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Analysis of the Ideal Phase-Doppler System: Limitations Imposed by the Single-Particle Constraint*

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Abstract

This paper explores the effects of particle statistics on the ability of a phase-Doppler system (or any single-particle diagnostic) to make accurate measurements of complex particle flows. This is accomplished by analyzing the response of an ideal phase-Doppler system to a postulated particle flux. The ideal system defined here senses particles of all sizes and velocities with perfect accuracy, but is subject to one constraint: in order for a measurement to be considered valid there must be only one particle in the probe volume at a time. A consequence of this constraint is that the measured flux of particles is similar to the true flux, but reduced by passage through two stages of filters. The first rejects particles for insufficient spacing and is controlled by a spatial Poisson process, while the second rejects particles for excessive residence time and is driven by a temporal Poisson process. The key filter parameters are the *expected values* of the number of particles in the probe volume and the number of particles entering the probe region during the residence time of a previous particle. Only if these values are kept below order 10^{-2} can the measured joint distribution function, flux rate, and derived quantities, be assumed to reflect the true nature of the flow.

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Introduction

Phase-Doppler Anemometry is a technique via which both the size and velocity of spherical particles may be determined with good accuracy, high speed and good spatial resolution. Because of this capability, it is possible to obtain particle size distributions, size-velocity correlations, and even liquid volume flux rate for spray applications. However, some controversy has arisen recently regarding its ability to provide accurate volume flux measurements in test flows designed specifically for that purpose. In other flows which are of more practical interest, there is no way to know how accurate the measured volume flux is, so this unknown must be carried as a caveat with the data set when it is used.

This issue was brought clearly into focus at a recent ILASS-Americas meeting in Hartford, CT. At this meeting a paper was presented by L. Dodge of Southwest Research Institute which showed a potential underaccounting of the liquid volume flux in a test spray (see Dodge and Schwalb, 1989, 1990). Other papers were presented which reported data for liquid volume flux, but which were unable to quantify the bands of uncertainty associated with the data (Edwards et al., 1990a).

During the discussion of these papers, Prof. D. Senger of Purdue University suggested that the major difficulty might simply be a result of the requirement that there be only a single particle in the probe volume. He also noted that in flows where a strong size-velocity correlation exists, errors might be introduced into the measured size distribution function.

This paper investigates the potential for accuracy in such measurements in light of Prof. Senger's suggestion. It outlines the sampling process of an ideal phase-Doppler system in terms of the Poisson statistics of particles in the probe volume. The results of this work show that even for an ideal processor, Poisson statistics impose severe constraints on the ability of any "single particle" diagnostic to accurately measure size and volume flux distributions.

In the first section, an ideal phase-Doppler system is constructed. This is a system which is able to measure particle size and velocity with perfect accuracy, but is subject to one constraint: in order for a measurement to be considered valid, there must be only one particle in the probe volume. This constraint leads to a sampling process in which the measured particle flux reflects the true flow of particles through the region, but subject to losses due to two filters. Since the keys to these filters are the distributions of interparticle spacing and arrival time, both of which are Poisson distributed, we refer to these as Poisson filters.

In the second section, the expressions for, and constraints on, the Poisson distributions are developed, and in the following section, they are applied to the analysis of the ideal system. Up to that point, the analysis is quite general and little need be specified about the characteristics of the probe volume. Further analysis requires, however, that a concrete choice be made for the size and shape of the probe volume. In the next section we particularize the analysis to a probe volume made up of concentric ellipsoids. The reason for this choice is two fold: first, the ellipsoid is the simplest realistic geometry which is

amenable to analysis and, second, under certain experimental conditions, the probe volume of a real phase-Doppler system is well-characterized as ellipsoidal.

Once the filter expressions have been developed for the ellipsoidal probe volume, we return to analysis of the filter characteristics in the following two sections. The response of the filters to the Poisson process intensities (number density and flux rate) are investigated using simple particle distribution functions (delta and uniform distributions). The figures contained in these sections are intended to provide some guidance to the phase-Doppler practitioner as to when a filter's impact on measured data will be severe and when it can be neglected.

The final section presents an illustration of the ideal system response for a joint distribution function drawn from experimental data in a spray flame. It shows the impact of the filters on the measured vs. true particle and volume flux rates, size distribution functions and mean particle diameters (D_{30} , D_{32} , etc). These data show that some characteristics of the measured data are quite sensitive to particle rejection by the filters (most notably, volume flux) while other characteristics remain relatively unaffected, even at high particle rejection rates (most notably, D_{32}).

An Ideal Phase-Doppler System

Consider the flow of particles through the probe volume of an ideal phase-Doppler system as depicted in Fig. 1. The ideal phase-Doppler system which we postulate is not connected with any hardware but is a region of space in which particles of any size are detected with perfect accuracy and in which they exist only as points (zero spatial extent) but are marked with a label which indicates size. Since the particles are of zero spatial extent, no two can occupy the same position at the same time and no interactions between particles are possible (e.g., collisions).

Anticipating that the probe volume of an actual phase-Doppler system is a function of the particle size, the probe volume considered here is formed by combining component probe volumes of various sizes, $\Delta V(\phi)$, corresponding to each particle diameter ϕ . Each of the component probe volumes senses only those particles which are of size ϕ associated with it. The probe volumes may be of arbitrary shape but all are subject to the constraint that ΔV cannot be reentrant, that is, the outer surface must be convex everywhere with respect to points outside the volume. The need for this constraint arises from the fact that without it, particles on certain trajectories could pass through the probe volume more than once and thus be overcounted (Fig. 2). Each of the probe volumes is assumed to be sufficiently small such that the particle statistics are constant over its dimensions and the velocity and size of a particle are constant during transit through it. Each has associated with it an area, $A_p(\mathbf{v}, \phi)$, which is its cross sectional area projected in the direction of a particle traveling with vector velocity $\mathbf{v}$ (Fig. 3).

The question to be answered then is a simple one. Given a known flux of particles through this region and the restriction that there must be only one particle sensed in the

composite probe volume at a given time for a measurement to be considered valid, what will be the relationship between the validated and true fluxes through the region?

The answer to this question is that the validated flux of particles is a measure of the true flux, but reduced by passage through two stages of filters. Since the characteristics of these filters are specified by spatial and temporal Poisson processes, we will refer to these as Poisson filters.

The first stage filter (Fig. 4) is formed by a spatial Poisson process. Since a particle can only be measured if it is the only one in the probe volume, then at the instant of entry of any particle into the probe volume there must be sufficient interparticle separation that the previous particle has already left the probe volume. If the separation is not sufficient, the particle will be invalidated.

Consider a flow with only a single class of monodisperse particles of diameter ϕ , traveling through the probe volume (Fig. 4a). Denote this class of particles by the subscript 1 . Since all particles within class 1 are the same size, the composite probe volume is the same as the component probe volume and is given by $\Delta V_1 = \Delta V(\phi_1)$. The probability that a test particle in this class will pass through the spatial filter is the same as the probability that the interparticle volume, $\hat{V}_1$, is greater than the probe volume, ΔV_1 . If the probability distribution of interparticle volume for this class is denoted by $p_{x1}(\hat{V}_1)$, normalized such that

$$\int_{\hat{V}_1} p_{x1}(\hat{V}_1) d\hat{V}_1 = 1 \quad (1)$$

then the probability of passage through the spatial filter is given by

$$P_{x1} = P(\hat{V}_1 > DV1) = 1 - P(\hat{V}_1 \leq DV1) = 1 - \int_0^{DV1} p_{x1}(\hat{V}_1) d\hat{V}_1 \quad (2)$$

Suppose that a second class of monodisperse particles is added to the flow (subscript 2, open circles in Fig. 4b). Now, in addition to requiring that ΔV_1 does not contain any particles of class 1 at the time the test particle enters, it is also necessary to ensure that ΔV_2 does not contain any particles of class 2. But since the particles are non-interacting, and each probe volume senses only the particles belonging to its own class, the processes occurring for each of the two classes are independent of each other. The probability of finding that neither has a particle in its probe volume is given by the product of the single class probabilities, that is, $P_{x12} = P_{x1} P_{x2}$. Generalizing this to a system containing multiple classes, each denoted by index i , the probability that the composite probe volume is empty at the time a test particle first enters can be written as

$$P_x = \prod_i P_{xi} = \prod_i \left\{ 1 - \int_0^{\Delta V_i} p_{xi}(\hat{V}_i) d\hat{V}_i \right\} \quad (3)$$

Once the test particle has passed through the spatial Poisson filter — that is, it has an empty probe volume at the time of its entrance — the particle must still pass through another Poisson filter before it can be validated. This filter corresponds to the particle being able to exit the probe volume before another particle enters and thereby invalidates it (Fig. 5). It is apparent that the probability that the test particle will be able to do this will depend both upon its residence time in the probe volume and upon the distribution of interparticle arrival times at the probe volume.

If the residence time of the test particle in the probe volume is denoted by τ , and the probability distribution of interparticle arrival times at the probe volume for class i is denoted by $p_{ii}(\hat{t}_i)$, normalized such that

$$\int_{\hat{t}_i} p_{ii}(\hat{t}_i) d\hat{t}_i = 1 \quad (4)$$

then the probability of passage of the test particle through this temporal filter for a single class i is given by

$$P_{ii} = P(\hat{t}_i > \tau) = 1 - P(\hat{t}_i \leq \tau) = 1 - \int_0^{\tau} p_{ii}(\hat{t}_i) d\hat{t}_i \quad (5)$$

Generalizing this to a system containing multiple classes, the probability that the test particle is able to exit before some other particle enters the composite probe volume can be written as

$$P_t = \prod_i P_{ii} = \prod_i \left\{ 1 - \int_0^{\tau} p_{ii}(\hat{t}_i) d\hat{t}_i \right\} \quad (6)$$

Figure 6 illustrates the serial nature of these filters. The true flow of particles through the region of space considered to be the probe volume is first reduced by the spatial filter (where particles are rejected for insufficient spacing) and then by the temporal filter (where particles are rejected for excessive residence time). The flow of particles measured by the ideal phase-Doppler system then corresponds to the true flow of particles but reduced by the losses through the filters. Since the filters represent independent (and serial) processes, the probability of passage through the composite filter (spatial and temporal) is given by the product of the filter probabilities, $P_x P_t$.

In order to complete this description we need expressions for the distribution functions $p_{xi}(\hat{V})$ and $p_{ii}(\hat{t})$. These expressions may be found by application of Poisson statistics to our point particle flux.

