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DEVELOPMENT REPORT

A THERMOMECHANICAL THEORY OF MATERIALS UNDERGOING LARGE ELASTIC AND VISCOPLASTIC DEFORMATION

(AWBA Development Program)

S. E. MARTIN
J. B. NEWMAN

MASTER

November 1980

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BETTIS ATOMIC POWER LABORATORY
WEST MIFFLIN, PENNSYLVANIA

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FOREWORD

The Shippingport Atomic Power Station located in Shippingport, Pennsylvania was the first large-scale, central-station nuclear power plant in the United States and the first plant of such size in the world operated solely to produce electric power. This program was started in 1953 to confirm the practical application of nuclear power for large-scale electric power generation. It has provided much of the technology being used for design and operation of the commercial, central-station nuclear power plants now in use.

Subsequent to development and successful operation of the Pressurized Water Reactor in the Atomic Energy Commission (now Department of Energy, DOE) owned reactor plant at the Shippingport Atomic Power Station, the Atomic Energy Commission in 1965 undertook a research and development program to design and build a Light Water Breeder Reactor core for operation in the Shippingport Station.

The objective of the Light Water Breeder Reactor (LWBR) program has been to develop a technology that would significantly improve the utilization of the nation's nuclear fuel resources employing the well-established water reactor technology. To achieve this objective, work has been directed toward analysis, design, component tests, and fabrication of a water-cooled, thorium oxide-uranium oxide fuel cycle breeder reactor for installation and operation at the Shippingport Station. The LWBR core started operation in the Shippingport Station in the Fall of 1977 and is expected to be operated for about 4 to 5 years. At the end of this period, the core will be removed and the spent fuel shipped to the Naval Reactors Expended Core Facility for a detailed examination to verify core performance including an evaluation of breeding characteristics.

In 1976, with fabrication of the Shippingport LWBR core nearing completion, the Energy Research and Development Administration, now DOE, established the Advanced Water Breeder Applications (AWBA) program to develop and disseminate technical information which would assist U. S. industry in evaluating the LWBR concept for commercial-scale applications. The program is exploring some of the problems that would be faced by industry in adapting technology confirmed in the LWBR program. Information being developed includes concepts for commercial-scale prebreeder cores which would produce uranium-233 for light water breeder cores while producing electric power, improvements for breeder cores based on the technology developed to fabricate and operate the Shippingport LWBR core, and other information and technology to aid in evaluating commercial-scale application of the LWBR concept.

All three development programs (Pressurized Water Reactor, Light Water Breeder Reactor, and Advanced Water Breeder Applications) are under the technical direction of the Division of Naval Reactors of DOE. They have the goal of developing practical improvements in the utilization of nuclear fuel resources for generation of electrical energy using water-cooled nuclear reactors.

Technical information developed under the Shippingport, LWBR, and AWBA programs has been and will continue to be published in technical memoranda, one of which is this present report.

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ABSTRACT

A Thermomechanical Theory of Materials Undergoing
Large Elastic and Viscoplastic Deformation
(AWBA Development Program)

S. E. Martin

J. B. Newman

A thermomechanical theory of large deformation elastic-inelastic material behavior is developed which is based on a multiplicative decomposition of the strain. Very general assumptions are made for the elastic and inelastic constitutive relations and effects such as thermally-activated creep, fast-neutron-flux-induced creep and growth, annealing, and strain recovery are compatible with the theory. Reduced forms of the constitutive equations are derived by use of the second law of thermodynamics in the form of the Clausius-Duhem inequality. Observer invariant equations are derived by use of an invariance principle which is a generalization of the principle of material frame indifference.

**A Thermomechanical Theory of Materials Undergoing
Large Elastic and Viscoplastic Deformation**
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I. Introduction

Intuitively, the deformation of many materials can be regarded as a combination of two deformations, one part called "elastic" which is determined by the current temperature and load, and a part called "inelastic" which displays a dependence on the history of the temperature and load. This intuitive concept can also be supported on a molecular basis. The elastic part of the deformation can be attributed to changes in intermolecular spacing and the inelastic part attributed to dislocation motion, recrystallization, twinning, grain boundary sliding and other microscopic phenomena.

In the small strain theory of plasticity [1]*, or the small strain theory of viscoplasticity [2], the strain at a point is decomposed into a sum of an elastic strain ϵ^e and a plastic strain ϵ^p . The elastic strain is determined from the infinitesimal theory of elasticity while the plastic strain is determined from an incremental flow law such as the von Mises flow law [1]. In some applications both the elastic and the inelastic strains are large and of approximately the same order of magnitude. For example, Lee [3] points out that elastic volume reductions of 25% or more are experienced in explosive wave propagation. In light water reactors, elastic thermal strains of 3% and inelastic strains of 5% are predicted to occur in ceramic fuels during normal operation. The use of the infinitesimal theory to analyze strains of this magnitude can result in unacceptable errors and inconsistencies.

* Throughout this paper, numbers in brackets denote items in the list of References. Numbers in parenthesis denote the equations included in the text.

In a theory of large deformation elastic-inelastic behavior it is natural, as in the small strain case, to decompose the strain into an elastic and an inelastic component. However, because the decomposition of the strain is not unique, the implementation of this concept for finite deformations is more complex than for small deformations. Both additive [4] and multiplicative [3] decompositions are possible. The kinematics associated with such decompositions have been discussed by Lee [5], Sideroff [6], Kleiber [7] and Freund [8].

Theories of material behavior which utilize finite-strain decompositions have been formulated by Lee [5], Green and Naghdi [4], and Freund [8] to model elastic-plastic behavior, and by Kratochvil [9] to model elastic-viscoplastic behavior. In the present paper, Kratochvil's theory is reformulated and extended by using an invariance requirement which combines the principle of material frame indifference and the Γ -invariance proposed by Sideroff [6]. A set of constitutive assumptions is used which allows a particularly straightforward application of the invariance requirement and, using the method developed by Coleman and Noll [10], the Clausius-Duhem inequality is used to further restrict the form of the constitutive assumptions. The resulting equations imply a finite strain decomposition similar to the one assumed at the outset by Lee [5]. Although no specific assumptions are made about the inelastic part of the strain, effects such as thermally-activated creep, fast-neutron flux-induced creep, fast-neutron flux-induced stress-free growth, annealing, and strain recovery are compatible with this theory and can be included in the constitutive equations for the inelastic

strain. The equations expressing material symmetry are developed and the concepts of elastic and inelastic material symmetry are shown to arise in a natural way. Finally, a simple form of the elastic constitutive equation is derived which is suitable for irrotational deformations of materials subject to large inelastic and thermal expansion strains, but small stress-induced elastic strains. This form of the elastic constitutive equation and the multiplicative elastic-inelastic strain decomposition has recently been incorporated into the CYGRO computer program [16] for the analysis of nuclear reactor fuel rod behavior.

II. Kinematics

In continuum mechanics a body B is a set of points $\{z_1, z_2, \dots\}$ called material points. A one-to-one mapping $\underline{\chi}$ from B into a three-dimensional vector space $\mathcal{V}^3$ is called a configuration of the body.* If $\underline{\chi}_1(z)$ and $\underline{\chi}_2(z)$ are two configurations, and the position vectors of material points in configuration 1 are denoted by $\underline{X}$ while position vectors in configuration 2 are denoted by $\underline{x}$, the mapping $\underline{\phi}$ from configuration 1 in $\mathcal{V}^3$ to configuration 2 in $\mathcal{V}^3$ defined by

$$\underline{x} = \underline{\chi}_2 \left\{ \underline{\chi}_1^{-1}(\underline{X}) \right\} = \underline{\phi}(\underline{X})$$

is called a deformation of the body B from configuration $\underline{\chi}_1$ to configuration $\underline{\chi}_2$. The derivative of the function $\underline{\phi}$ at the location $\underline{X}$ is the second-order tensor $\underline{F}$ called the deformation gradient. Intuitively, $\underline{F}$ is the linear function which most closely approximates the function $\underline{\phi}$ in the neighborhood of $\underline{X}$.

* Throughout this paper quantities which are vectors are indicated by placing a bar under the symbol for the quantity and tensor quantities are indicated by placing a tilde under the symbol for the quantity. Functions whose values are vectors or tensors are also distinguished in a corresponding manner.

As an illustration of these concepts, consider the behavior of a metal bar subjected to simple extension. This example is illustrated in Figure 1. Initially, the body is in the configuration $\underline{x}_1$ called the reference configuration. We assume that a system of forces deforms the body uniformly from the configuration $\underline{x}_1$ to the configuration $\underline{x}_3$ and that removal of the force system causes the body to occupy the stress-free configuration $\underline{x}_2$.

From Figure 1 it is evident that many relationships exist between the functions shown there. For example,

$$\underline{\beta}_t = \underline{x}_3 \left\{ \underline{x}_1^{-1}(\underline{x}) \right\} = \underline{x}_3 \cdot \underline{x}_1^{-1} ,$$

$$\underline{\beta}_1 = \underline{x}_2 \cdot \underline{x}_1^{-1} ,$$

$$\underline{\beta}_e = \underline{x}_3 \cdot \underline{x}_2^{-1} .$$

The deformations defined above are related by

$$\underline{\beta}_t = \underline{\beta}_e \cdot \underline{\beta}_1 . \quad (2.1)$$

The deformations $\underline{\beta}_t$, $\underline{\beta}_e$ and $\underline{\beta}_1$ will be called the total, the elastic, and the inelastic deformations, respectively. From the chain rule for multivariate derivatives, equation (2.1) gives

$$\underline{\beta}_t = \underline{\beta}_e \underline{\beta}_1 \quad (2.2)$$

FIGURE 1

Simple Extension of a Metal Bar

where $\underline{F}_t$, $\underline{F}_e$ and $\underline{F}_i$ are the deformation gradients corresponding to the total, the elastic, and the inelastic deformations.

