributions in the shell that have zero net toroidal current amounts to approximating the shell as a very thin annulus of very highly conducting material (keeping the ratio δ/η constant in this limit), with a very thin gap interrupting toroidal current. The presence of the gap actually makes the shell response problem three dimensional and hence intractable, although it is a reasonable approximation to neglect these three-dimensional effects.

The vertical field produced at the minor axis by liner or shell is given an expression of the form

$$B_z = \frac{\mu_0 \delta}{R_0 f \eta} D \dot{X}(r_l), \quad (28)$$

where δ is the thickness of the liner or shell, η is its resistivity, and f is the average bellows factor (note that $R_i f_i$ has been replaced by $R_0 f$). We obtain $X(r_l)$ from Eqs. (24). For the ZTH case given by Eq. (23),

$$D^{(liner)} = (0.109 \quad -0.190 \quad 0 \quad 0.008 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0). \quad (29)$$

The ZTH shell is shifted inward in major radius by amount ΔR with respect to the minor axis. Assuming

$$\begin{aligned} \Delta R &= 1.27 \text{ cm} \\ r^{(Shell)} &= 0.462 \text{ m} \end{aligned} \quad (30)$$

we obtain

$$D^{(Shell)} = (0 \quad -0.191 \quad 0 \quad 0.008 \quad 0 \quad -0.005 \quad -0.023 \quad 0 \quad -0.002 \quad 0). \quad (31)$$

Equation (28), using $D^{(liner)}$ and $D^{(Shell)}$ from Eqs. (29) and (31), gives accurate values of the liner and shell contributions to the vertical field. Note that in both cases Eq. (28) uses $\dot{X}(r_l)$ since this quantity is available, having already been used in the axisymmetric analysis [as opposed to $X(r_S)$]. The externally applied vertical field from the poloidal field windings can be calculated from the measured currents in these windings, using formulae given in Appendix E.

As a numerical example of Eq. (28), a loop voltage $[-\dot{\chi}_0(r_l)]$ of 100 V produces a vertical field from the liner of 28 G for nominal values of ZTH parameters [$\delta = 0.06$ inch, $R_0 = 2.4$ m, $f = 2.4$, $\eta = 1.3 \times 10^{-8}$ Ohm-m (Inconel 625)⁷]. This vertical field is reduced by the contribution from the shell, and on a fast time scale compared with the shell time constant, the vertical field would be 12 G for a loop voltage of 100 V [see Eq. (C7) of Appendix C].

Finally, we conclude this section with some useful approximate formulae. The current density in the liner or shell, considered as an annular conductor, may be represented as

$$j_z = \frac{1}{2\pi r_0 \delta} (I_0 + I_1 \cos \theta), \quad (32)$$

where r_0 is the liner or shell radius, and δ is its thickness. Such a current distribution produces a vertical field given by

$$B_z = \frac{\mu_0 I_0}{4\pi R_0} \left(\log \frac{8R_0}{r_0} - \frac{1}{2} \right) - \frac{\mu_0 I_1}{4\pi r_0}. \quad (33)$$

The externally applied vertical field from the poloidal field windings should have the proper index for stability. Toroidal effects naturally stabilize in-out shifts, and thus a small vertical index $\partial B_Z/\partial R$ (which equals $\partial B_R/\partial Z$ from $\nabla \times \mathbf{B} = 0$) can be used to give positive stability for up-down shifts ($\partial B_R/\partial Z > 0$), while still retaining in-out stability. The criterion, using $\Lambda \approx 0$ as expected in the Reversed Field Pinch, is³

$$0 < n < 1.3, \quad (34)$$

where

$$n \equiv -\frac{R_0}{B_Z^{ext}} \frac{\partial B_Z^{ext}}{\partial R}. \quad (35)$$

III. NONAXISYMMETRIC MAGNETIC FIELDS

In this section, for simplicity, the toroidal plasma geometry is simulated by a cylinder with identified ends, as shown in Fig. 2. The fact that the liner and the shell are not concentric is also ignored for simplicity. The proposed nonaxisymmetric field measurements are discussed in Appendix B.

The reversed-field-pinch magnetic field, although approximately axisymmetric, contains small nonaxisymmetric components, which will be denoted by $\tilde{\mathbf{B}}$. For analysis purposes, we expand $r\tilde{\mathbf{B}}$ as a Fourier series in θ, z , and t :

$$r\tilde{\mathbf{B}} = \text{Re} \left\{ \sum_s G_s(r) \exp[i(m\theta + kz + \omega t)] \right\}, \quad (36a)$$

where s denotes the set of mode indices $\{m, k, \omega\}$ and $G_s(r)$ is a complex mode amplitude.

From Maxwell's equations, $\nabla \cdot \tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{B}} = 0$ in the vacuum region outside the plasma. Neglecting toroidal effects, this implies⁴

$$\begin{aligned} \tilde{B}_\theta &= \text{Re} \left\{ \sum_s -iU \frac{m}{r} G'_s \exp[i(m\theta + kz + \omega t)] \right\} \\ \tilde{B}_z &= \text{Re} \left\{ \sum_s -iUk G'_s \exp[i(m\theta + kz + \omega t)] \right\}, \end{aligned} \quad (36b)$$

and that $G_s(r)$ satisfies

$$(UG'_s)' = UVG_s$$

with

$$\begin{aligned} U &\equiv \frac{r}{m^2 + k^2 r^2} \\ V &\equiv \frac{m^2 + k^2 r^2}{r^2}. \end{aligned}$$