Poisson Point Process

Poisson point processes are processes which describe the behavior of random events in space and time (Snyder, 1975). These events occur with intensity λ , which is the mean number of events per unit space or time. If the process to be described is continuous, then the statistics of the process are characterized completely by the process intensity. If λ is constant in space and time, then the statistics can be said to be fully homogeneous. If λ is constant in space or time only, then the statistics are considered to be spatially or temporally homogeneous, respectively. As applied to the present problem, a spatially homogeneous process is one for which the particle statistics do not vary over the extent of the probe volume. In other words, approximate spatial homogeneity requires that the probe volume be small in comparison to the mean spatial length scales in the physical system. Likewise, a temporally homogeneous process is one for which λ is constant in time. In the analysis which follows, all Poisson processes will be assumed to be homogeneous, that is, the probe volume is considered to be small with respect to length scales in the system and the mean particle flux rate is assumed to be constant.

Poisson point processes are considered to be orderly in the sense that time and space can be divided into sufficiently fine elements that the probability of finding more than one particle at the same point in space and time is negligible (no collisions), yet the probability of finding a single particle within any space or time element divided by the "size" of that element remains finite as the size of the element approaches zero. They are also Markov processes, that is, the statistics of the process have no dependence on past history.

The two results of Poisson particle statistics which are required for our analysis are the interparticle volume distribution, $p_{xi}(\hat{V})$, and the interparticle time distribution, $p_{ii}(\hat{t})$. Since the expressions to be cited apply to particles of all classes, the class index subscript, i , will be omitted in the following description.

The interparticle time distribution for a continuous, temporally homogeneous process is well-known and is given by Snyder (1975) as

$$p_i(\hat{t}) = \lambda_i(\mathbf{x}) e^{-\lambda_i(\mathbf{x})\hat{t}} \quad (7)$$

In a similar fashion, the interparticle volume distribution for a continuous, spatially homogeneous process can be shown to be given by

$$p_x(\hat{V}) = \lambda_x(t) e^{-\lambda_x(t) \hat{V}} \quad (8)$$

where $\lambda_t(x)$ and $\lambda_x(t)$ are the temporal and spatial process intensities. Note that, in general, the homogeneous temporal process intensity may be a function of space and the homogeneous spatial process intensity may be a function of time. However, for the present study, the particle process will be considered to be fully homogeneous so that $\lambda_t(x) = \lambda_t$ and $\lambda_x(t) = \lambda_x$, and both are constants. Also note that λ_x is the mean number of particles per unit volume or number density, ρ , of the spray.¹

Having given the expressions for the interparticle statistics of a homogeneous Poisson process, the question that remains to be answered is whether or not these statistics provide an accurate representation of the interdroplet statistics of a steady spray. If the interdroplet statistics are known to be random and homogeneous, then applicability of the Poisson expressions is limited only by the point-particle assumption. Under these circumstances, use of Poisson statistics to simulate those of the spray droplets is easily justified. Alternatively, tests may be performed on measured droplet statistics to determine the applicability of the Poisson distributions. In the work that follows we will assume applicability of the Poisson expressions in developing the response of the ideal phase-Doppler system. Examination of time-resolved experimental data (not reported here) shows that this is a reasonable assumption for steady spray flows.²

Application of Poisson Statistics to the Ideal System

Substituting the Poisson expressions for interparticle time and volume into the expressions for the spatial and temporal filters, P_x , P_t , gives the probability of measurement of a given test particle with residence time τ as

$$P = P_x P_t \quad (9)$$

$$P_x = \prod_i \left\{ 1 - \int_0^{\Delta V_i} \lambda_{xi} e^{-\lambda_{xi} \hat{V}_i} d\hat{V}_i \right\} = \prod_i e^{-\lambda_{xi} \Delta V_i} = \prod_i e^{-\rho_i \Delta V_i} = e^{-\sum_i \rho_i \Delta V_i} \quad (10)$$

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¹The fact that the interparticle volume distribution has the same functional form as the time distribution and that $\lambda_x = \rho$ may or may not be obvious to the reader. For this reason, Appendix A develops the expression for the interparticle volume distribution from direct application of Poisson counting statistics.

²These data will be published under separate cover. The reader who is interested in obtaining these data prior to publication is encouraged to contact the authors.

$$P_i = \prod_i \left\{ 1 - \int_0^\tau \lambda_{ii} e^{-\lambda_{ii} \hat{t}_i} d\hat{t}_i \right\} = \prod_i e^{-\lambda_{ii} \tau} = e^{-\sum_i \lambda_{ii} \tau} \quad (11)$$

These expressions give the probability of measurement of a test particle within the spray. What is required, however, is the response of the ideal phase-Doppler system to a distribution of droplets within the spray. This may be accomplished by introducing a particle distribution function in the two quantities measured by the phase-Doppler instrument: velocity and size. This distribution function may be cast in either spatial (particles per unit volume) or temporal (particles per unit area per unit time) terms. Both formulations have advantages and disadvantages. However, since the phase-Doppler instrument is temporally-based (measuring particle flow rate through the probe volume) we will adopt the temporal distribution function in the following discussion. (The reader who is interested in the spatial approach to the problem may refer to Appendix B which lists the parallel development of the two approaches and provides correspondence expressions between the two.)

Consider the true particle flux through the probe volume. The number of particles entering the probe volume per unit time in an element of phase space of size $d\mathbf{v}$ about $\mathbf{v}$, and $d\phi$ about ϕ , may be written as

$$\lambda_i^* A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\phi d\mathbf{v} \quad (12)$$

where λ_i^* is the composite flux rate, $f_i(\mathbf{v}, \phi)$ is the temporal joint distribution function in size and velocity, and $d\mathbf{v}$ and $d\phi$ are differential elements of phase space centered about the vector velocity, $\mathbf{v}$, and size, ϕ .

The distribution function $f_i(\mathbf{v}, \phi)$ is normalized such that

$$\int \int_{\phi \mathbf{v}} f_i(\mathbf{v}, \phi) d\mathbf{v} d\phi = 1 \quad (13)$$

and has distribution functions of velocity, $\hat{f}_i(\mathbf{v})$, and size, $\tilde{f}_i(\phi)$, defined by

$$\hat{f}_i(\mathbf{v}) = \int_{\phi} f_i(\mathbf{v}, \phi) d\phi \quad (14)$$

$$\tilde{f}_i(\phi) = \int_{\mathbf{v}} f_i(\mathbf{v}, \phi) d\mathbf{v} \quad (15)$$

which are also normalized to unity.

The probabilities P_x and P_i may be cast in terms of this distribution function by letting each class i comprise the population of particles within a differential element $d\phi$ wide,

centered about particle size ϕ . Under these conditions, the summations over i become integrals over ϕ , and the probabilities for a test particle may be rewritten as

$$P_x = e^{-\int_{\phi} \tilde{\rho}(\phi) \Delta V(\phi) d\phi} \quad (16)$$

$$P_t = e^{-\int_{\phi} \tilde{\lambda}_t(\phi) \tau d\phi} \quad (17)$$

where $\tilde{\rho}(\phi)$ and $\tilde{\lambda}_t(\phi)$ are the number density and occurrence rate of particles in size element $d\phi$ of phase space.

The occurrence rate $\tilde{\lambda}_t(\phi)$ may be found by integrating the joint distribution function over all velocity space:

$$\tilde{\lambda}_t(\phi) = \lambda_t^* \int_{\mathbf{v}} A_p(\mathbf{v}, \phi) f_t(\mathbf{v}, \phi) d\mathbf{v} \quad (18)$$

The density $\tilde{\rho}(\phi)$ can be written in terms of the spatial size distribution function (Appendix B) as

$$\tilde{\rho}(\phi) = \rho \tilde{f}_x(\phi) \quad (19)$$

Substitution into the probability expressions for the filters then yields

$$P_x = e^{-\rho \int_{\phi} \Delta V(\phi) \tilde{f}_x(\phi) d\phi} \quad (20)$$

$$P_t = e^{-\lambda_t^* \tau \int_{\phi} \int_{\mathbf{v}} A_p(\mathbf{v}, \phi) f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi} \quad (21)$$

But the integral in the expression for P_x is, by definition, the spatial mean volume of the probe region

$$\overline{\Delta V}_x = \int_{\phi} \Delta V(\phi) \tilde{f}_x(\phi) d\phi \quad (22)$$

In a similar fashion, the integral in the expression for P_t is, by definition, the temporal mean area of the probe region

$$\overline{A_{pt}} = \int \int_{\phi \mathbf{v}} A_p(\mathbf{v}, \phi) f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi \quad (23)$$

Correspondence between the temporal and spatial approaches (Appendix B) gives the number density in terms of temporal variables as

$$\rho = \lambda_t'' / \overline{S}_x \quad (24)$$

where $\overline{S}_x$ is the spatial mean particle speed given by

$$\overline{S}_x = \int \int_{\phi \mathbf{v}} |\mathbf{v}| f_x(\mathbf{v}, \phi) d\mathbf{v} d\phi = \left\{ \int \int_{\phi \mathbf{v}} \frac{1}{|\mathbf{v}|} f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi \right\}^{-1} \quad (25)$$

Using the definition of $\overline{S}_x$, $\overline{\Delta V}_x$ may be rewritten in terms of temporal variables as

$$\overline{\Delta V}_x = \overline{S}_x \int \int_{\phi \mathbf{v}} \frac{\Delta V(\phi)}{|\mathbf{v}|} f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi \quad (26)$$

Substituting into the expressions for P_x and P_t , and noting that the total particle rate is given by $\lambda_t = \lambda_t'' \overline{A_{pt}}$, the test particle probabilities become

$$P_x = e^{-\rho \overline{\Delta V}_x} = e^{-\lambda_t'' \overline{\Delta V}_x / \overline{S}_x} \quad (27)$$

$$P_t = e^{-\lambda_t'' \overline{A_{pt}} \tau} = e^{-\lambda_t \tau} \quad (28)$$

Figure 7 illustrates the dependence of the test particle filter probabilities on their arguments. Since both filters have the same functional form, each can be represented by the same function of its argument. The spatial filter depends on $\rho \overline{\Delta V}_x$ which is the expected value of the number of particles in the composite probe volume at any instant. Similarly, the temporal filter depends on $\lambda_t'' \overline{A_{pt}} \tau$ which is the expected value of the number of particles entering the probe volume during time interval τ . Note that when either expected value is 1, the probability of passage through the corresponding filter is only 0.36 (1/e). Even at values of 10^{-1} , the probability is still only 0.91. Not until the expected value drops below 10^{-2} can the effect of the filters be neglected (with < 2% error).