We remark that the above example was given in order to motivate the definitions which will be given presently. Our purpose is to formulate a local constitutive equation which relates variables such as stress and strain which are defined at a point in the material. Accordingly, we take the local strain decomposition given by

$$\underline{F}_t = \underline{F}_e \underline{F}_i \quad (2.3)$$

as our starting point. The reason for abandoning consideration of the global mappings χ and the global deformations $\underline{b}$ in favor of the local relation (2.3) is that a stress-free state will generally not exist for non-uniform deformations. If the stress-free state does not exist, $\underline{F}_0$ and $\underline{F}_1$ will not correspond to independent deformations of a continuous body. Although some authors (e.g. [7]) have chosen to regard the stress-free state as existing in a non-euclidean space and have extensively developed the corresponding kinematics, we do not believe that this approach is necessary. The consideration of global mappings should only motivate the form of the local constitutive equations.

III. Field Equations

A. Mechanical Effects for Quasi-Static Processes

In the absence of couple-stresses the equations of equilibrium are expressed by Cauchy's laws, given by (see e.g. [11])

$$\nabla \cdot \underline{T}(\underline{x}, t) + \rho(\underline{x}, t) \underline{b}(\underline{x}, t) = \underline{0} \quad (3.1)$$

and

$$\underline{T} = \underline{T}^T \quad (3.2)$$

where $\nabla \cdot$ is the divergence operator, $\underline{T}$ is the Cauchy (or true) stress tensor, $\underline{x}$ is the spatial position vector, t is time, ρ is the mass per unit volume, and $\underline{b}$ is the body force per unit mass. Conservation of mass is expressed by

$$\frac{\partial \rho(\underline{x}, t)}{\partial t} + \nabla \cdot [\rho \underline{v}(\underline{x}, t)] = 0 \quad (3.3)$$

where $\underline{v}$ is the particle velocity.

Equations (3.1) - (3.3) can be used to derive the mechanical energy transfer theorem:

$$\int_{\partial P_t} (\underline{T} \underline{v}) \cdot \underline{n} \, dA + \int_{P_t} \rho \underline{b} \cdot \underline{v} \, dV = \int_{P_t} \text{tr} \{ \underline{T} \underline{L} \} \, dV \quad (3.4)$$

where P_t is the region of space occupied by the body at time t , ∂P_t is the boundary of P_t , $\underline{n}$ is the outward normal to ∂P_t , $\text{tr} \{ \}$ denotes the trace and $\underline{L}$ is the spatial gradient of the velocity field. The three integrals in (3.4) have a simple interpretation. The first integral represents the rate of work done on the body by surface forces and the second integral represents the rate of work done on the body by body forces. If we were dealing with a rigid body, the work done by external forces would go into the kinetic energy of the body. However, kinetic energy is neglected in (3.4) and the third integral will be zero for rigid motions. Therefore, the third integral in (3.4) represents the rate of work expended in deforming the body. This integral will be denoted $\dot{\phi}(P_t)$. The term $\text{tr} \{ \underline{T} \underline{L} \}$ will be denoted by $\dot{w}$ and referred to as the stress power.

B. Thermal Effects

The rate of flow of energy into the body due to thermal effects is denoted $Q(P_t)$ and is given by

$$Q(P_t) = \int_{\partial P_t} \underline{q}(\underline{x}, t) \cdot \underline{n} \, dA + \int_{P_t} r(\underline{x}, t) \rho(\underline{x}, t) \, dV \quad (3.5)$$

where $\underline{q}$ is the heat flow per unit area per unit time into the body and r is the heat supply per unit mass. We assume that there exists a quantity $\epsilon(\underline{x}, t)$ called the specific internal energy such that

$$E(P_t) = \int_{P_t} \epsilon(\underline{x}, t) \rho(\underline{x}, t) \, dV \quad (3.6)$$

where $E(P_t)$ is the internal energy of the body. The energy balance axiom, or principle of conservation of energy, is expressed by

$$\frac{d}{dt} [E(P_t)] = \varphi(P_t) + Q(P_t) \quad (3.7)$$

The local form of (3.7) is given by

$$\rho \dot{\epsilon} = \text{tr} \{ \underline{T} \underline{L} \} + \nabla \cdot \underline{q} + \rho r \quad (3.8)$$

where a superposed dot denotes material time differentiation defined by

$$\dot{\epsilon}(\underline{x}, t) = \frac{\partial \epsilon}{\partial t} + \underline{v}(\underline{x}, t) \cdot \nabla \epsilon(\underline{x}, t) \quad (3.9)$$

The Clausius-Duhem dissipation inequality, expressing the second law of thermodynamics, takes the form

$$\frac{d}{dt} H(P_t) \geq \int_{\partial P_t} \left(\frac{q}{\theta} \right) \cdot n \, dA + \int_{P_t} \left(\frac{r}{\theta} \right) \rho \, dV \quad (3.10)$$

where θ is the absolute temperature and

$$H(P_t) = \int_{P_t} \eta(\underline{x}, t) \rho(\underline{x}, t) \, dV \quad (3.11)$$

is the entropy of the body and $\eta(\underline{x}, t)$ is the entropy per unit mass. The local form of equation (3.10) is given by

$$\rho \dot{\eta} - \rho \left(\frac{r}{\theta} \right) - \nabla \cdot \left[\frac{q}{\theta} \right] \geq 0.$$

In terms of the Helmholtz free-energy ψ , defined by

$$\psi = e - \eta \theta \quad (3.12)$$

equation (3.8) can be used to reduce the dissipation inequality to the form

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + \text{tr} \{ \underline{T} \underline{L} \} + \left(\frac{1}{\theta} \right) \underline{q} \cdot \nabla \theta \geq 0 \quad (3.13)$$

where, again, a dot denotes material time differentiation.

IV. Constitutive Assumptions

In the following sections we shall use the results of the preceding section with the principle of material frame indifference and the symmetries of the material to derive the general form of a constitutive equation describing simultaneous elastic and inelastic deformations. The resulting equations

will suggest the functional forms required to represent significant processes of viscoplasticity including annealing and strain recovery upon loading or reduced loading.

The derivation will make use of the concept that thermoelastic deformations occur without dissipation of energy, while viscoplastic deformations inevitably involve dissipation. Some of this dissipation results in local heating, while the remainder acts to cause distortions of the crystal structure that may prove to be partially recoverable under certain circumstances.

Accordingly, we make the following constitutive assumptions.

(A) The Helmholtz free-energy ψ is given by

$$\psi = \hat{\psi}(\underline{F}_e, \underline{F}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M). \quad (4.1)$$

The ζ 's are scalars representing externally controlled influences on the material behavior such as the effects of fast-neutron flux.

The ξ 's are scalar internal variables, such as dislocation density, which characterize the state of the material.

(B) The internal variables are determined from rate equations such as

$$\dot{\xi}_n = \hat{\xi}_n(\underline{F}_e, \underline{F}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M), \quad n = 1, \dots, M. \quad (4.2)$$

(C) We assume that the Cauchy stress tensor is determined by the two equations

$$\underline{T} = \hat{\underline{T}}_1(\underline{F}_e, \underline{F}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \quad (4.3)$$

and

$$\underline{T} = \hat{T}_2 (\dot{\underline{F}}_1, \underline{F}_e, \underline{F}_1, \theta, \underline{\zeta}_1, \dots, \underline{\zeta}_N, \underline{\xi}_1, \dots, \underline{\xi}_M). \quad (4.4)$$

The assumption that there are two independent equations which determine the stress is a generalization of the idea that the stress can be calculated either from a knowledge of the elastic strain or from a knowledge of the inelastic strain rate.

Alternatively, one could solve (4.3) for $\underline{F}_e$, solve (4.4) for $\dot{\underline{F}}_1$, and then refer to these as the elastic and inelastic stress relations respectively. As will be shown below, writing the stress relations in the form of (4.3) and (4.4) allows a particularly straightforward application of the principle of material frame indifference.

(D) The entropy per unit mass η is given by

$$\eta = \hat{\eta} (\underline{F}_e, \underline{F}_1, \theta, \underline{\zeta}_1, \dots, \underline{\zeta}_N, \underline{\xi}_1, \dots, \underline{\xi}_M) \quad (4.5)$$

(E) The heat flux per unit area $\underline{q}$ is given by

$$\underline{q} = \hat{\underline{q}} (\underline{F}_e, \underline{F}_1, \theta, \underline{\zeta}_1, \dots, \underline{\zeta}_2, \underline{\xi}_1, \dots, \underline{\xi}_M, \nabla \theta) \quad (4.6)$$

where $\nabla \theta$ is the spatial temperature gradient.