If the shell is effective, $G_s(r)$ in the liner-shell interspace must vanish at the shell radius $r = r_S$ so that $G(r) \approx G(r_l)(r_S - r)/\Delta r$, where $\Delta r = r_S - r_l$. Thus, using Eqs. (36), the relative size of the components of $\tilde{\mathbf{B}}$, outside the liner, is

$$\tilde{B}_r : \tilde{B}_\theta : \tilde{B}_z = 1 : \frac{im}{m^2 + k^2 r_l^2} \frac{r_l}{\Delta r} : \frac{ikr_l}{m^2 + k^2 r_l^2} \frac{r_l}{\Delta r}, \quad (37)$$

for a given k and m . The liner partially shields magnetic fluctuations, and inside the liner (see Appendix F)

$$\tilde{B}_r : \tilde{B}_\theta : \tilde{B}_z = 1 : F \frac{im}{m^2 + k^2 r_l^2} \frac{r_l}{\Delta r} : F \frac{ikr_l}{m^2 + k^2 r_l^2} \frac{r_l}{\Delta r},$$

where

$$F = \frac{i\omega\mu_0\Delta r f \delta}{\eta} \frac{m^2 + k^2 r^2}{f^2 m^2 + k^2 r^2} + 1, \quad (38)$$

with δ the liner thickness, η the liner resistivity, and f the bellows factor

$$f = \frac{\text{length of stretched out bellows}}{\text{actual compressed bellows length}}. \quad (39)$$

Note that Eq. (38) does not apply for very low frequencies having the time scale of the shell. Equation (38) assumes a perfect shell separated from the liner by Δr . Very low frequencies, however, penetrate the shell. The correct formula replacing Eq. (38) for very low frequencies involves Bessel functions to describe the external magnetic field.

It is known that in a reversed field pinch the spectrum of G_s , although mainly $m = 0$ and $m = 1$, contains many toroidal mode numbers n , where $k = n/R$, and a wide range of frequencies ω . As a practical matter it is impossible to measure G_s for all s and therefore completely characterize the nonaxisymmetric field at the plasma surface. More limited objectives for the nonaxisymmetric measurements must, therefore, be formulated. As examples of such objectives, the following are discussed (Appendix B describes the proposed diagnostics package).

A. Normalized Quantitative Measure of Fluctuations

External magnetic field measurements determine the nonaxisymmetric field at the plasma edge. Quantities of interest in comparing different operating modes of the machine or different machines are the normalized fluctuation amplitudes $\tilde{B}_r/B_0$, $\tilde{B}_\theta/B_0$, and $\tilde{B}_z/B_0$, where B_0 is the magnitude of the axisymmetric edge magnetic field.

Since m and particularly k are not known, Eq. (38) cannot be used to calculate F and thus obtain $\tilde{B}_\theta$ and $\tilde{B}_z$ inside the liner. These components can, however, be measured using a magnetic probe inserted just inside the liner. If the probe consists of a magnetic pickup coil inside a metal cylinder, there is shielding of magnetic signals caused by the metal cylinder. A useful formula for such geometry (see Appendix F) is

$$\frac{\tilde{B}^{(in)}}{\tilde{B}^{(\infty)}} = \frac{1}{1 + i\omega\mu_0 a\delta/(2\eta)}, \quad (40)$$

where a is the radius of the cylinder, δ its thickness, and η its resistivity. By having δ and a small and η large, the probe bandwidth can be much larger than the liner bandwidth. For example, using the parameters of a magnetic probe used on ZT-40M (0.030-inch wall, 0.375-inch o. d. stainless steel jacket) the probe bandwidth is $f_{1/2} \approx 100$ kHz. The liner bandwidth from Eq. (38), on the other hand, is $f_{1/2} \approx 2.2$ kHz, assuming $f = 2.4$, $\Delta r = 3.5$ cm, $\delta = 0.06$ inch, $\eta = 1.3 \times 10^{-6}$ Ohm-m (Inconel 625)⁷, and $kr_l \gg m$.

It is convenient to measure B_r with large loops of wire on the liner surface, where the large area of the loop compensates for the natural smallness of $\tilde{B}_r$ compared to $\tilde{B}_z$ for example, caused by small Δr (the shell-liner separation distance). An example of such an array is shown in Fig. 4. A large B_r loop is preferable to a small multiturn B_r , B_θ , or B_z coil because alignment of the large loop is not as critical as alignment of the small coil. Also, large loops, covering the whole torus as planned for ZTH, allow accurate monitoring for large-scale nonaxisymmetric errors in the equilibrium magnetic field. Furthermore, measurement of $\tilde{B}_r$ rather than $\tilde{B}_\theta$ or $\tilde{B}_z$ allows a simple physical interpretation of the measurement in terms of the displacement of the magnetic surface caused by the nonaxisymmetric field. If the outer magnetic surface is represented by the equation

$$r = r_p + \text{Re} \left\{ \sum_s \xi_s \exp[i(m\theta + kz + \omega t)] \right\},$$

then the magnetic surface displacement ξ_s is given by

$$\xi_s = \frac{G_s(r_p)}{i(mB_\theta + krB_z)},$$

with G_s defined by Eq. (36a), and B_θ and B_z the axisymmetric equilibrium fields. When the denominator $mB_\theta + krB_z$ vanishes at r_p , a resonant condition exists, and the presence of B_r for such a mode implies altered magnetic surface topology (island or ergodic region) at $r = r_p$.

Liner shielding seems unimportant for $\tilde{B}_r$, since $\tilde{B}_r$ is continuous across the liner. However, in order for $\tilde{B}_r$ to be nonzero, it must already have penetrated the liner. Therefore, inside the liner $\tilde{B}_r/\tilde{B}_z$ or $\tilde{B}_r/\tilde{B}_\theta$ is small because of the liner shielding factor [see Eq. (38)]. Thus, when the liner shielding factor becomes infinite (for example, when $\omega \rightarrow \infty$), nonaxisymmetric fields can exist inside the plasma even when $\tilde{B}_r = 0$. The magnitude of these fields is characterized by $\tilde{B}_\theta$ or $\tilde{B}_z$ inside the liner, which can be measured using a

magnetic probe inserted just inside the liner. Of these components, $\tilde{B}_z$ is usually larger since the effective kr usually exceeds the effective m in a reversed field pinch [see Eq. (38)].