Also note that the spatial filter probability does not depend upon the size or velocity class of the test particle being considered, but only on the composite characteristics of the particle flow. As such, this filter rejects particles uniformly over the size/velocity space of the particle flux, and therefore introduces no bias into the measured size/velocity distribution.

This is not true of the temporal filter, however. In this case the filter probability depends upon the residence time of the particle under consideration. Since the residence time is correlated inversely with the particle velocity and directly with the probe volume depth, it is anticipated that the temporal filter will preferentially suppress the measured populations of particles with low velocities or large component probe volumes and will therefore introduce a bias into the measured distribution function.

Armed with the expressions for P_x and P_t , we can now return to consideration of the relationship between the flux of particles which is measured by our ideal system and the true flux of particles.

Recall that the true number of particles passing through the probe volume per unit time in $dv d\phi$ was given by

$$\lambda_i A_p(v, \phi) f_i(v, \phi) dv d\phi \quad (29)$$

Combining this with the expression for P_x , the number of particles passing through the spatial filter per unit time in $dv d\phi$, is given by

$$\lambda_i A_p(v, \phi) P_x f_i(v, \phi) dv d\phi \quad (30)$$

These are the particles which are eligible for measurement if they can pass through the temporal filter. P_t gives the probability of passage through the temporal filter as a function of particle residence time in the probe volume. As noted previously, this residence time will certainly depend upon the velocity and may also depend upon the particle size. However, specification of particle velocity and size alone is not sufficient to determine particle residence time. This is because particles with identical size and velocity but traveling through different parts of the probe volume will yield different residence times since the depth of the probe volume is not constant but is a function of the position where the particle enters (Fig. 8).

In order to account for this distribution of residence times among the particles that populate $dv d\phi$, we introduce an auxiliary distribution function of residence time, $g(\tau; v, \phi)$, with parametric dependence upon v and ϕ . This distribution function is used, in effect, to spread the distribution function $f_i(v, \phi)$ into a new dimension τ in accordance with a particular probe geometry. It is normalized such that

$$\int_{\phi} g(\tau; v, \phi) d\tau = 1 \quad (31)$$

Using this, we can express the number of particles passing through the spatial filter per unit time in dv about v , $d\phi$ about ϕ , and $d\tau$ about τ as

$$\lambda_i^- A_p(v, \phi) P_x g(\tau; v, \phi) f_i(v, \phi) d\phi dv d\tau \quad (32)$$

Combining this with P_t gives the number of particles per unit time which pass through the composite Poisson filter in $dv d\phi d\tau$

$$\lambda_i^- A_p(v, \phi) P_x P_t g(\tau; v, \phi) f_i(v, \phi) d\phi dv d\tau \quad (33)$$

But we are only interested in the distribution of particles over size/velocity space. This may be recovered by integrating over the particle residence time. This yields the number of particles which pass through the composite filter per unit time in $dv d\phi$

$$\lambda_i^- A_p(v, \phi) P_x \bar{P}_t(v, \phi) f_i(v, \phi) d\phi dv \quad (34)$$

where $\bar{P}_t(v, \phi)$ is the mean temporal filter probability given by

$$\bar{P}_t(v, \phi) = \int_{\tau} e^{-\lambda_i^- \bar{A}_{pt} \tau} g(\tau; v, \phi) d\tau = \int_{\tau} e^{-\lambda_i^- \tau} g(\tau; v, \phi) d\tau \quad (35)$$

Stated another way, $\bar{P}_t$ is the effective probability that a particle with velocity v and size ϕ will make it through the temporal filter.

For an ideal phase-Doppler system, subject only to the single particle constraint, this is the measured number of particles per unit time in $dv d\phi$, which can be expressed as

$$\lambda_{im} f_{im}(v, \phi) d\phi dv \quad (36)$$

where λ_{im} is the measured occurrence rate (particles per unit time) and $f_{im}(v, \phi)$ is the measured joint distribution function with normalization

$$\int_{\phi} \int_{v} f_{im}(v, \phi) dv d\phi = 1 \quad (37)$$

The relationship between the measured and true distributions is then given by

$$f_{im}(v, \phi) = \frac{\lambda_i^- A_p(v, \phi)}{\lambda_{im}} P_x \bar{P}_t(v, \phi) f_i(v, \phi) \quad (38)$$

The relationship between the measured particle rate and the true flux rate may be determined by integrating over $\mathbf{v}$ and ϕ and using the normalization constraint on $f_{im}(\mathbf{v}, \phi)$:

$$\lambda_{im} = \lambda_i'' \int \int_{\phi \mathbf{v}} A_p(\mathbf{v}, \phi) P_x \bar{P}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v} d\phi \quad (39)$$

Finally, the relationships between the measured and true velocity and size distribution functions are given by

$$\hat{f}_{im}(\mathbf{v}) = \frac{\lambda_i''}{\lambda_{im}} P_x \int_{\phi} A_p(\mathbf{v}, \phi) \bar{P}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\phi \quad (40)$$

$$\tilde{f}_{im}(\phi) = \frac{\lambda_i''}{\lambda_{im}} P_x \int_{\mathbf{v}} A_p(\mathbf{v}, \phi) \bar{P}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v} \quad (41)$$

The relationship between the measured and true distribution functions can best be expressed in terms of a Poisson efficiency for each distribution. We define the Poisson efficiency as the number of particles measured per unit time in some element of phase space divided by the true number of particles that passed through the probe volume per unit time in that same element. Using this definition, the joint distribution efficiency is given by

$$\eta(\mathbf{v}, \phi) = \frac{\lambda_{im} f_{im}(\mathbf{v}, \phi)}{\lambda_i'' A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi)} = P_x \bar{P}_i(\mathbf{v}, \phi) \quad (42)$$

The efficiency of velocity measurement is given by

$$\hat{\eta}(\mathbf{v}) = \frac{\lambda_{im} \hat{f}_{im}(\mathbf{v})}{\lambda_i'' \int_{\phi} A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\phi} = P_x \frac{\int_{\phi} A_p(\mathbf{v}, \phi) \bar{P}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\phi}{\int_{\phi} A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\phi} \quad (43)$$

and the efficiency of size measurement is given by

$$\tilde{\eta}(\phi) = \frac{\lambda_{im} \tilde{f}_{im}(\phi)}{\lambda_i'' \int_{\mathbf{v}} A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v}} = P_x \frac{\int_{\mathbf{v}} A_p(\mathbf{v}, \phi) \bar{P}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v}}{\int_{\mathbf{v}} A_p(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v}} \quad (44)$$

Up to this point, the analysis has been conducted without reference to any particular probe geometry -- the shapes of the component probe volumes have been arbitrary, but non-reentrant. Further development of the analysis requires that the size and shape of the probe volume be specified so that the effect of $A_p(\mathbf{v}, \phi)$, $g(\tau; \mathbf{v}, \phi)$, and $\Delta V(\phi)$ on the filter probabilities may be evaluated concretely.

Development for an Ellipsoidal Probe Volume

In this section we particularize the expressions for the Poisson filters to a probe region made up of component probe volumes which are concentric ellipsoids, the size of which vary with ϕ . The choice of an ellipsoid was made for two reasons. The first is that this represents the simplest realistic three-dimensional geometry. The second is that under certain experimental conditions, the probe volume of a real phase-Doppler system may be considered to be approximately ellipsoidal.³

Figure 9 illustrates a component ellipsoidal probe volume $\Delta V(\phi)$. The ellipsoid has its axes aligned with the xyz coordinate directions and is taken to be axisymmetric about z.⁴ In terms of the lengths of its semi-axes, $a(\phi)$ and $c(\phi)$, the surface of the ellipsoid and its volume are given by

$$\frac{x^2 + y^2}{a^2(\phi)} + \frac{z^2}{c^2(\phi)} = 1 \quad (45)$$

$$\Delta V(\phi) = \frac{4}{3} \pi a^2(\phi) c(\phi) \quad (46)$$

Since $A_p(\mathbf{v}, \phi)$ depends upon the direction of approach of a particle to the probe volume, but not on the particle speed, it is convenient to treat the particle velocity vector in a spherical polar coordinate system sharing a common origin with the xyz rectangular Cartesian system. In terms of these coordinates (Fig. 10a), the velocity vector $\mathbf{v}$ is given by coordinates (v, θ, ψ) , where $v = |\mathbf{v}|$ is the speed of the particle, θ is the polar angle ($0 \leq \theta \leq \pi$), and ψ is the azimuthal angle ($0 \leq \psi \leq 2\pi$). Due to the symmetry of the probe volume about the z axis, $A_p(\mathbf{v}, \phi)$ is a function of θ and ϕ only, i.e., $A_p(\mathbf{v}, \phi) = A_p(\theta, \phi)$, and $g(\tau; \mathbf{v}, \phi)$ will vary with v (speed), θ , and ϕ only, i.e., $g(\tau; \mathbf{v}, \phi) = g(\tau; v, \theta, \phi)$.

In order to find the projected area and residence time distribution it is useful to define another rectangular Cartesian coordinate system, $\alpha\beta\gamma$, which again shares a common origin

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³The experimental conditions under which this assumption is realistic correspond to those where the collection optics are situated on the plane of symmetry between the transmitted beams and the collected light is not apertured before detection.

⁴The assumption that the probe volume has rotational symmetry about the z-axis is not essential for the tractability of the analysis which follows. However, it does yield a considerable reduction in algebraic complexity.

with the xyz system but which is rotated about the origin. If we take the γ axis to be aligned with the velocity vector v , and fix the location of the α and β axes by choosing β to lie in the xy plane (Fig. 10b), then the projected area of the ellipsoid $A_p(\theta, \phi)$ is bounded by a curve which lies entirely in the $\alpha\beta$ plane. Similarly, since the residence time τ is given by the ratio of the path length through the probe volume to the particle speed, L/v , it will depend only on the γ coordinates of the surface of the ellipsoid. Figure 11 shows the ellipsoid, its projected area, and its path lengths in the $\alpha\beta\gamma$ coordinate system.