V. The Principle of Material Frame Indifference

The principle of material frame indifference, which was first explicitly stated by Noll [12], asserts that if two observers who are rotating (and possibly translating) relative to one another observe the deformation of a material and calculate its stress (or other quantities such as heat flux) from the constitutive equations, they should get the same answer. Here, "same" must be interpreted in a physical sense because the relative rotation of the observers must be accounted for when interpreting the results. Although this principle seems obvious, it is surprising how easily it can be violated when constitutive equations are postulated. For the present analysis the implications of this principle are that the functions postulated in the constitutive assumptions (4.1) - (4.6) are not arbitrary, but must be subject to certain restrictions. Our objective is to re-write equations (4.1) - (4.6) in the most general form which will automatically satisfy Noll's principle of material frame indifference. Consequently, our equations will always display the desired observer independence.

If the orientation of observer A relative to observer B is characterized by a time-dependent orthogonal (rotation) tensor $\underline{Q}_1$, it can be shown that

$$\left. \begin{aligned} (\underline{F})_A &= \underline{Q}_1 (\underline{F})_B \\ (\dot{\underline{F}})_A &= \dot{\underline{Q}}_1 (\underline{F})_B \\ (\underline{T})_A &= \underline{Q}_1 (\underline{T})_B \underline{Q}_1^T \\ (\underline{\psi})_A &= (\underline{\psi})_B \\ (\theta)_A &= (\theta)_B \\ (\eta)_A &= (\eta)_B \end{aligned} \right\} \quad (5.1)$$

$$\left. \begin{aligned} (\zeta_n)_A &= (\zeta_n)_B & n &= 1, \dots, N \\ (\xi_m)_A &= (\xi_m)_B & m &= 1, \dots, M \\ (\nabla\theta)_A &= Q_1 (\nabla\theta)_B \\ (\underline{q})_A &= Q_1 (\underline{q})_B \end{aligned} \right\} \quad (5.1)$$

where $(\cdot)_A$ and $(\cdot)_B$ indicate the quantity as observed by observer A and observer B respectively. In other words, if observer A sees the quantities denoted by $(\cdot)_A$ then observer B should see the quantities $(\cdot)_B$ given by (5.1) if he is seeing the same physical event.

The next step in investigating the consequences of the principle of material frame indifference is to substitute the equivalences (5.1) into the constitutive equations (4.1) - (4.6) in order to determine the conditions that the equations must satisfy. Before doing this however, we observe that it is not $\underline{F}$, but $\underline{F}_e$ and $\underline{F}_1$ which appear in the constitutive equations. Recalling that $\underline{F} = \underline{F}_e \underline{F}_1$ we see that, in terms of $\underline{F}_e$ and $\underline{F}_1$, $\underline{F}$ can be replaced by $Q_1 \underline{F}$ in a variety of ways. For example, if Q_2 is any orthogonal tensor, replacing $\underline{F}_e$ by $Q_1 \underline{F}_e Q_2^T$ and replacing $\underline{F}_1$ by $Q_2 \underline{F}_1$ is easily seen to be equivalent to replacing $\underline{F}$ by $Q_1 \underline{F}$. The arbitrariness in the choice of Q_2 reflects the fact that the orientation of the local inelastic configuration is not uniquely determined when $\underline{F}$ is replaced by $Q_1 \underline{F}$. In other words, the inelastic strain consists of the original strain $\underline{F}_1$ followed by a rotation Q_2 . Thus Q_2 determines the orientation of the local inelastic configuration. Next, the elastic part of the strain consists of undoing the rotation Q_2 by application of Q_2^T , stretching the material by $\underline{F}_e$, and then applying a rotation Q_1 . Clearly, the net result of all of this is to replace the original $\underline{F}$ by $Q_1 \underline{F}$.

We will require that the constitutive equations be insensitive, or invariant, to an intermediate rotation of the local inelastic configuration. In other words, if $\underline{F}_e$ and $\underline{F}_1$ are replaced by $\underline{F}_e \underline{Q}_2^T$ and $\underline{Q}_2 \underline{F}_1$ respectively, the quantities ψ , ξ , $\underline{T}$, η and $\underline{q}$ predicted by the constitutive equations will be unchanged. This invariance to an intermediate rotation was used by Green and Naghdi [4] and by Sideroff in [6] where it is called Γ -invariance.

More generally, we observe that $\underline{F}$ can also be replaced by $\underline{Q}_1 \underline{F}$ by replacing $\underline{F}_e$ by $\underline{Q}_1 \underline{F}_e \underline{A}^{-1}$ and replacing $\underline{F}_1$ by $\underline{A} \underline{F}_1$, where $\underline{A}$ is any second-order tensor. For the principle of material frame indifference to hold, $\underline{Q}_1$ must be arbitrary. However, this is not the case for the tensor $\underline{A}$. The two stress relations (4.3) and (4.4) must yield the same value of the stress. However, when $\underline{A}$ is not an orthogonal tensor, in a certain sense the relative "amounts" of elastic and inelastic strain are changed. For example, choosing $\underline{A}$ equal to $\underline{F}_1^{-1}$ would eliminate the inelastic strain and make the strain completely elastic. Certainly under these conditions we would not, in general, expect the stress relations (4.3) and (4.4) to yield the same stress. Therefore, we will only require that the equations be invariant for the case in which $\underline{A}$ is orthogonal and argue that such invariance represents the limit of generality applicable to the present analysis.

VI. Consequences of the Principle of Material Frame Indifference and Γ -Invariance

Combining the two principles stated above, we obtain the following:

Invariance Requirement

If the set of time-dependent quantities

$$\{\underline{F}_e, \underline{F}_1, \underline{T}, \psi, \eta, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M, \nabla\theta, \underline{q}\}$$

satisfy the constitutive equations (4.1) - (4.6), then the quantities defined by

$$\left. \begin{aligned}
 \hat{\underline{F}}_e^* &= \underline{Q}_1 \underline{F}_e \underline{Q}_2^T \\
 \hat{\underline{F}}_1^* &= \underline{Q}_2 \underline{F}_1 \\
 \hat{\underline{T}}^* &= \underline{Q}_1 \underline{T} \underline{Q}_1^T \\
 \hat{\eta}^* &= \eta \\
 \hat{\theta}^* &= \theta \\
 \hat{\underline{\zeta}}_n^* &= \underline{\zeta}_n \quad n = 1, \dots, N \\
 \hat{\underline{\xi}}_m^* &= \underline{\xi}_m \quad m = 1, \dots, M \\
 \hat{\nabla}^* \theta &= \underline{Q}_1 \nabla \theta \\
 \hat{\underline{q}}^* &= \underline{Q}_1 \underline{q}
 \end{aligned} \right\} \quad (6.1)$$

must also satisfy the constitutive equations for arbitrary choices of the time dependent, orthogonal tensors $\underline{Q}_1$ and $\underline{Q}_2$.

For the first constitutive equation, equation (4.1), the invariance requirement implies

$$\begin{aligned}
 &\hat{\Psi}(\underline{F}_e, \underline{F}_1, \theta, \underline{\zeta}_1, \dots, \underline{\zeta}_N, \underline{\xi}_1, \dots, \underline{\xi}_M) \\
 &= \hat{\Psi}(\underline{Q}_1 \underline{F}_e \underline{Q}_2^T, \underline{Q}_2 \underline{F}_1, \theta, \underline{\zeta}_1, \dots, \underline{\zeta}_N, \underline{\xi}_1, \dots, \underline{\xi}_M).
 \end{aligned} \quad (6.2)$$

We now use the polar decomposition theorem to re-write $\underline{F}_e$ and $\underline{F}_1$. This theorem asserts that any tensor $\underline{F}$ can be written as a product of two unique tensors $\underline{R}$ and $\underline{U}$, where $\underline{R}$ is an orthogonal tensor and $\underline{U}$ is a positive-definite symmetric tensor. Accordingly, we decompose $\underline{F}_e$ and $\underline{F}_1$ as $\underline{R}_e \underline{U}_e$ and $\underline{R}_1 \underline{U}_1$ respectively. Using the decomposition for $\underline{F}_1$, the expression for calculating the free-energy looks like

$$\psi = \hat{\psi}(\underline{F}_e, \underline{R}_1, \underline{U}_1, \theta, \dots),$$

which can also be written

$$\psi = \hat{\psi}(\underline{F}_e, \underline{R}_1, (\underline{R}_1^T), (\underline{R}_1) \underline{U}_1, \theta, \dots) \quad (6.3)$$

since $\underline{R}_1 \underline{R}_1^T = \underline{I}$, the identity tensor. Now, comparing the form of the right side of (6.3) with the right side of (6.2), and letting $\underline{R}_1$ assume the role of $\underline{Q}_2$, we see that (6.2) implies that we can write

$$\begin{aligned} \hat{\psi}(\underline{F}_e, \underline{F}_1, \dots) &= \hat{\psi}(\underline{F}_e, \underline{R}_1, \underline{R}_1^T, \underline{R}_1 \underline{U}_1, \dots) \\ &= \hat{\psi}(\underline{F}_e, \underline{R}_1, \underline{U}_1, \dots). \end{aligned}$$

In other words, the invariance requirement expressed by (6.2) implies that if

$$\psi = \hat{\psi}(\underline{F}_e, \underline{F}_1, \dots), \quad (6.4)$$

then ψ must also be given by

$$\psi = \hat{\psi}(\underline{F}_e, \underline{R}_1, \underline{U}_1, \dots). \quad (6.5)$$

The next step is to use the polar decomposition of $\underline{F}_e \underline{R}_1$ in order to further reduce the equation. To obtain the polar decomposition of $\underline{F}_e \underline{R}_1$ we write it in the form

$$\underline{F}_e \underline{R}_1 = \underline{R}_e \underline{U}_e \underline{R}_1 = (\underline{R}_e \underline{R}_1) (\underline{R}_1^T \underline{U}_e \underline{R}_1).$$

Since $\begin{pmatrix} \underline{R}_e & \underline{R}_1 \end{pmatrix}$ is orthogonal and $\begin{pmatrix} \underline{R}_1^T & \underline{U}_e & \underline{R}_1 \end{pmatrix}$ is positive-definite symmetric, these terms represent the unique polar decomposition of $\underline{F}_e \underline{R}_1$.