B. Modal Analysis

A modal analysis in θ is feasible using a small number of measuring coils arranged in the θ direction, since only $m = 0$ and $m = 1$ are dominant modes. This can be checked by using enough coils to determine $\cos m\theta$ and $\sin m\theta$ modes for $m = 0, 1$ and 2 to see whether or not $m = 2$ is much smaller than $m = 1$ or $m = 0$.

In the z direction, a full modal analysis is, in practice, ruled out because of the very large number of measurements required. Some information about the spectrum of n numbers ($n = kR$) can be obtained with a partial array of measuring coils. For example, Fig. 5 shows the response of a four-element array of point measuring coils, spaced by Δz , mixed in external circuitry with relative signal phasings

	<i>coil</i> ₁	<i>coil</i> ₂	<i>coil</i> ₃	<i>coil</i> ₄
$S_a :$	1	1	1	1
$S_b :$	1	-1	1	-1
$S_c :$	1	0	-1	0
$S_d :$	1	1	-1	-1

for output signals S_a , S_b , S_c , and S_d . The maximum response n numbers for these signals are

$$\begin{aligned} S_a : \quad n_{max} &= 0 \\ S_b : \quad n_{max} &= \frac{\pi R}{\Delta z} \\ S_c, S_d : \quad n_{max} &\approx \frac{\pi R}{2\Delta z}. \end{aligned}$$

Thus, this type of array provides a three-point characterization of the spectrum of n numbers. This is a simple example of thinking that could be applied to a larger array of coils.

Figure 5 shows the response of such a phased array of *point coils* versus mode number n . An equal admixture of $\sin kz$ and $\cos kz$ signals is assumed (square root of sum of squares). Figure 6 shows the result of having large B_r loops rather than point coils. The loops are as shown in Fig. 4.

C. Space/Time Correlation Analysis

The objective here is to determine a convection time between different spatial points for various types of spatial signatures. For example, for measuring coils or loops at positions z_1 and z_2 detecting signals $S_1(t)$ and $S_2(t)$, one can evaluate the time correlation function

$$C(\tau) = \frac{1}{T} \int_0^T S_1(t) S_2(t - \tau) dt,$$

for a large block of data representing equivalent 'steady-state' conditions. If $C(\tau)$ versus τ has a peak at some time τ , this represents the convection time for a magnetic disturbance between z_1 and z_2 . With an array of measuring coils in the θ and z directions, one can Fourier analyze in one direction and measure the convection time in the other, obtaining convection velocity as a function of Fourier mode, for example, the velocity of $m = 1$ or $m = 0$ magnetic signals in the z direction.

D. Triggered Time Window (Boxcar) Analysis⁵

This method uses an event trigger defined in some way, for example, a signal above some threshold level on one measuring coil, to define a time window centered at the time of the event. The average signals from other detectors, shifted in time to be centered in the time window at the trigger time, are accumulated, that is,

$$B(t) = \frac{1}{N} \sum_{k=1}^N S(t - t_{trig}^{(k)}),$$

where k denotes the event number. As in the time-correlation analysis, a large block of data representing steady-state conditions is used.

Some examples of triggers are:

- 1) Signal amplitudes above some threshold in a particular coil;
- 2) $m = 0$ or 1 signal above threshold from an array of θ coils;
- 2) High, medium, or low n signals from a four-coil phased array of z coils above threshold;
- 4) Amplitude above threshold in coil 1 and amplitude of definite polarity above threshold in coil 2. The event time is defined in some definite way in terms of time 1 and time 2.

APPENDIX A—VACUUM MAGNETIC FIELD SOLUTIONS

For axisymmetry, the $\hat{\phi}$ component of Maxwell's equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, becomes

$$\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R} = \mu_0 j_\phi.$$

In terms of χ ,

$$\frac{\partial^2 \chi}{\partial R^2} - \frac{1}{R} \frac{\partial \chi}{\partial R} + \frac{\partial^2 \chi}{\partial Z^2} = -2\pi R \mu_0 j_\phi. \quad (\text{A1})$$

It is interesting to note how inside the plasma, Eq. (A1) becomes the Grad-Shafranov equation⁶

$$\frac{\partial^2 \chi}{\partial R^2} - \frac{1}{R} \frac{\partial \chi}{\partial R} + \frac{\partial^2 \chi}{\partial Z^2} = -\mu_0 \left(J \frac{dJ}{d\chi} + R^2 \frac{dp}{d\chi} \right), \quad (\text{A2})$$

where J is the poloidal current and p is the plasma pressure. In the vacuum region outside the plasma, which we are concerned with here, Eq. (A1) takes the form

$$\frac{\partial^2 \chi}{\partial R^2} + \frac{\partial^2 \chi}{\partial Z^2} = \frac{1}{R} \frac{\partial \chi}{\partial R}. \quad (\text{A3})$$

In what follows, we go over to the curved r, θ, z coordinate system shown in Fig. 1. The transformation between R, ϕ, Z and r, θ, z is given by

$$\begin{aligned} R &= R_0 - r \cos \theta \\ \phi &= \frac{z}{2\pi R} \\ Z &= r \sin \theta. \end{aligned} \quad (\text{A4})$$

The operator

$$\partial^2 / \partial R^2 + \partial^2 / \partial Z^2 \equiv \nabla^2$$

in Eq. (A3) is well known to transform under Eq. (A4) to

$$(1/r) \partial / \partial r (r \partial / \partial r) + (1/r^2) \partial^2 / \partial \theta^2.$$

Equation (A3) therefore becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} = \frac{1}{R} \left(-\cos \theta \frac{\partial \chi}{\partial r} + \frac{\sin \theta}{r} \frac{\partial \chi}{\partial \theta} \right), \quad (\text{A5})$$

where we have also transformed $\partial \chi / \partial R$ to r, θ, z coordinates.