While $A_p(\theta, \phi)$ and $g(\tau; v, \theta, \phi)$ may be found by direct manipulation of the equation of the ellipsoid transformed into the $\alpha\beta\gamma$ coordinate system, it is simpler and more intuitive to make an additional transformation which achieves the same end. This transformation is arrived at by noting that the projected area of the ellipsoid is bounded by an ellipse which lies in the $\alpha\beta$ plane and that the path length through the probe volume, $L = \gamma_2 - \gamma_1$, at any position (α, β) is the same as that of an equivalent (but different) ellipsoid whose semi-axes lie on the $\alpha\beta\gamma$ coordinate axes (see Appendix C). This transformed ellipsoid (Fig. 12) has the same projected area and path lengths as the original ellipsoid, but has semi-axes d , e , f , aligned with the γ , β , α , axes respectively. In terms of the original ellipsoid's semi-axes, and the polar approach angle, the semi-axes of the transformed ellipsoid are given by

$$d(\theta, \phi) = \sqrt{\left\{ \frac{\sin^2 \theta}{a^2(\phi)} + \frac{\cos^2 \theta}{c^2(\phi)} \right\}^{-1}} \quad (47)$$

$$e(\phi) = a(\phi) \quad (48)$$

$$f(\theta, \phi) = \sqrt{c^2(\phi) \sin^2 \theta + a^2(\phi) \cos^2 \theta} \quad (49)$$

The equation of the transformed ellipsoid is then specified by

$$\frac{\gamma^2}{d^2(\theta, \phi)} + \frac{\beta^2}{e^2(\phi)} + \frac{\alpha^2}{f^2(\theta, \phi)} = 1 \quad (50)$$

The projected area of the ellipsoid is bounded by the ellipse with semi-axes $f(\theta, \phi)$ and $e(\phi)$ in the $\alpha\beta$ plane ($\gamma = 0$) and is given by

$$A_p(\theta, \phi) = \pi f(\theta, \phi) e(\phi) = \pi a(\phi) \sqrt{c^2(\phi) \sin^2 \theta + a^2(\phi) \cos^2 \theta} \quad (51)$$

The distribution of residence times can be found by considering the flow of particles which pass through a differential element of A_p per unit time

$$\lambda_i'' f_i(v, \phi) dA_p d\phi dv \quad (52)$$

where $dA_p = d\alpha d\beta$.

If we make a change of variables from α to $\xi \equiv \alpha d(\theta, \phi)/c(\phi)$, then the equation of the equivalent ellipsoid becomes

$$\frac{\gamma^2}{d^2(\theta, \phi)} + \frac{\beta^2 + \xi^2}{a^2(\phi)} = 1 \quad (53)$$

It is then apparent that the lengths of the particle paths are only functions of a radial coordinate, $r^2 \equiv \beta^2 + \xi^2$, in the $\xi\beta$ plane. Choosing the dA_p to be circular annuli in the $\xi\beta$ plane, (elliptical in the $\alpha\beta$ plane) the flow rate of particles across area elements of width dr centered about r is

$$\lambda_i \ddot{f}_i(v, \phi) dv d\phi \frac{c(\phi)}{d(\theta, \phi)} 2\pi r dr \quad (54)$$

But since γ is a function of r only, this may be transformed into a flow rate of particles with values of γ in the element $d\gamma$ about γ . Since

$$\frac{\gamma^2}{d^2(\theta, \phi)} + \frac{r^2}{a^2(\phi)} = 1 \quad (55)$$

differentiating,

$$\frac{2\gamma d\gamma}{d^2(\theta, \phi)} + \frac{2r dr}{a^2(\phi)} = 0 \quad (56)$$

which leads to

$$r dr = - \frac{a^2(\phi)}{d^2(\theta, \phi)} \gamma d\gamma \quad (57)$$

Substituting into the expression for the particle flowrate per unit area, we obtain the particle flow rate with γ in interval $d\gamma$

$$2\pi \lambda_i \ddot{f}_i(v, \phi) dv d\phi \frac{a^2(\phi)c(\phi)}{d^3(\theta, \phi)} \gamma d\gamma \quad (58)$$

But since $\tau = 2\gamma/v$ and $d\tau = 2d\gamma/v$, this may be rewritten as the flow rate of particles per unit time with residence time in $d\tau$ about τ :⁵

$$\frac{\pi}{2} \lambda_i \ddot{f}_i(v, \phi) dv d\phi \frac{a^2(\phi)c(\phi)}{d^3(\theta, \phi)} v^2 \tau d\tau \quad (59)$$

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⁵In setting $L = 2\gamma$ we have chosen to use only positive values of γ . Note that the minus sign disappears by choosing $d\gamma > 0$.

Equating this with the previously derived expression for the flow of particles through this element of phase space (Eq. 32 with $P_x = 1$), and using the expression for A_p (Eq. 51), the result is

$$g(\tau; \mathbf{v}, \phi) = \begin{cases} \frac{1}{2} \frac{v^2}{d^2(\theta, \phi)} \tau; & 0 \leq \tau \leq \tau_{\max} \\ 0; & \tau > \tau_{\max} \end{cases} \quad (60)$$

where $\tau_{\max} = 2d(\theta, \phi)/v$ is the maximum residence time in the probe volume ($2d(\theta, \phi)$ is the maximum path length). Note that the normalization condition (31) is satisfied. We see that the only geometrical property on which $g(\tau; \mathbf{v}, \theta, \phi)$ depends is $d(\theta, \phi)$, which in turn is a function of the probe volume semi-axes $a(\phi)$, $c(\phi)$, and the polar angle θ .

All that is left to complete the description of the ellipsoidal composite probe volume is the size dependence of the semi-axes $a(\phi)$ and $c(\phi)$ of the component probe volumes. Since our objective in including this size dependence is to illustrate its effect on the filter probabilities, we would like to use expressions for the dependence which are as realistic as possible. To accomplish this we use an expression for $a(\phi)$ which is derived from the experimental data of Bachalo et al. (1988) for the effective "diameter" of the probe volume. This relationship between $a(\phi)$ and ϕ is shown in Fig. 13. In the figure $a(\phi_{\min})$ is the length of the semi-axis evaluated for the minimum particle size of interest $\phi_{\min}$.

Specification of $c(\phi)$ may be accomplished in either of two ways. If we are attempting to simulate the probe volume of an unapertured phase-Doppler system, then the ratio of c/a is a constant with respect to ϕ , but varies with the separation angle of the laser beams. However, in a typical system, the separation angle is small and the ratio of c/a can become quite large. In practice, the length of the probe volume is then reduced by use of an aperture to limit the field of view of the detection system. Since use of the aperture introduces no significant size dependence, a more realistic choice of $c(\phi)$ is to let this be a constant.

Evaluation of P_x for an Ellipsoidal Probe Volume

The behavior of P_x may now be illustrated for the ellipsoidal probe volume. In order to make the illustration concrete, we will choose a range of particle diameters from $2 \mu\text{m}$ ($\phi_{\min}$) to $50 \mu\text{m}$ ($\phi_{\max}$), a minimum probe "radius" of $50 \mu\text{m}$ ($a(\phi_{\min})$), and a fixed probe "length" of $600 \mu\text{m}$ ($c = 300 \mu\text{m}$). Figure 14 shows a scaled view of three of the component probe volumes, those corresponding to 2, 10, and $50 \mu\text{m}$.

Using this probe volume, the dependence of P_x on the intensity of the spatial Poisson process (i.e., the number density) and on the form of the distribution in particle size may now be evaluated. Although there are any number of distribution functions that may be chosen for this illustration, the limiting values of P_x will be found by choosing the distribution to be a delta function at $\phi_{\min}$ and $\phi_{\max}$. Figure 15 shows the dependence of P_x on

the number density for these extreme choices as well as for the intermediate particle size of 10 μm . As expected, the lowest probability of passage through the spatial filter is for the distribution function which results in the largest mean probe volume (50 μm particles). The smallest particle distribution allows for some increase in number density (at constant probability of passage) but less than an order of magnitude. Choosing the intermediate δ -function distribution as representative of the average behavior of the filter, we see that at a number density of 10^5 /cc, about 50% of the particles are lost in the spatial filter. At 10^4 /cc the losses drop to 10%, and below 10^3 /cc the losses might be considered negligible.

Evaluation of $\bar{P}_t$ for an Ellipsoidal Probe Volume

Using the expression for $g(\tau;v,\phi)$ for the ellipsoidal probe volume, Eq. (35) may be evaluated to find $\bar{P}_t$:

$$\bar{P}_t(v,\theta,\phi) = \frac{1}{2} \frac{v^2}{d^2(\theta,\phi)} \int_0^{\tau_{\max}} \tau e^{-\lambda_t \tau} d\tau = \frac{1}{2} \frac{v^2}{d^2(\theta,\phi)\lambda_t^2} \left[1 - (\lambda_t \tau_{\max} + 1) e^{-\lambda_t \tau_{\max}} \right] \quad (61)$$

Recalling that $\tau_{\max} = 2d(\theta,\phi)/v$, $\bar{P}_t$ may be written as a function of the single variable T,

$$\bar{P}_t(v,\theta,\phi) = \bar{P}_t(T) = \frac{2}{T^2} [1 - (1 + T) e^{-T}] \quad (62)$$

The new variable T is the ratio of the maximum crossing time of a particle through the probe volume to the mean time between particle arrival at the probe volume, and is given by

$$T(v,\theta,\phi) = \frac{\text{Max. crossing time of a particle}}{\text{Mean time between particles}} = \frac{2d(\theta,\phi)/v}{1/\lambda_t} \quad (63)$$

Figure 16 shows $\bar{P}_t$ as a function of T. At small values of T, $\bar{P}_t$ approaches 1, while as T goes to ∞ , $\bar{P}_t$ goes as $2/T^2$. At $T = 1$, the probability of passage through the temporal filter is 53%, while at 10^{-1} this rises to 94%. Not until the value of the T drops below order 10^{-2} can losses in the temporal filter be assumed to be negligible.

Substituting Eq. 47 for $d(\theta,\phi)$ in the expression for $T(v,\theta,\phi)$ and recalling the expressions for $\Delta V(\phi)$ and $A_p(\theta,\phi)$, T may be rewritten in a form which is more easily analyzed:

$$T(v, \theta, \phi) = \frac{3}{2} \lambda_t'' \frac{\Delta V(\phi)}{v} \frac{\bar{A}_{pt}}{A_p(\theta, \phi)} \quad (64)$$

In addition to the dependencies shown explicitly in this equation, T also depends upon the distribution function of the entire spray through $\bar{A}_{pt}$. (Note that this dependence existed in the previous formulation of T through the dependence of λ_t on the distribution function.)

Analysis of $\bar{P}_t(T)$ then requires specification of the distribution function $f_t(v, \phi)$ as well as the composite flux rate λ_t'' of the spray.