Inserting this decomposition into (6.5) we obtain

$$\psi = \hat{\psi} \left(\begin{pmatrix} \underline{R}_e & \underline{R}_1 \end{pmatrix} \begin{pmatrix} \underline{R}_1^T & \underline{U}_e & \underline{R}_1 \end{pmatrix}, \underline{U}_1, \dots \right). \quad (6.6)$$

Using the invariance requirement (6.2), this time with $\underline{Q}_2$ equal to $\underline{1}$ and $\underline{Q}_1$ equal to $\underline{R}_e \underline{R}_1$, we see that (6.6) can be reduced to

$$\begin{aligned} \psi &= \hat{\psi} \left(\begin{pmatrix} \underline{R}_1^T & \underline{U}_e & \underline{R}_1 \end{pmatrix}, \underline{U}_1, \dots \right) \\ &= \hat{\psi} \left(\begin{pmatrix} \hat{\underline{U}}_e & \underline{U}_1 \end{pmatrix}, \dots \right) \end{aligned} \quad (6.7)$$

where

$$\hat{\underline{U}}_e = \underline{R}_1^T \underline{U}_e \underline{R}_1. \quad (6.8)$$

The main purpose of the preceding derivation has been to show that if the free-energy is a function of the set $\{\underline{F}_e, \underline{F}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$, the invariance requirement implies that it is only necessary to have a knowledge of the set $\{\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$. This represents a significant reduction in complexity since $\hat{\underline{U}}_e$ and $\underline{U}_1$ are symmetric, and thus represent six independent components each, whereas $\underline{F}_e$ and $\underline{F}_1$ represent nine independent components each.

Since the set of quantities $\{\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$ is always sufficient to determine the free-energy, we could have assumed at the outset that the free-energy was a function $\tilde{\psi}$ of this set. We will show that if this assumption is made the invariance requirement places no restrictions on the form of $\hat{\psi}$.

Assume that the free-energy is given by

$$\psi = \tilde{\psi}(\hat{U}_e, U_1, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (6.9)$$

where $\tilde{\psi}$ is an arbitrary function of its arguments. Since the class of arbitrary functions contains the class of functions satisfying the invariance requirement, equation (6.9) does not represent any loss of generality. Next we apply the invariance requirement to (6.9). From (6.1) we have

$$\tilde{F}_e^* = Q_1 \tilde{F}_e Q_2^T \quad (6.10)$$

$$\tilde{F}_1^* = Q_2 \tilde{F}_1. \quad (6.11)$$

To investigate the invariance requirements of (6.9) we need to calculate $\tilde{U}_e^*$ and $\tilde{U}_1^*$. To do this, we write equations (6.10) and (6.11) in the forms

$$\tilde{F}_e^* = Q_1 \tilde{R}_e U_e Q_2^T = (Q_1 \tilde{R}_e Q_2^T)(Q_2 U_e Q_2^T) \quad (6.12)$$

and

$$\tilde{F}_1^* = Q_2 \tilde{F}_1 = (Q_2 \tilde{R}_1)(U_1). \quad (6.13)$$

Thus, using the fact that the polar decomposition is unique, we see that

$$\tilde{R}_e^* = Q_1 \tilde{R}_e Q_2^T \quad (6.14)$$

$$\tilde{U}_e^* = Q_2 U_e Q_2^T \quad (6.15)$$

$$\tilde{R}_1^* = Q_2 \tilde{R}_1 \quad (6.16)$$

$$\tilde{U}_1^* = U_1 \quad (6.17)$$

Having calculated $\hat{\underline{U}}_e^*$ and $\hat{\underline{R}}_1^*$, we calculate $\hat{\underline{U}}_e^*$ as (see equation (6.8),

$$\begin{aligned}\hat{\underline{U}}_e^* &= \hat{\underline{R}}_1^{*T} \hat{\underline{U}}_e^* \hat{\underline{R}}_1^* = \left(\hat{\underline{R}}_1^T \hat{\underline{Q}}_2^T \right) \left(\hat{\underline{Q}}_2 \underline{U}_e \hat{\underline{Q}}_2^T \right) \left(\hat{\underline{Q}}_2 \hat{\underline{R}}_1 \right) \\ &= \hat{\underline{R}}_1^T \underline{U}_e \hat{\underline{R}}_1 = \hat{\underline{U}}_e \quad .\end{aligned}\quad (6.18)$$

Therefore, $\underline{U}_1$ and $\hat{\underline{U}}_e$ are invariant. This means that the free-energy equation (6.9) will automatically satisfy the invariance principle for an arbitrary function ψ .

In summary, we have shown that if the free-energy is expressed as a function of the set $\{\underline{F}_e, \underline{F}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$, the function must be of a special type and only a knowledge of the set $\{\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$ is required to determine the free-energy. Alternatively, if we assume at the outset that the free-energy is a function of the set $\{\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots\}$ then no restrictions are imposed upon the function by the invariance requirement.

Going through a similar analysis for the remainder of the constitutive equations, we obtain the following set of reduced equations.

$$\psi = \tilde{\psi}(\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \quad (6.19)$$

$$\dot{\xi}_n = \tilde{\xi}_n(\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \quad n = 1, \dots, M \quad (6.20)$$

$$\underline{T} = \underline{R}_e \underline{R}_1 \tilde{\underline{T}}_1(\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \underline{R}_1^T \underline{R}_e^T \quad (6.21)$$

$$\underline{T} = \underline{R}_e \underline{R}_1 \tilde{\underline{T}}_2(\hat{\underline{U}}_1, \hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \underline{R}_1^T \underline{R}_e^T \quad (6.22)$$

$$\eta = \tilde{\eta}(\hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M) \quad (6.23)$$

$$\underline{q} = \underline{R}_e \underline{R}_1 \tilde{\underline{q}}(\hat{\underline{U}}_e, \underline{U}_1, \zeta_1, \dots, \zeta_N, \xi_1, \dots, \xi_M, \underline{R}_1^T \underline{R}_e^T \nabla \theta) \quad (6.24)$$

Although it may at the time have seemed like just a nicety to include $\underline{F}_e$ and $\underline{F}_1$ in the constitutive equation for the heat flux (4.6), the reduced form of the heat conduction equation, equation (6.24), shows that the material rotations are very important in determining the material response. In hindsight, it is easy to see that $\underline{F}_e$ and $\underline{F}_1$ must enter the constitutive equation for the heat flux if we are to consider materials which are anisotropic with regard to heat flow. For example, if the material is thermally anisotropic, a finite rotation of the material will affect its response to a temperature gradient. Equation (6.24) does not tell us exactly how the material response is altered, but instead, tells us how the material rotations must enter the equation if all observers are to see the same response.

It is interesting to note that the original strain decomposition given by

$$\underline{F} = \underline{F}_e \underline{F}_1$$

can be written

$$\begin{aligned} \underline{F} &= \underline{F}_e \underline{F}_1 = \underline{R}_e \underline{U}_e \underline{R}_1 \underline{U}_1 \\ &= \begin{pmatrix} \underline{R}_e & \underline{R}_1 \end{pmatrix} \begin{pmatrix} \underline{R}_1^T & \underline{U}_e & \underline{R}_1 \end{pmatrix} \begin{pmatrix} \underline{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} \underline{R}_e & \underline{R}_1 \end{pmatrix} \hat{\underline{U}}_e \underline{U}_1 \\ &= \underline{R} \hat{\underline{U}}_e \underline{U}_1 \end{aligned} \tag{6.25}$$

where $\underline{R} = \underline{R}_e \underline{R}_1$. Thus, the use of $\hat{\underline{U}}_e$ and $\underline{U}_1$ in the constitutive equations is equivalent to using a strain decomposition of the form (6.25). The strain decomposition (6.25) corresponds to a normalization of the rotation occurring

between the inelastic and elastic strains. The strain is decomposed into a pure inelastic stretch, followed by a pure elastic stretch, followed by a rotation. In developing his theory of plasticity, Lee [5] assumed at the outset a strain decomposition similar to (6.24). We have shown that if the more fundamental strain decomposition

$$\tilde{F} = \tilde{F}_e \tilde{F}_1$$

is assumed, then the stretches $\hat{\tilde{U}}_e$ and $\tilde{U}_1$ in the decomposition

$$\tilde{F} = \tilde{R} \hat{\tilde{U}}_e \tilde{U}_1$$

arise naturally as a consequence of the invariance requirement.