The Fourier-analyzed version of Eq. (A5) is a set of coupled, linear, second-order ordinary differential equations involving $\chi_m(r)$. Retaining only the lowest-order terms in $1/R_0$ and truncating for $m > 2$,

$$\begin{aligned} \chi_0'' + \frac{\chi_0'}{r} &= -\chi_0' \frac{r}{2R_0} - \chi_{1c}' \frac{1}{2R_0} - \chi_{1c} \frac{1}{2R_0 r} \\ \chi_{1c}'' + \frac{\chi_{1c}}{r} - \frac{\chi_{1c}}{r^2} &= -\chi_0' \frac{1}{R_0} \\ \chi_{1s}'' + \frac{\chi_{1s}'}{r} - \frac{\chi_{1s}}{r^2} &= 0 \\ \chi_{2c}'' + \frac{\chi_{2c}'}{r} - \frac{4\chi_{2c}}{r^2} &= -\chi_0' \frac{r}{2R_0} - \chi_{1c}' \frac{1}{2R_0} + \frac{\chi_{1c}}{2R_0 r} \\ \chi_{2s}'' + \frac{\chi_{2s}'}{r} - \frac{4\chi_{2s}}{r^2} &= -\chi_{1s}' \frac{1}{2R_0} + \chi_{1s} \frac{1}{2R_0 r}. \end{aligned} \quad (\text{A6})$$

Equations (A6) are solved using an aspect-ratio expansion about the zero-order solution

$$\chi^{(0)} = -IR_0 \log r + A. \quad (\text{A7})$$

The complete solution is given by

$$\begin{aligned} \chi = & -IR_0 \log r + A + \left(\frac{3}{16} \frac{I}{R_0} - \frac{C}{4R_0} - \frac{E}{4R_0} \right) r^2 - \frac{I}{8R_0} r^2 \log r \\ & + \left(Cr + \frac{D}{r} + \frac{I}{2} r \log r \right) \cos \theta \\ & + \left(Er + \frac{F}{r} \right) \sin \theta \\ & + \left(Gr^2 + \frac{H}{r^2} - \frac{D}{4R_0} + \frac{I}{16R_0} r^2 \log r \right) \cos 2\theta \\ & + \left(Pr^2 + \frac{Q}{r^2} - \frac{F}{4R_0} \right) \sin 2\theta, \end{aligned} \quad (\text{A8})$$

with 10 free linear parameters: $I, A, C, D, E, F, G, H, P,$ and Q . These 10 parameters are used as the fit parameters to fit the poloidal fluxes and poloidal magnetic fields ($\sim \partial\chi/\partial r$) measured at different poloidal locations. The more physically meaningful parameters defined by the vector X in Eq. (24b) and used to state the final result are related to $\{I, A, C, D, E, F, G, H, P, Q\}$ by a straightforward linear transformation.

APPENDIX B—PROPOSED ZTH ELECTRICAL DIAGNOSTICS

The electrical diagnostics planned for ZTH will be located primarily in the region between the liner and the shell (interspace diagnostics). There are also some diagnostics (external diagnostics) located outside the shell. In addition, currents in the toroidal and poloidal field windings will be measured and used in the equilibrium calculations. The table below lists the interspace and external diagnostics.

Interspace

I_ϕ Rogowskii coils	4 coils equally spaced toroidally.
χ loops	8 loops equally spaced poloidally.
ψ loops	16 two-turn loops, equally spaced toroidally.
B_ϕ partial Rogowskii coils	2 coils, one on top and one on bottom of torus, at same toroidal location.
B_θ and B_ϕ coil arrays	8 B_θ and 8 B_ϕ coils equally spaced poloidally per array, 4 arrays equally spaced toroidally.
B_r arrays	136 two-turn loops covering entire liner surface, 4 loops ($m = 0, 1$) at each toroidal location.

External

I_ϕ Rogowskii coils	2 coils equally spaced toroidally.
B_θ coil arrays	8 B_θ coils equally spaced poloidally per array, 2 arrays equally spaced toroidally.

The ψ and B_r loops are two turns in order to have a grounded center tap. The layout of the interspace diagnostics on the torus is shown in plan view in Fig. b1 (except for B_r loops) and Fig. b2 (B_r loops).

The B_r loop layout covers the entire liner surface so as to allow detection of flux penetration of the liner anywhere on the torus surface. (Severe limitations on the number of fiber optic links available, however, will allow monitoring only a few loops at a time.) Coarse and fine loop sizes have been chosen to allow monitoring of toroidal n -numbers for 0 to 64, with $m = 0$ or 1. The coarse B_r loop size is 1/16 of the toroidal circumference, and the fine loop size is 1/8 of that. The χ loops and B_θ coil arrays are used to

measure the axisymmetric equilibrium field, as described in the text. The ψ loops measure the toroidal flux and are also useful in following traveling flux disturbances around the torus. The I_ϕ Rogowskii coils measure the plasma plus liner current at four locations, which should all be equal unless a breakdown occurs between the liner and the shell. The B_ϕ partial Rogowskii coils provide local measurement of the toroidal field outside the plasma. The B_θ and B_ϕ coil arrays consist of 8 B_θ and 8 B_ϕ coils each and will be capable of analyzing $m = 0, 1, 2$ and $n = 0, 1, 2$ mode numbers. This information will be complemented by information from the B_r loops, which will be better at distinguishing higher n numbers.

Any difference between the external and internal I_ϕ Rogowskii coils implies a fault condition with current conduction across the poloidal field gap. The external B_θ coils will be located immediately over the interspace B_θ coils. The difference between the value of B_θ inside and outside the shell will provide a direct measure of the $m = 1$ shell current. Shell current, because of the poloidal field gap in the shell, causes nonaxisymmetric error fields to be generated at the gap.

APPENDIX C—VACUUM SHOTS

We consider vacuum shots where 1) only the toroidal field windings are driven, and 2) where only the poloidal field windings are driven. Vacuum shots of type 1 are useful to experimentally measure the poloidal resistance of the liner R_θ , if the toroidal field inside the liner is measured (say by an inserted magnetic probe). Also, this type of shot checks the calibration of ψ loops and B_ϕ coils and checks for misalignment of B_θ coils and B_r loops.