In the remainder of this section we will restrict our attention to a series of simple distribution functions to analyze the filter behavior. The figures contained in these sections are intended to provide some guidance to the phase-Doppler practitioner as to when a filter's impact on measured data will be significant and when it can be neglected. An illustration of the complete system response to a more realistic joint distribution function may be found in the following section.

Single speed, unidirectional, monodisperse flow:

If we take the distribution to be a δ -function in all four variables,

$$f_t(v, \phi) = \frac{1}{J} \delta(v - v_o) \delta(\theta - \theta_o) \delta(\psi - \psi_o) \delta(\phi - \phi_o) \quad (65)$$

(where $J = v^2 \sin\theta$ is the Jacobian of transformation from rectangular Cartesian to spherical velocity coordinates) the flow which is represented is one which has a single speed class, is unidirectional, and monodisperse in particle size. Given this distribution, the expression for T is significantly simplified because the ratio of projected areas drops out. That is, T becomes

$$T(v, \theta, \phi) = \frac{3}{2} \lambda_t'' \frac{\Delta V(\phi_o)}{v_o} \quad (66)$$

But note that because of this, T is no longer a function of the angle of approach to the probe volume, i.e., $T(v, \theta, \phi) = T(v, \phi)$ only. This unexpected result, that the probability of passage through the temporal filter is independent of the orientation relative to the probe volume (for δ -function flows), is a direct consequence of the geometrical properties of the ellipsoid. As the polar angle is varied from 0 to $\pi/2$ (Fig. 9), the increase in the total particle rate by virtue

of the projected area ($\lambda_t = \lambda_t'' \bar{A}_{pt}$) is just balanced by a decrease in the mean passage time through the probe volume by virtue of a decreased mean path length. The importance of this finding is that, for flows which are well-characterized by δ -functions, there is no preferred orientation of the probe volume with respect to the direction of the flow.

Response of the temporal filter to the intensity of the particle flux (λ_i'') can be seen to take the same form as Fig. 16, λ_i'' differing from T only by a scaling factor. Figure 17 shows the response of the filter vs. the particle flux rate for different values of $\Delta V(\phi_o)/v_o$.

Distributed speed, unidirectional, monodisperse flow:

Response of the temporal filter to a flow which is unidirectional and monodisperse, but has a distribution of particle speeds, i.e.,

$$f_i(v, \phi) = \frac{1}{J} f_i^*(v) \delta(\theta - \theta_o) \delta(\psi - \psi_o) \delta(\phi - \phi_o) \quad (67)$$

allows us to focus on the speed dependence of the temporal filter. The response to this choice of distribution function is similar to that for the complete δ -function case since, as in the previous case, the ratio of projected areas drops out of the expression for T , leaving the simple dependence of Eq. (66). As a result, there is no need to specify the functional form of

the speed distribution explicitly. The value of $\bar{P}_i$ for any element dv of $f_i^*(v)$ is the same as if the flow were a δ -function at v . Similarly, there is no dependence of the filter probability on the angle of approach to the probe volume, but there is an *inverse* dependence of T on v .

Figure 18 shows the dependence of $\bar{P}_i$ on v for various values of the product $\lambda_i'' \Delta V(\phi_o)$.

Single speed, distributed angle, monodisperse flow:

Allowing the flow to be distributed over some range of angles while holding the speed and particle size fixed (i.e., a circular line segment in velocity space),

$$f_i(v, \phi) = \frac{1}{J} \delta(v - v_o) h_i(\theta) \delta(\psi - \psi_o) \delta(\phi - \phi_o) \quad (68)$$

where $h_i(\theta)$ is the polar angle distribution function, allows us to investigate the directional dependence of the filter response. This may be explored by taking $h_i(\theta)$ to be a uniform distribution over a prescribed angle interval $\Delta\theta$, centered at the polar angle θ_o . That is,

$$h_i(\theta) = \begin{cases} 1/\Delta\theta & ; \theta_o - \Delta\theta/2 \leq \theta \leq \theta_o + \Delta\theta/2 \\ 0 & ; \text{elsewhere} \end{cases} \quad (69)$$

Figure 19 shows $\bar{P}_i$ as a function of the polar angle θ , for various values of $\Delta\theta$, with $\Delta\theta$ values of $\pi/100$, $\pi/4$, $\pi/2$, and π , centered at θ_o values of $\pi/2$ and $\pi/4$. Focusing on the family of curves centered about $\theta_o = \pi/2$, for $\lambda_i'' \Delta V(\phi_o)/v_o = 1$, it can be seen that the

centerline (θ_o) probability gradually increases with $\Delta\theta$, but that the drop in $\bar{P}_i$ at the extreme values of θ results in a mean value which is about constant with respect to $\Delta\theta$. Because of

this drop in moving away from θ_0 , a directional bias is introduced into measurements of flows with a distributed flow angle. Note that for a flow with angular interval of π , the probability of measuring a particle traveling near the z axis ($\theta = 0$) is less than 1/3 the probability of measuring a particle traveling along the x or y axes ($\theta = \pi/2$).

Changing the center polar angle θ_0 to $\pi/4$ causes the probability curves for different values of $\Delta\theta$ to merge together and into the curve $\Delta\theta = \pi$. (Note that the curve $\Delta\theta = \pi$ is independent of θ_0 since all trajectories are included regardless of the nominal center angle.)

Decreasing $\lambda_r^* \Delta V(\phi_0)/v_0$ to 0.1 increases the probability of measurement substantially while reducing directional bias effects. Conversely, increasing this value to 10 has the opposite effect -- the overall measurement probability is severely reduced and bias effects are augmented.

Single speed, unidirectional, polydisperse flow:

Choosing the flow to be polydisperse but unidirectional, and at a fixed speed, i.e.,

$$f_i(\mathbf{v}, \phi) = \frac{1}{J} \delta(v - v_0) \delta(\theta - \theta_0) \delta(\psi - \psi_0) \tilde{f}_i(\phi) \quad (70)$$

allows the size dependence of the temporal filter to be investigated. The limiting behaviors of the filter may be estimated by considering the flow to be a δ -function in size at the upper and lower limits using the curves of Fig. 17. Establishing a more general response again requires that some choice be made for the distribution function $\tilde{f}_i(\phi)$. In keeping with the simple approach we are attempting to take in this analysis, we choose the distribution function to be uniform over some size interval $\Delta\phi$ centered about the mid-range particle size $\phi_0 = 26 \mu\text{m}$ chosen for construction of the probe volume of Fig. 14, i.e.,

$$\tilde{f}_i(\phi) = \begin{cases} 1/\Delta\phi ; & \phi_0 - \Delta\phi/2 \leq \phi \leq \phi_0 + \Delta\phi/2 \\ 0 ; & \text{elsewhere} \end{cases} \quad (71)$$

Figure 20 shows the response of the filter as a function of particle size for various values of λ_r^*/v_0 , and $\Delta\phi$ values of 2, 10, 20, and 48 μm , centered about $\phi_0 = 26 \mu\text{m}$. Focusing on the data for $\lambda_r^*/v_0 = 10^{11}/\text{m}^3$, it is apparent that the curves for $\Delta\phi = 2$ and 10 μm are completely indistinguishable from that for $\Delta\phi = 20 \mu\text{m}$, and that at none of these values does a significant size bias appear. Not until $\Delta\phi = 48 \mu\text{m}$ is there a sufficient population of small particles that the size bias effect of $a(\phi)$ begins to be significant (recall Fig. 13). As with the flow angle dependence, these effects are vitiated or augmented at lower or higher particle fluxes, respectively.

Response of the Ideal Phase-Doppler System to a Realistic Particle Flow

The effect of the Poisson filters on the data measured by an ideal phase-Doppler system may be illustrated by choosing a joint distribution function which is characteristic of data obtained under experimental conditions. We accomplish this by choosing a joint distribution which replicates that measured in a steady kerosene spray flame by Edwards et al. (1990b).

Figure 21 shows the joint distribution function used to simulate the experimental data, along with the size and speed distribution functions from which it is composed. In a decelerating flow like this (the near-injector region of a spray flame), one expects to find that particles up to some maximum size will follow the gas phase velocity. Above this size droplets remain ballistic to the flow, increasingly so with size. As shown in the figure, droplets of 10 μm and less are equilibrated with the gas phase speed (in the mean) while those above 10 μm have excess speed. The mean speed at any droplet size continues to rise until it reaches the original injection speed of the droplets. In the current illustration, droplets above 40 μm retain their original injection speed, not having felt any influence from the gas phase deceleration. The distribution of particle speeds observed in the experiment was approximately Gaussian, with the mean value ($\bar{v}$) being a function of the droplet size. The standard deviation of the speed distribution (σ) was (surprisingly) independent of size and was ~ 3 m/s. Using these characteristics, the speed distribution may be approximated as a Gaussian which depends parametrically on droplet size, but is clipped at $\pm 3\sigma$ to prevent the occurrence of negative speeds, i.e.,

$$m_t(v;\phi) = \begin{cases} \frac{C(\phi)}{\sqrt{2\pi}\sigma} e^{-\frac{(v - \bar{v}(\phi))^2}{2\sigma^2}} & ; \bar{v}(\phi) - 3\sigma \leq v \leq \bar{v}(\phi) + 3\sigma \\ 0 & ; \text{elsewhere} \end{cases} \quad (72)$$

$C(\phi)$ is a normalization constant (at fixed ϕ) which accounts for the tails of the distribution clipped at $\pm 3\sigma$.

Since the measurements were taken along the peak density of the spray cone, there was little spread in the particle trajectory angles. We approximate this by choosing the angular dependence to be a δ -function at $\theta_0 = \pi/2$.

The measured size distribution of the flow extends from approximately 2 to 50 μm and is slightly skewed to small particle sizes ($\phi_0 = 20 \mu\text{m}$). Since this skew might tend to mask (or exaggerate) the effects of the Poisson filters, we have chosen not to use the experimental distribution but to use a symmetric (non-skewed) distribution which approximates the data. This distribution,

$$\tilde{f}_i(\phi) = \begin{cases} \frac{6}{\Delta\phi} \left[\frac{1}{4} - \frac{(\phi - \phi_o)^2}{\Delta\phi^2} \right] ; & \phi_o - \Delta\phi/2 \leq \phi \leq \phi_o + \Delta\phi/2 \\ 0 ; & \text{elsewhere} \end{cases} \quad (73)$$

is an inverted parabola, centered about $\phi_o = 26 \mu\text{m}$, with $\Delta\phi = 48 \mu\text{m}$, as shown at the top of the figure.