VII. Restrictions Imposed by the Clausius-Duhem Inequality

Equations (6.19) - (6.24) are now substituted into the Clausius-Duhem inequality (3.13). In order to do this we must first calculate $\dot{\psi}$ and to facilitate this we regard all quantities as functions of their position in a fixed reference configuration. Differentiating (4.26) gives

$$\begin{aligned} \dot{\psi} = \text{tr} \left[\left(\frac{\partial \psi}{\partial \hat{\tilde{U}}_e} \right)^T \dot{\hat{\tilde{U}}}_e + \left(\frac{\partial \psi}{\partial \tilde{U}_1} \right)^T \dot{\tilde{U}}_1 \right] + \left(\frac{\partial \psi}{\partial \theta} \right) \dot{\theta} \\ + \sum_n \left(\frac{\partial \psi}{\partial \xi_n} \right) \dot{\xi}_n + \sum_n \left(\frac{\partial \psi}{\partial \zeta_n} \right) \dot{\zeta}_n \end{aligned} \quad (7.1)$$

Substituting (7.1) into (3.13) and re-arranging terms gives

$$\begin{aligned}
 & \text{tr} \left\{ \left[\tilde{T}^* - \rho \hat{U}_e \left(\frac{\partial \psi}{\partial \hat{U}_e} \right)^T \right] \tilde{L}_e \right\} - \rho \left[\frac{\partial \psi}{\partial \theta} + \eta \right] \dot{\theta} \\
 & + \text{tr} \left[\tilde{T}^* \left(\hat{U}_e \tilde{L}_1 \hat{U}_e^{-1} \right) - \rho \hat{U}_1 \left(\frac{\partial \psi}{\partial \hat{U}_1} \right) \tilde{L}_1 \right] - \rho \sum_n \left(\frac{\partial \psi}{\partial \xi_n} \right) \dot{\xi}_n \\
 & - \rho \sum_n \left(\frac{\partial \psi}{\partial \zeta_n} \right) \dot{\zeta}_n + \left(\frac{1}{\theta} \right) \underline{q} \cdot \nabla \theta \geq 0
 \end{aligned} \tag{7.2}$$

where

$$\tilde{T}^* = \tilde{R}_1^T \tilde{R}_e^T \tilde{T} \tilde{R}_e \tilde{R}_1 \tag{7.3}$$

$$\tilde{L}_e = \dot{\hat{U}}_e \hat{U}_e^{-1} \tag{7.4}$$

$$\tilde{L}_1 = \dot{\hat{U}}_1 \hat{U}_1^{-1} . \tag{7.5}$$

The tensors $\tilde{L}_e$ and $\tilde{L}_1$ will be called the elastic and inelastic velocity gradients respectively.

We now use the fact that it is possible, in principle, to vary $\tilde{L}_e$ and $\dot{\theta}$ arbitrarily while holding all the remaining quantities in (7.2) fixed. It is easy to show that this implies:

$$\tilde{T}^* = \rho \hat{U}_e \left(\frac{\partial \psi}{\partial \hat{U}_e} \right)^T \tag{7.6}$$

and

$$\eta = - \left(\frac{\partial \psi}{\partial \theta} \right) . \tag{7.7}$$

These equations are of the same forms as the stress and entropy relations obtained in the theory of thermoelasticity with $\hat{U}_e$ assuming the role of the total stretch. These relations to a certain extent justify calling $\tilde{F}_e$ the elastic component of the deformation gradient. We remark that (7.6) does

not, in general, give a symmetric stress tensor. However, if, as in the theory of nonlinear thermoelasticity, we make the assumption that ψ is a function of $\hat{U}_e^2$, then the stress will always be symmetric.

Since it is physically reasonable to assume that $\underline{q}$ is bounded as $\nabla \theta$ goes to zero, we can set $\nabla \theta$ to zero in (7.2) to obtain

$$\dot{w}_1 = \rho \operatorname{tr} \left\{ \underline{U}_1 \left(\frac{\partial \psi}{\partial \underline{U}_1} \right) \underline{L}_1 \right\} = \rho \sum_n \left(\frac{\partial \psi}{\partial \xi_n} \right) \dot{\xi}_n = \rho \sum_n \left(\frac{\partial \psi}{\partial \xi_n} \right) \dot{\xi}_n \geq 0 \quad (7.8)$$

where $\dot{w}_1$ is defined by

$$\dot{w}_1 = \operatorname{tr} \left[\underline{T}^* \left(\hat{\underline{U}}_e \underline{L}_1 \hat{\underline{U}}_e^{-1} \right) \right] \quad (7.9)$$

Recalling that the stress power $\dot{w}$ is defined by

$$\dot{w} = \operatorname{tr} \left\{ \underline{T} \underline{L} \right\} ,$$

it is easy to show that the stress power can be written as a sum of two terms $\dot{w}_1$ and $\dot{w}_e$, where $\dot{w}_1$ is defined by (7.9) and $\dot{w}_e$ is defined by

$$\dot{w}_e = \operatorname{tr} \left\{ \underline{T}^* \underline{L}_e \right\} .$$

Since w_e is that portion of the stress power associated with the elastic strain, and $\dot{w}_1$ is that portion associated with inelastic strain, it is natural to refer to $\dot{w}_e$ and $\dot{w}_1$ as the elastic and inelastic stress powers respectively.

From the inequality (7.8) it is clear that strain recovery upon a reduction in stress, which implies a negative inelastic stress power, is possible only because of the dependence of $\dot{\psi}$ on either U_1 , ξ_n or ζ_n . That is, if $\dot{\psi}$ did not depend upon U_1 , ξ_n and ζ_n , the inequality (7.8) would require that the inelastic stress power be greater than or equal to zero at all times. For example, for a bar being inelastically stretched by a uniaxial force, the inequality (7.8) would be violated if the bar inelastically shortened when the magnitude of the force was reduced. This type of inelastic shortening, or strain recovery, has been observed in real materials [13]. From (7.8) we see that allowing $\dot{\psi}$ to depend on the ζ 's admits the possibility of strain recovery only when the ζ 's are non-zero, i.e. when the external variables are changing. From the inequality (7.8) it can be shown that allowing the free-energy to depend upon the inelastic strain U_1 does not properly model all aspects of strain recovery. By considering a one-dimensional case it can be shown that if $\frac{\partial \dot{\psi}}{\partial U_1}$ is zero at some values of U_1 , then strain recovery will not be allowed for certain ranges of values of U_1 . On the other hand, if we assume that $\frac{\partial \dot{\psi}}{\partial U_1} \neq 0$ for all values of U_1 , the stress would always have to be greater than some positive value before the material could have a positive strain rate. The first of these possibilities is not general enough and the second possibility is not desirable. Therefore, in order to properly model the material behavior associated with strain recovery we assume that the free-energy also depends upon the set of internal variables ξ_n . We believe that many of the aspects of strain recovery can be modelled by an appropriate choice of the rate equations (7.8) which, in turn, determine the values of the internal variables.

VIII. Material Symmetry

A rotation $\underline{Q}$ of a body is an element of the symmetry group of the body if the rotation does not affect the response of the material to subsequent deformations. As an example of this, consider a cube of material whose faces are aligned with a set of rectangular cartesian coordinate planes. If this material is subjected to a given stretch $\underline{U}$ in the x-direction, a stress $\underline{T}$ will be produced. Now suppose that instead of being simply stretched, the cube had been first rotated by 90° about the z-axis and then stretched in the x-direction by an amount $\underline{U}$. If the stress that results is the same as the stress $\underline{T}$ resulting when the material is stretched without prior rotation, then the material is behaving the same in two different orientations. The set of all orientations which have the same response is known as the symmetry group of the material.

Mathematically, this concept is expressed by the requirement that the material response be unchanged when the deformation gradient $\underline{F}$ is replaced by $\underline{F} \underline{Q}$, where $\underline{Q}$ is an orthogonal tensor. In other words the straining, represented by $\underline{F}$, is preceded by the rotation $\underline{Q}$. It is important to recognize the logical difference between material symmetry, where the rotation precedes the deformation, and the principle of material frame indifference, where the rotation $\underline{Q}$ follows the deformation.

When the decomposition

$$\underline{F} = \underline{F}_e \underline{F}_1 \quad (8.1)$$

is used, we can write

$$\underline{\underline{F}} \underline{\underline{Q}} = (\underline{\underline{F}}_e \underline{\underline{A}}^{-1})(\underline{\underline{A}} \underline{\underline{F}}_1 \underline{\underline{Q}}) \quad (8.2)$$

where $\underline{\underline{Q}}$ is orthogonal and $\underline{\underline{A}}$ is any second-order tensor. Thus, replacing $\underline{\underline{F}}_e$ by $\underline{\underline{F}}_e \underline{\underline{A}}^{-1}$ and replacing $\underline{\underline{F}}_1$ by $\underline{\underline{A}} \underline{\underline{F}}_1 \underline{\underline{Q}}$ is equivalent to replacing $\underline{\underline{F}}$ by $\underline{\underline{F}} \underline{\underline{Q}}$. As in the section on material indifference, we will restrict the tensor $\underline{\underline{A}}$ to the class of orthogonal tensors. With this assumption, it is easy to show that if

$$\underline{\underline{F}} \rightarrow \underline{\underline{F}} \underline{\underline{Q}}$$

then

$$\underline{\underline{U}}_1 \rightarrow \underline{\underline{Q}}^T \underline{\underline{U}}_1 \underline{\underline{Q}} \quad (8.3)$$

$$\underline{\underline{\hat{U}}}_e \rightarrow \underline{\underline{Q}}^T \underline{\underline{\hat{U}}}_e \underline{\underline{Q}} \quad (8.4)$$

$$\underline{\underline{R}} \rightarrow \underline{\underline{R}} \underline{\underline{Q}} \quad (8.5)$$

$$\underline{\underline{\dot{U}}}_1 \rightarrow \underline{\underline{Q}}^T \underline{\underline{\dot{U}}}_1 \underline{\underline{Q}} \quad (8.6)$$