Vacuum shots in category 2 provide known signals to check the Rogowskii coils, χ loops, B_θ coils, and B_r loops. Assuming the liner and shell resistivities are uniform and known (Appendix D discusses measurements of the toroidal and poloidal liner resistances), the liner and shell can be conceptually divided up into a large number of discretized current loops [see Eq. (27)], and an exact calculation for the resulting L-R system carried out. An even better calculation would include the effect of the poloidal field gap in the shell.

It is also useful to have an approximate calculation, which gives the basic idea of system behavior. Such a calculation is given in what follows.

A solution for χ in the three regions, (I) inside liner, (II) interspace between liner and shell, and (III) outside shell, is given by

$$\begin{aligned}\chi^{(I)} &= a + br \cos \theta \\ \chi^{(II)} &= a - \mu_0 I_l R_0 \log \frac{r}{r_l} + \left(cr + \frac{d}{r} + \frac{\mu_0 I_l r}{2} \log \frac{r}{r_l} \right) \cos \theta \\ \chi^{(III)} &= a - \mu_0 I_l R_0 \log \frac{r}{r_l} + \left[b_{ex} r + \frac{e}{r} - \frac{\mu_0 I_l r}{2} \left(\log \frac{8R_0}{r} - 1 \right) \right] \cos \theta,\end{aligned}\tag{C1}$$

where a , b , c , d , e , b_{ex} , and I_l are constants. The constants a , b_{ex} , and I_l are imagined to be known, representing the poloidal flux on axis, the applied vertical field, and the liner current. This solution is a special case of Eq. (A8), obtained by neglecting terms down by two orders in aspect ratio. Also, the $\cos \theta$ term in $\chi^{(III)}(r)$ is designed to have the limit

$$\chi_{\cos}^{(III)}(r) \rightarrow -\frac{\mu_0 I_l r}{2} \left(\log \frac{8R_0}{r} - 1 \right),$$

as $r \rightarrow \infty$, for the case $b_{ex} = 0$, which agrees with the flux far from a current loop carrying current I_l [see Eq. (E14)]. Thus, b_{ex} represents the externally applied vertical field. On a long time scale, where liner and shell are unimportant, χ is given by $a + b_{ex} r \cos \theta$, with a produced by the Ohmic poloidal field windings, and b_{ex} produced by the equilibrium windings. From Eq. (4),

$$B_z = -\frac{b_{ex}}{2\pi R_0},$$

and this relation can be used to define b_{ex} in terms of currents in the poloidal field windings.

There are four boundary conditions that determine the four unknowns in Eq. (C1), the unknowns being b , c , d , and e . These boundary conditions are Eq. (21) (using $1/\eta \sim 1 - r \cos \theta / R_0$, as discussed following

Eq. (22)] and the continuity condition $[\chi] = 0$, applied at the liner and shell radii r_l and r_s (these radii are to the center of the finite thickness liner and shell; the fact that the shell is offset with respect to the liner is ignored). Doing the algebra, one finds the following differential equation giving b in terms of b_{ex} and I_l :

$$b + \left(\frac{\mu_0 \delta_l r_l}{2\eta_l} + \frac{\mu_0 \delta_s r_s}{2\eta_s} \right) \dot{b} + \frac{\mu_0 \delta_l r_l}{2\eta_l} \frac{\mu_0 \delta_s r_s}{2\eta_s} \left(1 - \frac{r_l^2}{r_s^2} \right) \ddot{b} = b_{ex} - \frac{\mu_0 I_l}{2} \left(\log \frac{8R_0}{r_l} - \frac{1}{2} \right) - \frac{\mu_0 \dot{I}_l}{4} \frac{\mu_0 \delta_s r_s}{2\eta_s} \left(1 - \frac{r_l^2}{r_s^2} + 2 \log \frac{r_s}{r_l} \right). \quad (C2)$$

The additional unknowns are given in terms of b by the equations

$$\begin{aligned} c &= b + \frac{\mu_0 I_l}{4} + \frac{\mu_0 \delta_l r_l}{2\eta_l} \dot{b} \\ d &= -r_l^2 \left(\frac{\mu_0 I_l}{4} + \frac{\mu_0 \delta_l r_l}{2\eta_l} \dot{b} \right) \\ e &= -r_s^2 \left[b_{ex} - b - \frac{\mu_0 I_l}{2} \left(\log \frac{8R_0}{r_l} - 1 \right) \right]. \end{aligned} \quad (C3)$$

In Eq. (C2), $I_l(t)$ is thought of as given, although actually it is obtained from the equation

$$I_l = -\frac{\dot{a}}{R_\phi}, \quad (C4)$$

where R_ϕ is the liner resistance in the toroidal direction. Similarly, $b_{ex}(t)$ is a given quantity, representing the externally applied vertical field. For "steady state" conditions ($\dot{a}$ is imagined to be constant),

$$b = b_{ex} - \frac{\mu_0 I_l}{2} \left(\log \frac{8R_0}{r_l} - \frac{1}{2} \right). \quad (C5)$$

On a fast time scale compared to the shell time constant, Eq. (C2) may be time integrated to obtain the approximate relation

$$b + \frac{\mu_0 \delta_l r_l}{2\eta_l} \left(1 - \frac{r_l^2}{r_s^2} \right) \dot{b} = -\frac{\mu_0 I_l}{4} \left(1 - \frac{r_l^2}{r_s^2} + 2 \log \frac{r_s}{r_l} \right). \quad (C6)$$

Thus, the "steady state" result on this time scale is

$$b = -\frac{\mu_0 I_l}{4} \left(1 - \frac{r_l^2}{r_s^2} + 2 \log \frac{r_s}{r_l} \right) \quad (C7)$$

instead of Eq. (C5).

Equations (C2) and (C3) are the result sought. By solving them one obtains χ and hence any poloidal field quantity of interest for a vacuum shot with given Ohmic $[a(t)]$ and equilibrium field $[b_{ex}(t)]$ programming. Such a calculation (perhaps using a more exact model) can be used to check the poloidal field diagnostics as well as monitoring the liner and shell electrical characteristics.