Combining these expressions for the speed, angle, and droplet size dependence, the joint distribution of our realistic flow case can be specified as

$$f_i(\mathbf{v}, \phi) = \frac{1}{J} m_i(\mathbf{v}; \phi) \delta(\theta - \theta_o) \delta(\psi - \psi_o) \tilde{f}_i(\phi) \quad (74)$$

This joint distribution can be integrated over θ and ψ and depicted as a joint distribution in speed and size by isopleths in the speed/size plane as shown in Fig. 21. The outer boundaries of the joint distribution ($f_i = 0$) are given by the straight lines in the $\mathbf{v}\phi$ plane. Four isopleths have been chosen to illustrate the population density. These have values of .001, .002, .003, and .004 $\mu\text{m}^{-1} \text{m}^{-1} \text{s}$, moving in towards the center of the plot, respectively. The dashed lines in the $\mathbf{v}\phi$ plane show $\bar{v}(\phi)$, $\bar{v}(\phi) \pm 3\sigma$, and ϕ_o , and are used as reference lines in the plots which follow.

In addition to the joint distribution function, it is also necessary to postulate a particular probe geometry and particle flux rate to complete the posing of the problem. As in the previous sections we adopt an ellipsoidal probe volume with $a(\phi)$ as specified by Fig. 13 ($a(\phi_{\min}) = 50 \mu\text{m}$) and constant $c(\phi)$ ($300 \mu\text{m}$) as illustrated by Fig. 14. Exploration of the response of the ideal phase-Doppler system can be accomplished by varying the particle flux rate (λ_i) parametrically over the range of values of interest in most spray applications (10^{10} to $10^{13} \text{m}^{-2} \text{s}^{-1}$).

Table I summarizes the relationship between the measured and true spray characteristics over this range of particle flux rates. In order to help orient the reader as to the meaning of these flux rates, the corresponding number density and measured particle rate through the probe volume are shown to the left of the table. To the right are shown the spatial filter probability P_x , a quantity $\bar{\bar{P}}_i$ to be defined below, the measured fractions of the particle and volume fluxes, and the ratio of the measured to true values of the mean diameters D_{10} , D_{30} and D_{32} . The quantity $\bar{\bar{P}}_i$ is derived from the measured fraction of particle flux,

$$\frac{\lambda_{im}''}{\lambda_i''} = \frac{1}{\lambda_i''} \int \frac{\lambda_{im} f_{im}}{A_p(\mathbf{v}, \phi)} d\mathbf{v} d\phi = \int P_x \bar{\bar{P}}_i(\mathbf{v}, \phi) f_i(\mathbf{v}, \phi) d\mathbf{v} d\phi = P_x \bar{\bar{P}}_i \quad (75)$$

$$\bar{P}_i = \frac{\lambda_{im}''/\lambda_i''}{P_x} \quad (76)$$

and expresses the overall efficiency of the temporal Poisson filter. Knowing the values of $\lambda_{im}''/\lambda_i''$ and P_x (and therefore $\bar{P}_i$), the relative contributions of each filter to the rejection rate may be examined.

At flux rates of $10^{10} \text{ m}^{-2} \text{ s}^{-1}$ and below, there is little rejection by either filter and the measured quantities are determined accurately to within 1%. Increasing the flux by an order of magnitude to 10^{11} , equal losses begin to occur for each filter ($\sim 8\%$), and the fraction of the flux which is measured drops to $\sim 86\%$. Note that the measured particle rate at which these losses occur is 7400 /s which is well within the range of particle rates normally found in steady spray applications. But it is also interesting to note that although almost 13% of the particle flux has been lost, the mean size parameters remain virtually unaffected. This remains the case even when the particle flux is increased by another order of magnitude to $10^{12} \text{ m}^{-2} \text{ s}^{-1}$. Although almost 80% of the particle and volume fluxes are lost at this condition, the error in mean diameters is less than 5%. The relative contributions of the two filters to the rejection rate remains about the same at $\sim 50\%$. Increasing the flux rate by another order of magnitude to 10^{13} , even the ideal phase-Doppler system is overwhelmed, measuring only one particle out of every 10^5 which traverse the probe volume. Despite this unrealistically high flux rate, the mean diameters are still determined with reasonable accuracy.

Since particle flux rates of 10^{10} and below result in (effectively) no particle rejections, and flux rates of 10^{13} and above result in (effectively) complete particle rejection, the only conditions under which it is interesting to examine the joint distribution efficiency, $\eta(v, \phi)$, are flux rates of 10^{11} and $10^{12} \text{ m}^{-2} \text{ s}^{-1}$. These data are shown in Figs. 22 and 23. These figures show regions of the $v\phi$ plane where $\eta(v, \phi)$ lies within the intervals indicated. The joint efficiency is always least at the low end of the velocity scale because of the increased residence time in the probe volume, but there is some remediation of this effect at small particle sizes because of the reduced "diameter" of the probe volume.

Figures 24 and 25 show the size measurement efficiency and distribution at flux rates from 10^{10} to $10^{12} \text{ m}^{-2} \text{ s}^{-1}$. As apparent from these curves and the mean diameter data of Table I, there is little size bias introduced into the measured data under these conditions. The dominant effect of the Poisson filters is an overall reduction of particle validation.

It should be noted that high particle flux rate is not the only condition under which the Poisson filters can have a substantial impact on measured data. An alternative case of practical interest is that of intermediate flux rate but long particle residence time. This may be achieved by considering the same decelerating flow depicted in Fig. 21, but with a much lower carrier velocity.

The joint distribution shown in Fig. 21 is representative of the data taken for an air-assist atomizer with mass-based atomizing-air to fuel ratio of one. The result of this high atomizing-air flow rate is that the carrier (air) stream for this flow maintains a significant

speed (~ 10 m/s) even as the spray decelerates. Instead, consider the case where there is no carrier stream, but the spray is into quiescent air. This may be simulated by reducing the mean speed of the equilibrated particles from 10 m/s, as shown in Fig. 21, to ~ 1 m/s and then allowing the mean speed to increase linearly to 20 m/s at $40\text{ }\mu\text{m}$. Allowing the width of the speed distribution to be scaled in a similar fashion (from .3 m/s to 3 m/s) we obtain the joint distribution function for this reduced carrier speed case as shown in Fig. 26.

Table II gives the results of passing a spray with this characteristic through the ideal phase-Doppler system. Now even at $10^{10}\text{ m}^{-2}\text{ s}^{-1}$ effects are beginning to be seen in the measured particle flux (4% loss). Note that this particle flux rate corresponds to a measured particle rate of only 830 /s which is quite low in practice. The results at higher flux rates are similar to those shown in Table I but with greater losses occurring at low flux rates and a significant increase in the ratio of the measured to true mean diameters.

Figures 27 and 28 show the joint distribution efficiencies for the intermediate particle flux rates, while Figs. 29 and 30 show the size measurement efficiency and distribution functions. It is apparent that with a reduced carrier speed, a significant size bias is introduced into the measured data. Note that if, instead of the symmetric distribution function for $\tilde{f}_r(\phi)$, we had chosen to use the experimentally determined distribution (which is skewed toward the smaller size range) this effect would have been even more dramatic.

Summary and Conclusion

In this paper we have outlined the effects of particle statistics on the ability of a phase-Doppler system to make accurate measurements of complex particle flows. We have done this by analyzing the response of an ideal phase-Doppler system to a postulated particle flux. The ideal system defined here senses particles of all sizes and velocities with perfect accuracy but is subject to only one constraint: in order for a measurement to be considered valid there must be only one particle in the probe volume at a time. The consequence of this constraint is that the measured flux of particles resembles the true flux, but only after having passed through two stages of filters. The first stage rejects particles for insufficient spacing and is controlled by a spatial Poisson process while the second rejects particles for excessive residence time and is driven by a temporal Poisson process. The keys to the filters are the expected values of the number of particles in the probe volume $\rho\Delta V_x$, and the number of particles entering the probe region during residence time τ , which is given by $\lambda_i \bar{A}_{pi} \tau$. Only if these values are kept below order 10^{-2} can the measured joint distribution function and flux rate be assumed to reflect the true distribution and rate of the flow. Since the spatial Poisson filter has no dependence on particle size or velocity, it rejects particles uniformly over the distribution and therefore introduces no bias into the form of the measured distribution function. The temporal filter, on the other hand, depends very strongly on the particle size, speed, and direction, and therefore has the potential to introduce significant bias into the measured distribution functions.

Particularizing the analysis to a specific ellipsoidal probe geometry, some observations about the response of the filters to various simple flow distributions were possible. These showed that below a number density of about 10^3 /cc, the effect of the spatial filter on the data could be neglected, but that for values of the order of 10^4 and 10^5 , losses in the range of 10 to 50% could be expected in the spatial filter. The response of the temporal filter was seen to depend upon a single dimensionless group (denoted T) which is the ratio of the maximum transit time of a particle through the probe volume to the mean interparticle time of the flux through the probe volume. Values of T below 10^{-2} ensure that the measured data is largely unaffected while values of order 1 result in a ~ 50 % loss through the filter. Restricting attention to flows that are near monodisperse and unidirectional, it was shown that the temporal filter response is independent of the relative orientation of the flow to the probe volume and that for particle flux rates below 10^9 m⁻² s⁻¹ the effects of the filter may be considered negligible. Considering flows which are uniformly distributed in angle or size, but δ -functions in the other flow variables, it was shown that the bias characteristics of the filters are not strong until the mean rejection rate of the filter reaches about 50%. The importance of this result is that it indicates that the filter does not introduce any insidious effects -- by the time bias effects become significant there is a sufficient increase in the rejection rate that the data is known to be suspect.

An illustration conducted using a realistic joint distribution function for a decelerating flow showed that the composite filter system may begin to effect measured particle and volume flux rates at validation rates as low as 1000 /s ($\rho \sim 2000$ /cc), and that measurements in flows with validation rates above 10^4 /s are probably suspect. However, particle size distributions and mean values were less strongly affected. In one example, the measured value of D_{32} was still accurate to within 10% even with a particle rejection rate in excess of 90%.