Making the substitutions (8.3) - (8.6) into the reduced constitutive equations (6.19) - (6.24) gives

$$\psi = \hat{\psi}(\underline{\underline{Q}}^T \underline{\underline{\hat{U}}}_e \underline{\underline{Q}}, \underline{\underline{Q}}^T \underline{\underline{U}}_1 \underline{\underline{Q}}, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (8.7)$$

$$\dot{\xi}_n = \tilde{\xi}_n(\underline{\underline{Q}}^T \underline{\underline{\hat{U}}}_e \underline{\underline{Q}}, \underline{\underline{Q}}^T \underline{\underline{U}}_1 \underline{\underline{Q}}, \theta, \zeta_1, \dots, \xi_1, \dots) \quad n = 1, \dots, M \quad (8.8)$$

$$\underline{\underline{T}} = \underline{\underline{R}}_e \underline{\underline{R}}_1 \underline{\underline{Q}} \underline{\underline{T}}_1 (\underline{\underline{Q}}^T \underline{\underline{\hat{U}}}_e \underline{\underline{Q}}, \underline{\underline{Q}}^T \underline{\underline{U}}_1 \underline{\underline{Q}}, \theta, \zeta_1, \dots, \xi_1, \dots) \underline{\underline{Q}}^T \underline{\underline{R}}_1^T \underline{\underline{R}}_e^T \quad (8.9)$$

$$\underline{\underline{T}} = \underline{\underline{R}}_e \underline{\underline{R}}_1 \underline{\underline{Q}} \underline{\underline{T}}_2 (\underline{\underline{Q}}^T \underline{\underline{\dot{U}}}_1 \underline{\underline{Q}}, \underline{\underline{Q}}^T \underline{\underline{\hat{U}}}_e \underline{\underline{Q}}, \underline{\underline{Q}}^T \underline{\underline{U}}_1 \underline{\underline{Q}}, \theta, \zeta_1, \dots, \xi_1, \dots) \underline{\underline{Q}}^T \underline{\underline{R}}_1^T \underline{\underline{R}}_e^T \quad (8.10)$$

$$\eta = \tilde{\eta}(\hat{Q}^T \hat{U}_e Q, Q^T U_1 Q, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (8.11)$$

$$\underline{q} = \underline{R}_e \underline{R}_1 Q \tilde{q}(\hat{Q}^T \hat{U}_e Q, Q^T \hat{U}_1 Q, \theta, \zeta_1, \dots, \xi_1, \dots, \nabla \theta) \quad (8.12)$$

In general, the values of the time-dependent quantities $\dot{\nu}$, $\dot{\xi}$, $\dot{T}$, η , and $\underline{q}$ calculated from (8.7) - (8.12) will depend upon the choice of the orthogonal tensor $\underline{Q}$. The set of all orthogonal tensors which leave the predicted values of a given equation unchanged from the value predicted when $\underline{Q} = \underline{1}$ is called the symmetry group of that equation.

In the theory presented here, the stress is determined by two independent equations which define the elastic and inelastic response of the material. In general, we would expect the symmetries of these two equations to be different. For example, a material could conceivably be shown to be elastically isotropic during short-time tensile tests and shown to be inelastically anisotropic during long-time creep tests. Since the stress is determined from the free-energy by the stress relation (7.6), it seems reasonable to expect the isotropy group for the elastic stress equation (6.21) to be related to the isotropy group of the free-energy. For hyperelastic materials (i.e. elastic materials which have a strain-energy function), Truesdell [14] has shown that the orthogonal members of the isotropy groups of the stress and free-energy are the same. A modified form of Truesdell's proof will be used to show that the isotropy group for the elastic stress relation (6.21) contains the isotropy group for the free-energy.

Suppose that $\underline{Q}$ is an element of the isotropy group of the stress. Substituting the stress relation (7.6) into (8.9) and simplifying yields

$$\left(\frac{\partial \psi}{\partial \hat{\underline{U}}_e} \right) = \underline{Q} \left(\frac{\partial \psi}{\partial \hat{\underline{U}}_e} \right) \underline{Q}^T, \quad (8.13)$$

$$\begin{cases} \hat{\underline{U}}_e = \hat{\underline{U}}_e' \\ \underline{U}_1 = \underline{U}_1' \end{cases} \quad \begin{cases} \hat{\underline{U}}_e = \underline{Q}^T \hat{\underline{U}}_e' \underline{Q} \\ \underline{U}_1 = \underline{Q}^T \underline{U}_1' \underline{Q} \end{cases}$$

for the symmetry condition. This can be written as

$$\frac{\partial \psi}{\partial \hat{\underline{U}}_e} (\hat{\underline{U}}_e', \underline{U}_1', \dots) = \frac{\partial \psi}{\partial \hat{\underline{U}}_e} [\psi(\underline{Q}^T \hat{\underline{U}}_e' \underline{Q}, \underline{Q}^T \underline{U}_1' \underline{Q}, \dots)] ,$$

or

$$\frac{\partial \psi}{\partial \hat{\underline{U}}_e} [\psi(\underline{Q}^T \hat{\underline{U}}_e' \underline{Q}, \underline{Q}^T \underline{U}_1' \underline{Q}, \dots) - \psi(\hat{\underline{U}}_e', \underline{U}_1', \dots)] = 0$$

which implies that the quantity in brackets does not depend on $\hat{\underline{U}}_e$. Therefore, setting $\hat{\underline{U}}_e' = \underline{1}$ gives

$$\begin{aligned} \psi(\underline{Q}^T \hat{\underline{U}}_e' \underline{Q}, \underline{Q}^T \underline{U}_1' \underline{Q}, \dots) &= \psi(\hat{\underline{U}}_e', \underline{U}_1', \dots) + \psi(\underline{1}, \underline{U}_1', \dots) \\ &\quad - \psi(\underline{1}, \underline{Q}^T \underline{U}_1' \underline{Q}, \dots) \end{aligned} \quad (8.14)$$

Since all the steps leading to (8.14) are reversible, an orthogonal tensor $\underline{Q}$ is an element of the isotropy group of the stress if and only if (8.14) is satisfied.

Now, suppose $\underline{Q}$ is an element of the isotropy group of ψ . Then, by the definition of the isotropy group of ψ ,

$$\psi(\underline{Q}^T \hat{\underline{U}}_e \underline{Q}, \underline{Q}^T \underline{U}_1 \underline{Q}, \dots) = \psi(\hat{\underline{U}}_e, \underline{U}_1, \dots) .$$

and it is seen that (8.14) will be satisfied. Thus $\underline{Q}$ will be an element of the isotropy group of the elastic stress equation (6.21). In the case where ψ either does not depend on $\underline{U}_1$ or depends only upon the invariants of $\underline{U}_1$, equation (8.14) reduces to

$$\psi(\underline{Q}^T \underline{\hat{U}}_e \underline{Q}, \underline{Q}^T \underline{U}_1 \underline{Q}, \dots) = \psi(\underline{\hat{U}}_e, \underline{U}_1, \dots)$$

which shows that $\underline{Q}$ is an element of the isotropy group of the free-energy. Therefore, in this case the isotropy groups coincide. In general, however, it seems that all we can say is that the isotropy group for the elastic stress relation contains the isotropy group for the free-energy.

IX. The Inelastic Strain Rate

From equations (6.21) and (6.22), we can write the elastic and inelastic stress relations in the forms

$$\underline{\hat{T}} = \underline{\hat{T}}_1(\underline{\hat{U}}_e, \underline{U}_1, \theta, \epsilon_1, \dots, \xi_1, \dots) \quad (9.1)$$

and

$$\underline{\hat{T}} = \underline{\hat{T}}_e(\dot{\underline{U}}_1, \underline{\hat{U}}_e, \underline{U}_1, \theta, \epsilon_1, \dots, \xi_1, \dots) \quad (9.2)$$

where

$$\underline{\hat{T}} = \underline{R}_1^T \underline{R}_e^T \underline{T} \underline{R}_e \underline{R}_1 = \underline{R}^T \underline{T} \underline{R} \quad (9.3)$$

Although (9.2) is a good form of the inelastic stress relation for theoretical calculations, in practice it is usually the inelastic strain rate which is known as a function of stress. Therefore, we assume that (9.2) can be solved for $\dot{\underline{U}}_1$ to obtain

$$\dot{\underline{U}}_1 = \underline{H}'(\hat{\underline{T}}, \hat{\underline{U}}_e, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (9.4)$$

Next, if the elastic stress relation (9.1) can be solved for $\hat{\underline{U}}_e$, then $\hat{\underline{U}}_e$ can be eliminated from (9.4) to obtain

$$\dot{\underline{U}}_1 = \underline{H}''(\hat{\underline{T}}, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (9.5)$$

Post-multiplying by $\underline{U}_1^{-1}$ gives

$$\underline{L}_1 = \underline{H}(\hat{\underline{T}}, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots) \quad (9.6)$$

where $\underline{L}_1$ is the inelastic velocity gradient. In one dimension and in the absence of elastic strain, the inelastic velocity gradient $\underline{L}_1$ represents the rate of change of length per unit current length of the material and is the measure of strain rate usually used by metallurgists. Using equations (8.3)-(8.6) and (8.10), it is easy to show that the inelastic symmetry condition becomes

$$\underline{H}(\hat{\underline{T}}, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots) = \underline{Q} \underline{H}(\underline{Q}^T \hat{\underline{T}} \underline{Q}, \underline{Q}^T \underline{U}_1 \underline{Q}, \theta, \zeta_1, \dots, \xi_1, \dots) \underline{Q}^T \quad (9.7)$$

For isotropic inelastic response, equation (9.7) implies that $\underline{H}$ must be an isotropic function of $\hat{\underline{T}}$ and $\underline{U}_1$.