APPENDIX D—ANALYSIS FORMULAE FOR AXISYMMETRIC EQUILIBRIUM

In this appendix, explicit formulae are given for the axisymmetric equilibrium quantities of greatest interest. The column vector $X(r_p)$ defined in Eq. (24b) is given by

$$\begin{aligned} X(r_p) &= CX_-(r_l) \\ X_-(r_l) &= X_+(r_l) - E\dot{X}_+(r_l) \\ X_+(r_l) &= AY + BZ, \end{aligned} \quad (D1)$$

where Y and Z are column vectors containing the 8 measured flux signals and 5 measured B_θ mode signals respectively [see Eq. (24b)]; A , B , and C are as given in Eqs. (24c) and (25b); and E is given by [see Eq. (21)]

$$E = \frac{\mu_0 \delta}{r_l f \eta} \begin{pmatrix} 0 & \mathbf{I} \\ 0 & 0 \end{pmatrix} \quad (\text{D2})$$

in terms of 5×5 null, and unit matrices 0 and $\mathbf{I}$. $X_-(r_l)$ and $X_+(r_l)$ refer to X just inside and just outside the liner.

The equilibrium signals are $\chi_{1c}(r_p)$, $\chi_{1s}(r_p)$, $\chi_{2c}(r_p)$, and $\chi_{2s}(r_p)$. These quantities are the second through fifth components of $X(r_p)$.

The formulae for I_p , V_ϕ , $B_{\phi p}$, ψ_p , and $\Lambda(r_p)$ are as follows. The plasma current I_p is given by

$$I_p = I - I_l, \quad (\text{D3})$$

where I is the total plasma plus liner current measured by the Rogowskii, and the liner current I_l is given by

$$I_l = -\frac{\dot{\chi}_0(r_l)}{R_\phi}. \quad (\text{D4})$$

In Eq. (D4), $\dot{\chi}_0(r_l)$ is the time derivative of the first component of $X_+(r_l)$ [or $X_-(r_l)$], and R_ϕ is the measured liner resistance in the ϕ direction. We measure R_ϕ by driving the liner with a known voltage during a vacuum shot, measuring I_l with the Rogowskii, and using Eq. (D4).

The toroidal loop voltage at the plasma surface is given by

$$V_\phi = -\dot{\chi}_0(r_p), \quad (\text{D5})$$

where $\chi_0(r_p)$ is the first component of $X(r_p)$.

The toroidal magnetic field just outside the plasma $B_{\phi p}$ is given by

$$B_{\phi p} = \frac{\mu_0(J - J_l)}{2\pi R}, \quad (\text{D6})$$

where J is the total poloidal current in the external windings, and J_l is the poloidal current in the liner. By letting $R = R_0$ in Eq. (D6), we obtain an average value of $B_{\phi p}$. The poloidal current in the liner is given by

$$J_l = -(\dot{\psi} - \Delta\dot{\psi}_{out}) \frac{1}{R_\theta}, \quad (\text{D7})$$

where ψ is the flux measured by the flux loop, and $\Delta\psi_{out}$ is the correction for flux between flux loop and liner, given by

$$\Delta\psi_{out} = \pi(r_\psi^2 - r_l^2) \frac{\mu_0 J}{2\pi R_0}, \quad (\text{D8})$$

where r_ψ is the radius of the flux loop. In Eq. (D7), R_θ is the liner resistance in the poloidal direction, given, approximately, by

$$R_\theta = \frac{\eta r_l}{f R_0 \delta}, \quad (\text{D9})$$

where η is the resistance of the liner, R_l is the liner radius, f is the bellows factor, and δ is the thickness of the liner material. We might measure R_θ by using a vacuum shot with an internal B_ϕ probe inside the liner. Alternatively, R_θ can be calculated. Equation (D9) gives $R_\theta = 60 \mu\Omega$ for nominal ZTH parameters ($\eta = 1.3 \times 10^{-3}$ Ohm-m, $r_l = 0.409$ m, $f = 2.4$, $R_0 = 2.4$ m, $\delta = 0.06$ inch). Equation (D9) neglects the effect of the poloidal rings used to support the liner in ZTH. These will approximately halve the poloidal resistance.

The toroidal flux at the plasma surface is given by

$$\psi_p = \psi - \Delta\psi_{out} - \Delta\psi_{in}, \quad (\text{D10})$$

with $\Delta\psi_{in}$ an additional correction for the flux inside the liner, between the plasma surface and the liner,

$$\Delta\psi_{in} = \pi(r_l^2 - r_p^2)B_{\phi p}. \quad (D11)$$

Shafranov Λ is given by

$$\Lambda(r_p) = -\frac{r_p\chi'_{1c}(r_p)}{r_p\chi'_0(r_p)}\frac{R_0}{r_p}, \quad (D12)$$

where $r_p\chi'_{1c}(r_p)$ and $r_p\chi'_0(r_p)$ are the seventh and sixth components of $X(r_p)$. Alternatively, Eq. (26) can be solved for Λ , giving

$$\Lambda(r_p) = -B_Z \frac{4\pi R_0}{\mu_0 I_p} - \log \frac{8R_0}{r_p} + \frac{1}{2}. \quad (D13)$$

In Eq. (D13), B_Z must be calculated from the poloidal winding currents and from Eq. (28) for the liner and shell contributions. If Λ from Eq. (D13) differs from that of Eq. (D12), then the assumption of an axisymmetric equilibrium becomes suspect.