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Nomenclature

a	probe ellipsoid semi-axis aligned with x or y
A_p	probe region projected area
$\overline{A_{pt}}$	temporal mean probe area
c	probe ellipsoid semi-axis aligned with z
C	normalization constant
d	transformed ellipsoid semi-axis aligned with γ
D_{10}	arithmetic mean diameter
D_{30}	volume mean diameter
D_{32}	Sauter mean diameter
e	transformed ellipsoid semi-axis aligned with β
f	transformed ellipsoid semi-axis aligned with α
f	joint distribution function in size and velocity
$\hat{f}$	distribution function in velocity
$\tilde{f}$	distribution function in size
f^*	distribution function in speed
g	auxiliary distribution function in residence time
h	distribution function in polar angle
J	Jacobian of transformation
L	path length through probe region
m	auxiliary speed distribution function
p	probability density
P	probability
$\bar{P}_t$	mean temporal filter probability
$\bar{\bar{P}}_t$	average value of $\bar{P}_t$
r	radial coordinate in the $\xi\beta$ plane
$\bar{S}_x$	spatial mean particle speed
T	dimensionless temporal filter number
$\hat{t}$	interparticle time
$\mathbf{v}$	particle velocity
$v = \mathbf{v} $	particle speed
$\bar{v}$	mean particle speed
V''	volume flux rate
$\hat{V}$	interparticle volume
xyz	rectangular Cartesian coordinates
$\alpha\beta\gamma$	rotated rectangular Cartesian coordinates

δ	delta function
ΔV	probe region volume
$\Delta\phi$	width of size distribution
$\Delta\theta$	width of angle distribution
$\overline{\Delta V}_x$	spatial mean probe volume
ϕ	particle diameter
η	joint distribution efficiency
$\hat{\eta}$	velocity measurement efficiency
$\tilde{\eta}$	size measurement efficiency
ρ	number density
$\tilde{\rho}$	number density in size element $d\phi$
λ	Poisson process intensity
λ_i	particle flux rate
$\tilde{\lambda}_i$	particle rate in size element $d\phi$
θ	polar approach angle
σ	standard deviation in particle speed
τ	residence time
ξ	transformed ellipse coordinate
ψ	azimuthal approach angle

Subscripts:

x	spatial
t	temporal
m	measured by the ideal system
o	center value

Appendix A. Development of the Interparticle Volume Distribution

In the following, we sketch the derivation of the interparticle volume distribution given by Eq. (8). For present purposes, it is easiest to derive the integral of this distribution (the spatial filter probability P_x) directly. (See Eqs. (10) and (27).) One can then obtain Eq. (8) by differentiation.

The reader is referred to Sokolnikoff and Redheffer (1958), where two approaches to the derivation of the Poisson distribution formula are given. The present development is based on the first of these; it has the virtue of being concrete and explicit. While the second is more directly applicable to our discussion of particle statistics, it is more abstract. We have chosen the first as a better heuristic derivation of the formula we wish to obtain. These points will be discussed further at the end of this appendix.

The probability P_x is the probability that a particle that enters the probe volume will be rejected because at least one other particle is already in the volume. Consider a large volume V_T containing N particles. The average number density is then

$$\rho = \frac{N}{V_T} \quad (A1)$$

Now consider a probe volume ΔV (see Fig. 4(a)) inside V_T . (For clarity, we will drop the explicit indication of the dependence of the probe volume geometry on the particle size ϕ .) Consider any one particle. Assuming that the statistical distribution of particles is spatially homogeneous, the probability that the particle is inside the probe volume is

$$p = \frac{\Delta V}{V_T} \quad (A2)$$

Then the probability of finding r particles in the probe volume at any time is given by the binomial law for the probability of r successes in N events, each with probability p of success:

$$P_r(p) = \frac{N!}{r! (N-r)!} p^r (1-p)^{N-r} \quad (A3)$$

We assume $r \ll N$; following the usual procedure for the derivation of the Poisson distribution, this results in the approximation

$$P_r(p) \cong \frac{(Np)^r e^{-Np}}{r!} \quad (A4)$$

From (A1) and (A2), we have

$$Np = \rho \Delta V \quad (A5)$$

Hence, (A4) becomes

$$P_r(\Delta V) \cong \frac{(\rho \Delta V)^r e^{-\rho \Delta V}}{r!} \quad (\text{A6})$$

In particular, the probability that there are *no* other particles in the probe volume at the time that any particle just enters it is $P_0(\Delta V)$, which is approximately (compare to Eqs. (10) and (27))

$$P_x \cong e^{-\rho \Delta V} \quad (\text{A7})$$

In the limit that N and V_T become infinite (with $\rho = N/V_T$ finite), the above approximation becomes exact. However, it is not necessary to take that limit to establish equality in Eqs. (A6) and (A7). The reason that these formulas are only approximate in the above derivation is that N is considered fixed and the binomial distribution then correctly describes the system. The second approach given by Sokolnikoff and Redheffer (1958) is to consider Poisson processes as consisting of sequences of events in which it is unnecessary to impose fixed external volumes and fixed numbers of particles (using the nomenclature of the present paper). Consider particles passing through a probe volume as shown in Fig. 4(a). For notational simplicity, we temporarily denote the probe volume with the symbol V , rather than ΔV . It is taken to be axiomatic that the following assumptions are valid:

1. The probability that a given number of particles is in a given probe volume V depends *only* on the size of the volume V (and not on any information regarding particles outside of V , such as the number left in that part of V_T which is outside of V in the derivation above).
2. If $P_2(\delta V)$ is the probability of 2 or more points in a incremental volume δV , then $P_2(\delta V)/\delta V \rightarrow 0$ as $\delta V \rightarrow 0$.
3. If $P_1(\delta V)$ is the probability of *one particle* in δV , then $P_1(\delta V)/\delta V \rightarrow \rho$ as $\delta V \rightarrow 0$, where ρ is a constant (which is defined to be the number density).

Under these assumptions, by considering δV to be a differential quantity and considering the functional variation of $P_r(V)$ with V , Sokolnikoff and Redheffer show that the probability of r particles in V is given *exactly* by the Poisson law. We can therefore rewrite Eqs. (A6) and (A7) without approximation (reverting back to the ΔV notation):

$$P_r(\Delta V) = \frac{(\rho \Delta V)^r e^{-\rho \Delta V}}{r!} \quad (\text{A6}')$$

$$P_x = e^{-\rho \Delta V} \quad (\text{A7}')$$

To complete the derivation of Eq. (8), we write Eq. (10) in the form

$$\int_0^{\Delta V} p_x(\hat{V}) d\hat{V} = 1 - P_x$$

$$= 1 - e^{-\rho \Delta V}$$
(A8)

after substituting Eq. (A7). Differentiating with respect to ΔV , we obtain

$$p_x(\Delta V) = \rho e^{-\rho \Delta V}$$
(A9)

which becomes identical to Eq.(8) when we make the identification $\lambda_x = \rho$, and substitute the independent variable $\hat{V}$ for ΔV .

Appendix B. Spatial and Temporal Distribution Functions

In the following, we demonstrate the relationship between the temporal and spatial distribution functions $f_t(\mathbf{v}, \phi)$ and $f_x(\mathbf{v}, \phi)$ and derive Eqs. (24) and (25).

We start with the spatial distribution function $f_x(\mathbf{v}, \phi)$. It is defined and applied in velocity space according to the usual procedures of statistical mechanics (see, e.g., Reif (1965)). It is assumed to be normalized to unity:

$$\int_{\phi} \int_{\mathbf{v}} f_x(\mathbf{v}, \phi) d\mathbf{v} d\phi = 1$$
(B1)

If the density (average number of particles per unit spatial volume) is ρ , the number of particles in a phase space element $d\mathbf{v}d\phi$ is equal to $\rho f_x(\mathbf{v}, \phi) d\mathbf{v}d\phi$.

Let an arbitrary direction be defined by the direction of some velocity vector $\mathbf{v}$. Consider a differential element of area dA normal to that direction. The number of particles in $d\mathbf{v}d\phi$ which pass through dA per unit time is calculated from $f_x(\mathbf{v}, \phi)$ as follows:

$$d\dot{N} = \rho f_x(\mathbf{v}, \phi) |\mathbf{v}| d\mathbf{v}d\phi dA$$
(B2)

The temporal distribution function $f_t(\mathbf{v}, \phi)$ is *defined* in terms of area *fluxes*, rather than volume *densities*. This means that the velocity factor is not to be included when formulating the above rate in terms of $f_t(\mathbf{v}, \phi)$. Specifically, we write

$$d\dot{N} = \lambda_t f_t(\mathbf{v}, \phi) d\mathbf{v}d\phi dA$$
(B3)

where λ_t'' is a constant. Equating the right sides of (B2) and (B3), it is seen that the two distribution functions are related by

$$f_t(\mathbf{v}, \phi) = \frac{\rho |\mathbf{v}|}{\lambda_t''} f_x(\mathbf{v}, \phi) \quad (\text{B4})$$

Recall the normalization constraint given by Eq. (13):

$$\int_{\phi} \int_{\mathbf{v}} f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi = 1 \quad (13)$$

In view of this constraint, it is seen from Eq. (B3) that the constant λ_t'' is the composite flux rate, defined as $1/dA$ multiplied by the total rate at which particles pass through dA as $\mathbf{v}$ is rotated through 4π steradians and the integration over all velocities and particle sizes is carried out. Another way of putting this is that λ_t'' is the total rate at which particles from all directions enter a sphere, divided by the cross-sectional area of the sphere, in the limit that the size of the sphere goes to zero.

The relation between ρ and λ_t'' is determined by integrating both sides of Eq. (B4) over $\mathbf{v}$ and ϕ and applying Eq. (13):

$$\rho = \lambda_t'' / \overline{S_x} \quad (24)$$

where $\overline{S_x}$ is the spatial mean particle speed, defined by

$$\overline{S_x} \equiv \int_{\phi} \int_{\mathbf{v}} |\mathbf{v}| f_x(\mathbf{v}, \phi) d\mathbf{v} d\phi = \left\{ \int_{\phi} \int_{\mathbf{v}} \frac{1}{|\mathbf{v}|} f_t(\mathbf{v}, \phi) d\mathbf{v} d\phi \right\}^{-1} \quad (25)$$

Hence, we arrive at the principal relations between the temporal and spatial distributions as given in the main text of the paper.

Appendix C. Transformation to an Equivalent Ellipsoid Aligned with the Velocity Vector

In this appendix, we derive the equations for the transformed ellipsoid defined by Eqs. (47)-(50) and illustrated in Fig. 12. For clarity, we will drop the explicit indication of the dependence of the probe volume geometry on the particle size ϕ .