X. A Form of the Free-Energy Function

We now specialize the foregoing theory and derive a form of the free-energy function which is suitable for irrotational deformations of materials subject to large thermal expansion strains. For this case, it is assumed that the free-energy depends only upon the elastic strain $\underline{U}_e$ and the temperature θ and that the material has orthotropic symmetry.* This implies that ψ is of the form (see e.g. [15])

$$\psi = \hat{\psi}(e_1, e_2, e_3, \theta) \quad (10.1)$$

where the e 's are the principal engineering thermoelastic strains defined by

$$e_i = \hat{U}_{e_{ii}} - 1 \quad (10.2)$$

If the temperature is fixed, the free-energy function will be of the form

$$\psi = \tilde{\psi}_\theta(e_1, e_2, e_3) = \tilde{\psi}_\theta(\underline{e}) \quad (10.3)$$

where we can regard the e_k as components of a vector $\underline{e}$ in principal strain space. In general, the form of the function $\tilde{\psi}$ will depend upon the temperature. Figure 2 shows what ψ might look like if there were only one strain component.

Fig. 2

* A material has orthotropic symmetry if it has 180° symmetry about three mutually orthogonal axes.

In general, ψ will have a minimum at some strain e_0 . From the stress relation

$$\tilde{T}^* = \rho \hat{U}_e \left(\frac{\partial \psi}{\partial U_e} \right) \quad (10.4)$$

it is seen that e_0 is the strain corresponding to zero stress, i.e. it is thermal expansion strain. Intuitively, changing the temperature causes the point e_0 to move. This effect can be modelled by a lateral translation of the curve ψ_0 with the amount of translation depending on θ . To accomplish this we first assume that the temperature is held at the reference temperature θ_0 and express ψ as a complete second-degree polynomial in the strains.

$$\begin{aligned} \psi = & a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_1^2 + a_5 e_2^2 \\ & + a_6 e_3^2 + a_7 e_1 e_2 + a_8 e_1 e_3 + a_9 e_2 e_3 \end{aligned} \quad (10.5)$$

The magnitude of ψ is arbitrary to within an additive constant and we can therefore choose a_0 equal to zero. Next, we assume that ψ has a minimum at $e_1 = e_2 = e_3 = 0$. In other words, the stress will be zero whenever the body is at the reference temperature and the elastic strains are zero. This implies $a_1 = a_2 = a_3 = 0$. Therefore

$$\begin{aligned} \psi(\underline{e}) = & a_4 e_1^2 + a_5 e_2^2 + a_6 e_3^2 + a_7 e_1 e_2 + a_8 e_1 e_3 \\ & + a_9 e_2 e_3. \end{aligned} \quad (10.6)$$

A translation of $\tilde{\psi}(\underline{e})$ is given by $\tilde{\psi}(\underline{e} - \underline{\lambda})$ where $\underline{\lambda}$ is a vector whose components depend on the temperature. The minimum of the function $\tilde{\psi}(\underline{e} - \underline{\lambda})$ will be at $\underline{e} = \underline{\lambda}$. Therefore, the components of $\underline{\lambda}$ represent the stress free thermal strains.

Another way of seeing this is to substitute $\underline{e} - \underline{\lambda}$ for $\underline{e}$ in (10.6). To do this we define $\underline{\epsilon} = \underline{e} - \underline{\lambda}$. Equation (10.6) then becomes

$$\begin{aligned} \tilde{\psi}(\underline{e}) = \tilde{\psi}(\underline{\epsilon}) = & a_4 \epsilon_1^2 + a_5 \epsilon_2^2 + a_6 \epsilon_3^2 \\ & + a_7 \epsilon_1 \epsilon_2 + a_8 \epsilon_1 \epsilon_3 + a_9 \epsilon_2 \epsilon_3 \end{aligned} \quad (10.7)$$

Because we expect the stress-induced elastic deformation $\underline{\epsilon}$ to be small, equation (10.7) should be a good approximation even for large strains.

Finding the minimum for $\hat{\psi}(\underline{e})$ is accomplished by setting the partial derivatives of $\hat{\psi}(\underline{e})$ with respect to the e_i equal to zero. From (10.7), this gives

$$\left. \begin{aligned} \frac{\partial \hat{\psi}}{\partial e_1} &= 2 a_4 (e_1 - \lambda_1) + a_7 (e_2 - \lambda_2) + a_8 (e_3 - \lambda_3) = 0 \\ \frac{\partial \hat{\psi}}{\partial e_2} &= a_7 (e_1 - \lambda_1) + 2 a_5 (e_2 - \lambda_2) + a_9 (e_3 - \lambda_3) = 0 \\ \frac{\partial \hat{\psi}}{\partial e_3} &= a_8 (e_1 - \lambda_1) + a_9 (e_2 - \lambda_2) + 2 a_6 (e_3 - \lambda_3) = 0 \end{aligned} \right\} \quad (10.8)$$

In matrix form these equations become

$$\begin{bmatrix} 2a_4 & a_7 & a_8 \\ a_7 & 2a_5 & a_9 \\ a_8 & a_9 & 2a_6 \end{bmatrix} \begin{bmatrix} e_1 - \lambda_1 \\ e_2 - \lambda_2 \\ e_3 - \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (10.9)$$

or

$$\underline{A} [\underline{e} - \underline{\lambda}] = \underline{0} \quad (10.10)$$

Therefore, $\underline{e} = \underline{\lambda}$ is a solution of (10.10). If $\det \underline{A} \neq 0$ then $\underline{e} = \underline{\lambda}$ will be the only solution.

Next, suppose that in addition to translating, the $\tilde{\psi}(\underline{e})$ curve changes its shape as the temperature varies. We observe that allowing the components of the matrix $\underline{A}$ in (10.10) to depend upon the temperature θ does not affect the location of the minimum of $\tilde{\psi}(\underline{e})$. At any temperature, $\tilde{\psi}(\underline{e})$ will still have a minimum at $\underline{e} = \underline{\lambda}$. Therefore the a_i in the free-energy function (10.7) can be regarded as arbitrary functions of θ .

We observe that adding an arbitrary function of θ to the free-energy function does not affect any of the preceding considerations. From (7.7) the entropy function is given by

$$\eta = - \frac{\partial \psi}{\partial \theta} \quad (10.11)$$

which shows that adding an arbitrary function of θ to the free-energy will affect the entropy. Also, it is easily shown that the specific heat at constant strain is given by

$$k = -\theta \frac{\partial^2 \psi}{\partial \theta^2} = -\theta \frac{\partial \eta}{\partial \theta} . \quad (10.12)$$

Thus, the addition of an arbitrary function of θ to the free-energy can be used as a means of specifying the specific heat of the material.

To simplify the calculations we will write the free-energy function in the form

$$\psi = \sum_{\alpha=1}^3 \sum_{\beta=1}^3 b_{\alpha\beta} \epsilon_{\alpha} \epsilon_{\beta} + g(\theta) , \quad (10.13)$$

where ϵ_{α} is defined by

$$\epsilon_{\alpha} = e_{\alpha} - \lambda_{\alpha}(\theta) , \quad \alpha = 1, 2, 3, \quad (10.14)$$

$g(\theta)$ is a suitably defined function of θ , and $b_{\alpha\beta}$ is a matrix whose components are functions of θ . From (7.6), the stress will be given by

$$T_{\alpha} = \rho(U_e)_{\alpha} \sum_{\beta=1}^3 (b_{\alpha\beta} + b_{\beta\alpha}) \epsilon_{\beta}$$

where $(U_e)_{\alpha}$ denotes the α th component of $\underline{U_e}$. This can be written

$$T_{\alpha} = \rho(U_e)_{\alpha} \sum_{\beta=1}^3 c_{\alpha\beta} \epsilon_{\beta} , \quad \alpha = 1, 2, 3, \quad (10.15)$$

where $c_{\alpha\beta}$ is the symmetric matrix defined by

$$c_{\alpha\beta} = b_{\alpha\beta} + b_{\beta\alpha} \quad , \quad \alpha = 1, 2, 3 \quad , \quad \beta = 1, 2, 3 \quad . \quad (10.16)$$

Using the definition of $\underline{\epsilon}$, equation (10.15) can be put into the form

$$T_{\alpha} = \rho(U_e)_{\alpha} \sum_{\beta=1}^3 c_{\alpha\beta} \left[(U_e)_{\beta} - 1 - \lambda_{\beta}(\theta) \right] . \quad (10.17)$$

Equation (10.17) is the elastic stress relation corresponding to the free-energy function (10.7).