APPENDIX E—VERTICAL FIELD FROM CURRENT LOOP

The vector potential from a current loop carrying current I is given by

$$\mathbf{A}(\mathbf{r}) = \mu_0 I \int \frac{d\mathbf{r}'}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (E1)$$

using Biot and Savart's law. We use a Cartesian X , Y , and Z coordinate system to express $\mathbf{r}$ and $\mathbf{r}'$ in the integral, assuming the loop is located at $R = R_l$ and $Z = Z_l$,

$$\mathbf{r} = R \cos \phi \hat{\mathbf{X}} + R \sin \phi \hat{\mathbf{Y}} + Z \hat{\mathbf{Z}}, \quad (E2)$$

and

$$\mathbf{r}' = R_l \cos \phi' \hat{\mathbf{X}} + R_l \sin \phi' \hat{\mathbf{Y}} + Z_l \hat{\mathbf{Z}}. \quad (E3)$$

Substituting Eqs. (E2) and (E3) into (E1) one finds that only the $\hat{\phi}$ component of $\mathbf{A}$ is nonzero and that this component is given by

$$A_\phi = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R_l \cos \alpha d\alpha}{\sqrt{R^2 + R_l^2 - 2RR_l \cos \alpha + (Z - Z_l)^2}}. \quad (E4)$$

Equation (E4) can be written in the form

$$A_\phi = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R_l \cos \alpha d\alpha}{\sqrt{2RR_l[h^2 + 1 - \cos \alpha]}}, \quad (E5)$$

using the positive quantity h^2 defined as

$$h^2 \equiv \frac{(R - R_l)^2 + (Z - Z_l)^2}{2RR_l}. \quad (E6)$$

We now define a special function $\mathcal{I}(h)$, as

$$\mathcal{I}(h) \equiv \int_0^{2\pi} \frac{\cos \alpha d\alpha}{\sqrt{h^2 + 1 - \cos \alpha}}. \quad (E7)$$

A_ϕ is then given in terms of $\mathcal{I}(h)$ by

$$A_\phi = \frac{\mu_0 I}{4\pi} \sqrt{\frac{R_l}{2R}} \mathcal{I}(h). \quad (E8)$$

Note that $2\pi RA_\phi$ is symmetric in the coordinates of the loop and the coordinates of the spatial point, as it must be (symmetry of the inductance matrix). The vertical field, from Eq. (1), is given by

$$B_z = \frac{\mu_0 I}{8\pi R\sqrt{2}} \sqrt{\frac{R_l}{R}} \left[I(h) + \frac{I'(h)}{h} \left(\frac{R}{R_l} - 1 - h^2 \right) \right]. \quad (\text{E9})$$

The function $I(h)$ is evaluated in the table below:

h	$I(h)$	$I_0(h)$	$I_\infty(h)$
0.03	9.167	9.162	
0.1	5.796	5.757	
0.3	2.896	2.650	
0.7	1.128		
1.0	$6.336 \cdot 10^{-1}$		
1.5	$2.808 \cdot 10^{-1}$		
2.0	$1.432 \cdot 10^{-1}$		
3.0	$4.991 \cdot 10^{-2}$		$5.818 \cdot 10^{-2}$
10.0	$1.548 \cdot 10^{-3}$		$1.571 \cdot 10^{-3}$
30.0	$5.810 \cdot 10^{-5}$		$5.818 \cdot 10^{-5}$

The fact that I depends on only one parameter h (even though the problem is two dimensional) means that $I(h)$ can be rapidly calculated using a one-dimensional table lookup algorithm.

Also shown in the table are the asymptotic expansions

$$I_0(h) = 2\sqrt{2} \left(\log \frac{4\sqrt{2}}{h} - 2 \right), \quad (\text{E10})$$

and

$$I_\infty(h) = \frac{\pi}{2h^3}, \quad (\text{E11})$$

valid for $h \rightarrow 0$, and $h \rightarrow \infty$.

An important limiting case is the large-aspect-ratio limit. If $R_l = R_0$, and $Z_l = 0$,

$$h = \frac{r}{R_0\sqrt{2}} \frac{1}{\sqrt{1 - \frac{r}{R_0} \cos \theta}}. \quad (\text{E12})$$

Since h is small in the large-aspect-ratio limit, we use Eq. (E10) and obtain

$$I(h) \approx 2\sqrt{2} \left(\log \frac{8R_0}{r} - 2 - \frac{r}{2R_0} \cos \theta \right).$$

Equation (E8) then gives

$$A_\phi = \frac{\mu_0 I}{2\pi} \left[\log \frac{8R_0}{r} - 2 + \frac{r}{2R_0} \left(\log \frac{8R_0}{r} - 3 \right) \cos \theta \right], \quad (\text{E13})$$

and $\chi = 2\pi RA_\phi$ is given by

$$\chi = \mu_0 I R_0 \left[\log \frac{8R_0}{r} - 2 - \frac{r}{2R_0} \left(\log \frac{8R_0}{r} - 1 \right) \cos \theta \right]. \quad (\text{E14})$$

APPENDIX F—SHIELDING OF MAGNETIC FLUCTUATIONS BY METAL LINER

Toroidal curvature is ignored in this appendix, and the straight r, θ, z coordinate system shown in Fig. 2 is used.

The bellows configuration of the liner results in increasing the effective resistivity in the z direction, while decreasing the effective resistivity in the θ direction. Ohm's law applied to an annular liner with such an anisotropic resistivity gives

$$\begin{aligned} E_z &= f\eta j_z \\ E_\theta &= \frac{\eta}{f} j_\theta, \end{aligned} \quad (\text{F1})$$

where j_θ and j_z are the current density in the liner, η is the liner resistivity, and f is the bellows factor [see Eq. (39)]. Because of the presence of poloidal support rings in ZTII, a more accurate approximation (not used here) is to assume $f_\theta \approx 2f_z$.

From the radial component of $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$,

$$r \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t}. \quad (\text{F2})$$

The additional relationships required are obtained from the r -integral of $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ across the liner:

$$\begin{aligned} [B_\theta] &= \mu_0 j_z \delta \\ [B_z] &= -\mu_0 j_\theta \delta, \end{aligned} \quad (\text{F3})$$

where the square bracket denotes the jump in the enclosed quantity and δ is the liner thickness.