Differentiating (10.13) with respect to θ and substituting into (10.10) gives

$$\eta = - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \left(b'_{\alpha\beta} \epsilon_{\alpha} \epsilon_{\beta} - c_{\alpha\beta} \lambda'_{\alpha} \epsilon_{\beta} \right) - g' \quad (10.18)$$

where $c_{\alpha\beta}$ is defined by (10.16) and a prime denotes differentiation with respect to θ . Substituting (10.18) into (10.12) gives

$$c = - \theta \left\{ \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \left(b''_{\alpha\beta} \epsilon_{\alpha} \epsilon_{\beta} - 2 c'_{\alpha\beta} \lambda'_{\alpha} \epsilon_{\beta} - c_{\alpha\beta} \lambda''_{\alpha} \epsilon_{\beta} + c_{\alpha\beta} \lambda'_{\alpha} \lambda'_{\beta} \right) + g'' \right\}$$

From (10.19) it is seen that in the general case the specific heat will depend upon the strain. Equation (10.19) can be written in terms of the strains ϵ_{α} as

$$c = -\theta \left\{ \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \left[b_{\alpha\beta}'' (e_{\alpha} - \lambda_{\alpha}) (e_{\beta} - \lambda_{\beta}) - 2 c_{\alpha\beta}' \lambda_{\alpha}' (e_{\beta} - \lambda_{\beta}) - c_{\alpha\beta} \lambda_{\alpha}'' (e_{\beta} - \lambda_{\beta}) + c_{\alpha\beta} \lambda_{\alpha}' \lambda_{\beta}' \right] + g'' \right\} \quad (10.20)$$

When the strains are zero (or any fixed value) c will be a function of θ and the function g'' can be chosen to make c any required function of temperature. Although the specific heat function at zero strain can be chosen arbitrarily, the dependence of c on the strain is determined by the stress strain response and the free-thermal-expansion functions.

Returning now to the elastic stress relation (10.17), we will further linearize this equation. In terms of ϵ_{α} defined by (10.14), equation (10.17) can be written

$$T_{\alpha} = \left[\frac{\rho_0}{(U_1)_1 (U_1)_2 (U_1)_3} \right] \left[\frac{\epsilon_{\alpha} + 1 + \lambda_{\alpha}}{(\epsilon_1 + 1 + \lambda_1)(\epsilon_2 + 1 + \lambda_2)(\epsilon_3 + 1 + \lambda_3)} \right] \times \sum_{\beta=1}^3 c_{\alpha\beta} \epsilon_{\beta} \quad (10.21)$$

where the density has been expressed in terms of the density ρ_0 in the reference configuration and the stretches $\underline{U}_1$ and $\underline{U}_e$. Equation (10.21) can now be written in the form

$$T_{\alpha} = \left(\frac{1}{V_1} \right) A_{\alpha} \sum_{\beta=1}^3 c_{\alpha\beta} \epsilon_{\beta} \quad (10.22)$$

where V_1 is the inelastic volume ratio defined by

$$v_1 = (u_1)_1 (u_1)_2 (u_1)_3 , \quad (10.23)$$

A_α is given by

$$A_\alpha = \frac{\epsilon_\alpha + 1 + \lambda_\alpha}{(\epsilon_1 + 1 + \lambda_1)(\epsilon_2 + 1 + \lambda_2)(\epsilon_3 + 1 + \lambda_3)} , \quad (10.24)$$

and the matrix $\hat{c}_{\alpha\beta}$ is given by

$$\hat{c}_{\alpha\beta} = \rho_0 c_{\alpha\beta} . \quad (10.25)$$

Because the ϵ 's are small, we can expand the terms in the denominator of (10.24) in power series to obtain

$$\begin{aligned} A_\alpha = & \left[\epsilon_\alpha + 1 + \lambda_\alpha \right] \left\{ \left[\frac{1}{(1+\lambda_1)} \right] \left[1 - \left(\frac{\epsilon_1}{1+\lambda_1} \right) + \dots \right] \right\} \\ & \times \left\{ \left[\frac{1}{(1+\lambda_2)} \right] \left[1 - \left(\frac{\epsilon_2}{1+\lambda_2} \right) + \dots \right] \right\} \left\{ \left[\frac{1}{(1+\lambda_3)} \right] \left[1 - \left(\frac{\epsilon_3}{1+\lambda_3} \right) + \dots \right] \right\} \end{aligned} \quad (10.26)$$

Substituting (10.26) into (10.22) and retaining only first-order terms in the ϵ 's yields

$$T_\alpha = \left(\frac{1}{V_1} \right) (1 + \lambda_\alpha) \sum_{\beta=1}^3 \tilde{c}_{\alpha\beta} \epsilon_\beta \quad (10.27)$$

where $\tilde{c}_{\alpha\beta}$ is a new matrix of temperature dependent elastic moduli defined by

$$\tilde{c}_{\alpha\beta} = \left[\frac{1}{(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)} \right] \hat{c}_{\alpha\beta} . \quad (10.28)$$

Equation (10.27) can now be written

$$e_{\beta} = \sum_{\alpha=1}^3 C_{\beta\alpha} T_{\alpha} + \lambda_{\beta} \quad (10.29)$$

where $C_{\beta\alpha}$ is a matrix of temperature-dependent material moduli defined by

$$C_{\beta\alpha} = \frac{V_1}{(1+\lambda_{\alpha})} (\tilde{c}_{\alpha\beta})^{-1} \quad (10.30)$$

It is interesting to note the similarity between the constitutive equation (10.29) and the stress-strain relation from linear thermoelasticity given by

$$e_{\beta} = \sum_{\alpha=1}^3 B_{\beta\alpha} T_{\alpha} + a_{\beta}(\theta - \theta_0) \quad (10.31)$$

where $B_{\beta\alpha}$ is a symmetric matrix of material constants and a_{β} is a constant known as the coefficient of thermal expansion in the direction β . It is easy to see that equation (10.29) reduces to (10.30) when the thermal strains are small. However, in order to obtain an equation which more accurately predicts the behavior of materials subject to large temperature changes it has been the usual practice to assume (without an analysis of the approximations involved) that the material "constants" in (10.31) are functions of temperature. Equations (10.29) and (10.30) show that, in the general case, this generalization of the linear equation (10.31) is not correct. Because the matrix $(\tilde{c}_{\alpha\beta})^{-1}$ is symmetric, equation (10.30) shows that $C_{\beta\alpha}$ cannot be symmetric unless the thermal expansion is isotropic. In addition, equation (10.30) shows that the elastic moduli should depend linearly upon the inelastic volume strain V_1 . For example, if the local

inelastic volume doubles, the elastic stiffness constants should be reduced by half. With these differences in mind, we conclude that the practice of allowing the material constants in the constitutive equation of linear thermoelasticity to depend on temperature represents a valid generalization of (10.31) provided the inelastic volume change is zero and the thermal expansion is isotropic. However, whenever the thermal expansion is large and anisotropic or the inelastic volume change is significant the usual form is insufficient. When the thermal expansion is small, the effect of anisotropy on the symmetry of the compliance matrix can be neglected.

XI. Summary

We have presented a consistent mathematical theory for describing the large deformation of materials exhibiting elastic and viscoplastic behavior. This theory represents a general mathematical framework upon which theories of the behavior of particular materials can be built. The equations have been presented in a form that guarantees that the second law of thermodynamics and the requirement that the material response be observer invariant are automatically satisfied. This ensures that these fundamental principles will never be inadvertently violated. Although the development of a general invariant theory is necessarily more complex than simply making ad hoc assumptions about a material's behavior, the subsequent formulation of particular theories within the framework of the general theory is considerably simplified.

From a mechanical standpoint, the fundamental equations which have been derived are the stress relation

$$\underline{T}^* = \rho \hat{\underline{U}}_e \left(\frac{\partial \hat{\underline{U}}}{\partial \underline{U}_e} \right)^T$$

which characterizes the elastic response of the material, and the inelastic strain rate equation

$$\dot{L}_1 = \dot{L}_1(\hat{T}, \underline{U}_1, \theta, \zeta_1, \dots, \xi_1, \dots)$$

which characterizes the inelastic behavior. The elastic behavior of the material is determined by the form of the free-energy function ψ . In other words, instead of having elastic constants, a nonlinear elastic material is defined by its energy function ψ . We can choose ψ to be any function that is appropriate for our analysis, and in so doing, describe a particular elastic material. In Section X a form of the function ψ is derived which is particularly appropriate for irrotational deformations of materials subject to large inelastic and thermal expansion strains and accompanied by small stress-induced strains. When this function is substituted into the stress relation given above we obtain

$$T_\alpha = \rho \left(\frac{\partial \psi}{\partial U_e} \right)_\alpha \sum_{\beta=1}^3 b_{\alpha\beta} \left[(U_e)_\beta - 1 - \lambda_\beta \right]$$

for the thermoelastic stress-strain relation. This equation can be linearized further to obtain

$$e_\beta = \sum_{\alpha=1}^3 C_{\beta\alpha} T_\alpha + \lambda_\beta(\theta)$$

where

$$C_{\beta\alpha} = \frac{V_1}{(1+\lambda_\alpha)} (b_{\alpha\beta})^{-1}$$

At this time we have formulated the theory and considered one possible application. We have not explored all of the implications and possible applications of the theory. For example, it should be practical to derive a free-energy function suitable for describing the thermoelastic response of metals undergoing general large three-dimensional inelastic deformation. In addition, the implications of the inelastic strain rate equations have not been fully explored. It is possible that by using an inelastic strain rate equation such as the one given above we could arrive at a more productive mathematical structure for describing creep deformation.

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