Equation (F2) is applied to $\tilde{\mathbf{B}}$, substituting the representation given by Eq. (36a), and using Eqs. (F1) and (F3). The result gives the jump in G' across the liner (for notational simplicity, the subscript s denoting the mode numbers is omitted in this section):

$$\frac{[G']}{G} \equiv g = \frac{-i\omega\mu_0 f\delta(m^2 + k^2 r_l^2)}{\eta(m^2 f^2 + k^2 r_l^2)}. \quad (\text{F4})$$

If the shell is effective, $G(r_S)$ vanishes at $r = r_S$. Using the vacuum solutions for $G(r)$ in the region $r_l < r < r_S$, we can write

$$G \sim r \left[I'_m(|k|r) K'_m(|k|r_S) - K'_m(|k|r) I'_m(|k|r_S) \right], \quad (\text{F5})$$

which is a linear combination of vacuum solutions $[rI'_m(|k|r)]$ and $[rK'_m(|k|r)]$ satisfying the boundary condition $G(r_S) = 0$. Using the Bessel function identity $(xI')' = (m^2 + x^2)I/x$,

$$\frac{G'(r_l)}{G(r_l)} \equiv h = \frac{m^2 + k^2 r_l^2}{k r_l^2} \frac{I_m(|k|r_l) K'_m(|k|r_S) - K_m(|k|r_l) I'_m(|k|r_S)}{I'_m(|k|r_l) K'_m(|k|r_S) - K'_m(|k|r_l) I'_m(|k|r_S)}. \quad (\text{F6})$$

If r_l is close to r_S , we can expand $K'_m(|k|r_S)$ and $I'_m(|k|r_S)$ as

$$\begin{aligned} K'_m(|k|r_S) &= K'_m(|k|r_l) + |k|(r_S - r_l) K''_m(|k|r_l) \\ I'_m(|k|r_S) &= I'_m(|k|r_l) + |k|(r_S - r_l) I''_m(|k|r_l). \end{aligned} \quad (\text{F7})$$

Substituting Eq. (F7) into Eq. (F6) and using the above-mentioned Bessel function identity to express $I''_m(|k|r)$ and $K''_m(|k|r)$ in terms of 0th and 1st order derivatives gives the result

$$h \approx -\frac{1}{r_S - r_l}, \quad (\text{F8})$$

as is clearly true for $r_l \approx r_S$.

Equation (F8) (applied for $r > r_l$) and Eq. (F4) are the results sought. These results express $G'(r_l)$ outside and inside the liner in terms of $G(r_l)$, which is continuous across the liner, giving the result:

$$\begin{aligned} G'^{(out)} &= hG \\ G'^{(in)} &= (h - g)G. \end{aligned}$$

Using Eqs. (36) we may write this result in terms of the magnetic field inside and outside the liner, obtaining

$$\begin{aligned} \tilde{B}_r^{(in)} &= \tilde{B}_r^{(out)} \\ \tilde{B}_\theta^{(in)} &= F \tilde{B}_\theta^{(out)} \\ \tilde{B}_z^{(in)} &= F \tilde{B}_z^{(out)}, \end{aligned} \tag{F9}$$

where the liner shielding factor F is given by

$$F = \frac{G'^{(in)}}{G'^{(out)}} = -g/h + 1, \tag{F10}$$

and by Eq. (38), when the approximation given by Eq. (F8) is used for h .

The shielding of magnetic signals by a cylindrical metal jacket around a magnetic probe is not given simply by a limit of the above formulae, since the desired quantity is $\tilde{B}^{(in)}/\tilde{B}^{(\infty)}$ rather than $\tilde{B}^{(in)}/\tilde{B}^{(out)}$, where $\tilde{B}^{(\infty)}$ is the magnetic field far from the probe.

First consider a transverse field. General solutions for G outside and inside the probe are given by

$$\begin{aligned} G^{(out)} &= Cr + \frac{D}{r} \\ G^{(in)} &= Er. \end{aligned} \tag{F11}$$

The desired ratio E/C is obtained using Eq. (F4) and the continuity condition $G^{(out)} = G^{(in)}$, applied at the probe jacket radius, $r = a$. The result is

$$\frac{\tilde{B}^{(in)}}{\tilde{B}^{(\infty)}} = \frac{E}{C} = \frac{1}{1 + ga/2}, \tag{F12}$$

which for $f = 1$ in Eq. (F4) gives Eq. (40).

For a longitudinal field of strength $\tilde{B}^{(\infty)}$ outside the cylinder and $\tilde{B}^{(in)}$ inside the cylinder, the B_z jump condition from Eq. (F3) gives

$$\tilde{B}^{(\infty)} - \tilde{B}^{(in)} = -\mu_0 j_\theta \delta, \tag{F13}$$

while equating the Ohmic voltage resulting from j_θ to the rate of change of flux gives

$$-i\omega \tilde{B}^{(in)} \pi a^2 = 2\pi \eta j_\theta. \tag{F14}$$

Equations (F13) and (F14) again give Eq. (40).

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Fig. 1. Coordinate systems used for toroidal geometry.

Fig. 2. Straight cylindrical coordinate system with identified ends, used to approximate toroidal geometry.

Fig. 3.

Axisymmetric equilibrium measuring loop and coils at the i^{th} poloidal location. The poloidal field coils are actually multiturn. The poloidal field coil signals are toroidally averaged using signals from several different toroidal locations. The flux or field signal is the time integral of the loop or coil voltage.

Fig. 4. Array of B_r loops used to obtain information about the n -number spectrum.

Fig. 5. Response versus n -number for phased array of point coils, for different signal combinations. The coil spacing is $\Delta z = 10$ cm, with $R_0 = 2.4$ m.

Fig. 6. Response versus n -number for array of B_r loops instead of point coils. The loop size is $\Delta z = 10$ cm and $R_0 = 2.4$ m.

Fig.B1. Plan view of interspace electrical diagnostics layout, exclusive of B_r loops.

Fig. B2. Plan view of B_r loop layout.