Consider the velocity vector defined by the spherical coordinates (v, θ, ψ) and the rotated coordinate system $\alpha\beta\gamma$ shown in Fig. 10. The equations of transformation from the xyz system to the $\alpha\beta\gamma$ system are

$$x = \alpha \cos\theta \cos\psi - \beta \sin\psi + \gamma \sin\theta \cos\psi \quad (C1)$$

$$y = \alpha \cos\theta \sin\psi + \beta \cos\psi + \gamma \sin\theta \sin\psi \quad (C2)$$

$$z = -\alpha \sin\theta + \gamma \cos\theta \quad (C3)$$

Substituting into Eq. (45), we obtain the equation of the ellipsoid in $\alpha\beta\gamma$ coordinates

$$P \gamma^2 + 2Q \gamma + R = 0 \quad (C4)$$

where

$$P = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2} \quad (C5)$$

$$Q = \alpha \sin\theta \cos\theta \left(\frac{1}{a^2} + \frac{1}{c^2} \right) \quad (C6)$$

$$R = \frac{\alpha^2 \cos^2\theta + \beta^2}{a^2} + \frac{\alpha^2 \sin^2\theta}{c^2} - 1 \quad (C7)$$

Equation (C4) is written in the form of a quadratic equation in γ to facilitate evaluation of the entry and exit points γ_1 and γ_2 (see Fig. 11). These points are given by

$$\gamma_{(1)}(\alpha, \beta) = \frac{-Q \pm \sqrt{Q^2 - PR}}{P} \quad (C8)$$

The path length through the ellipsoid is equal to $2d$, where

$$\begin{aligned} L(\alpha, \beta) &= \gamma_1 - \gamma_2 \\ &= \frac{2 \sqrt{Q^2 - PR}}{P} \end{aligned} \quad (C9)$$

Since the path length is the quantity of interest, it is not necessary to deal with the longer formula (C8).

From Eqs. (C5)-(C7) the quantity inside the radical is

$$Q^2 - PR = P \left(1 - \frac{\beta^2}{a^2} \right) - \frac{\alpha^2}{a^2 c^2} \quad (C10)$$

We now observe that the path lengths given by Eq. (C9) are the same as the path lengths for an ellipsoid which has the defining equation

$$P \gamma^2 + \frac{\alpha^2}{a^2 c^2 P} + \frac{\beta^2}{a^2} = 1 \quad (C11)$$

or

$$\frac{\gamma^2}{d^2} + \frac{\beta^2}{e^2} + \frac{\alpha^2}{f^2} = 1 \quad (50)$$

where

$$d = \frac{1}{\sqrt{P}} = \sqrt{\left(\frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{c^2} \right)^{-1}} \quad (47)$$

$$e = a \quad (48)$$

$$f = ac\sqrt{P} = \sqrt{c^2 \sin^2 \theta + a^2 \cos^2 \theta} \quad (49)$$

We have thereby derived Eqs. (47)-(50). This represents a transformation of the do-main from the original ellipsoid to an ellipsoid with one axis aligned in the γ -direction, and with the other two orthogonal axes in the α - and β -directions.

Table I
Response of the ideal phase-Doppler system to a flow with a joint distribution function representative of experimentally observed data for a kerosene spray flame.

λ_t'' (m ² s ⁻¹)	ρ (/cc)	λ_{tm} (/s)	P_x	$\bar{P}_t$	$\frac{\lambda_{tm}''}{\lambda_t''}$	$\frac{V_{tm}''}{V_t''}$	$\frac{D_{10,tm}}{D_{10,t}}$	$\frac{D_{30,tm}}{D_{30,t}}$	$\frac{D_{32,tm}}{D_{32,t}}$
10 ¹⁰	730	850	0.99	0.99	0.99	0.99	1.00	1.00	1.00
10 ¹¹	7.3x10 ³	7.4x10 ³	0.93	0.92	0.86	0.87	1.01	1.00	1.00
10 ¹²	7.3x10 ⁴	2.1x10 ⁴	0.47	0.51	0.24	0.26	1.04	1.03	1.02
10 ¹³	7.3x10 ⁵	9.6	6x10 ⁻⁴	2x10 ⁻²	1x10 ⁻⁵	1x10 ⁻⁵	1.12	1.08	1.05

Table II
Response of the ideal phase-Doppler system to a flow with a joint distribution function representative of a reduced carrier gas speed.

λ_t'' (m ² s ⁻¹)	ρ (/cc)	λ_{tm} (/s)	P_x	$\bar{P}_t$	$\frac{\lambda_{tm}''}{\lambda_t''}$	$\frac{V_{tm}''}{V_t''}$	$\frac{D_{10,tm}}{D_{10,t}}$	$\frac{D_{30,tm}}{D_{30,t}}$	$\frac{D_{32,tm}}{D_{32,t}}$
10 ¹⁰	2100	830	0.98	0.98	0.96	0.97	1.01	1.00	1.00
10 ¹¹	2.1x10 ⁴	6.1x10 ³	0.83	0.84	0.70	0.77	1.06	1.03	1.01
10 ¹²	2.1x10 ⁵	4.9x10 ³	0.15	0.40	0.06	0.08	1.21	1.13	1.06
10 ¹³	2.1x10 ⁶	9x10 ⁻⁵	8x10 ⁻⁹	1x10 ⁻²	9x10 ⁻¹¹	2x10 ⁻¹⁰	1.33	1.21	1.12

Fig. 1 A model of the ideal phase-Doppler instrument. Particles traveling through the probe volume are idealized as points moving with constant velocity and marked by a size variable ϕ . The composite probe volume is made up of a series of component probe volumes, each associated with a particular particle size. Particles are sensed with perfect accuracy when inside their respective component probe volumes, but are not sensed by the other component probe volumes. The volume and orientation of each of the probe volumes is assumed to be a function of particle size.

Fig. 2 Example of a reentrant probe volume which is not valid for this analysis. Particles travelling on certain trajectories will be able to enter, leave and reenter the probe volume during passage.

Fig.3 . The projected area of a component probe volume, $A_p(v, \phi)$, depends upon both the particular component probe volume under consideration and on the direction of approach of the particle as determined by the velocity vector v . Note that two particles (1 and 2) of the same size ϕ share a common component probe volume, but have different projected areas corresponding to their velocities v_1 and v_2 .

(a)

(b)

Fig. 4 Passage through the spatial Poisson filter requires that there be sufficient interparticle volume that the probe volume is empty at the instant a test particle enters. Part (a) illustrates this for a flow with a single size class ϕ_1 . Part (b) illustrates this when a second class of particles, ϕ_2 , (depicted by open circles) is added. Generalization to a system with any number of size classes and component probe volumes is straightforward.

(a)

(b)

Fig. 5 Passage through the temporal Poisson filter requires that there be sufficient time between the arrival of particles such that the test particle leaves the probe volume before the next particle enters. Part (a) illustrates this for a flow with a single size class ϕ_1 . Part (b) illustrates this when a second class of particles, ϕ_2 , (depicted by open circles) is added. Generalization to a system with any number of size classes and component probe volumes is straightforward.

Fig. 6 Illustration of the serial nature of the Poisson filters. The particle flux measured by the ideal phase-Doppler system corresponds to the true particle flux, but reduced by losses through the two filters. The spatial filter rejects particles for insufficient spacing, while the temporal filter rejects particles for excessive residence time.

Fig . 7 Test particle probability of passage through the spatial or temporal filter. Both probabilities have the same functional form in terms of their argument. The spatial filter depends on the expected value of the number of particles in the composite probe volume. Similarly, the temporal filter depends on the expected value of the number of particles entering the probe volume during time interval τ .

Fig. 8 The test particle residence time depends not only on its size and velocity, but also on its point of entry into the probe volume. Two particles, P_1 , and P_2 , with identical size and velocity are shown. The residence time τ of P_1 is less than that of P_2 because its path length through the probe volume (L_1) is less than that for P_2 (L_2).

Fig. 9 Illustration of a component ellipsoidal probe volume. The ellipsoid is symmetric about the z axis and has particle size-dependent semi-axes $a(\phi)$ and $c(\phi)$. Particles approach the probe volume with a velocity vector $\mathbf{v}$ which can be expressed in terms of the spherical coordinate system given in Fig. 10a. Since the probe volume is symmetric about the z axis, only the polar angle θ is significant in specifying the direction of approach to the probe volume.

(a)

(b)

Fig. 10 (a) Spherical polar coordinate system used to define the direction of approach to the probe volume. In terms of this coordinate system, the velocity vector $\mathbf{v}$ has components (v, θ, ψ) , where v is the particle speed, θ is the polar angle, and ψ is the azimuthal angle. (b) Rotated rectangular Cartesian coordinate system $\alpha\beta\gamma$. The γ direction is chosen to coincide with the direction of travel of a particle with velocity vector $\mathbf{v}$. The position of the α and β axes is fixed by choosing β to lie in the xy plane.

Fig. 11 Illustration of a component probe volume, its projected area A_p , and path length L , in the $\alpha\beta\gamma$ coordinate system. Regardless of the direction of approach, the projected area is bounded by an ellipse in the $\alpha\beta$ plane, and the path length is given by the difference in γ values of the surface of the ellipsoid, $\gamma_2 - \gamma_1$.

Fig. 12 The transformed component probe volume is an ellipsoid which preserves the projected cross section and path lengths of the original probe volume, but which has its semi-axes (f, e, d) aligned with the $\alpha\beta\gamma$ coordinate system. The projected area of the probe volume is the same as that of an ellipse with semi-axes f and e .

Fig. 13 Assumed dependence of the probe volume semi-axis, a , on particle size, ϕ , based on the data of Bachalo et al. (1988). $a(\phi_{\min})$ is the length of the semi-axis for the smallest particle size of interest $\phi_{\min}$.

Fig. 14 Scale drawing of three component ellipsoidal probe volumes with a radial size dependence as illustrated in Fig. 13 and constant length. The component probe volumes shown correspond to particle sizes of 2, 10, and 50 μm .

Fig. 15 Probability of passage through the spatial filter as a function of number density for the composite probe volume illustrated in Fig. 14. The particle size distribution has been taken as a δ -function at the limits 2 and 50 μm , and at the intermediate value of 10 μm .

Fig. 16 Probability of passage through the temporal filter for an ellipsoidal probe volume as a function of the dimensionless group $T(v, \theta, \phi)$ -- the ratio of the maximum crossing time of a particle to the mean time between particles. A significant number of particles with values of T of order 1 are rejected by the filter while particles with values of $< 10^{-2}$ are not effected.

Fig. 17 Dependence of the mean temporal filter probability on particle flux rate for a single speed (v_o), unidirectional (θ_o), monodisperse (ϕ_o) flow.

Fig. 18 Dependence of the mean temporal filter probability on particle speed for a unidirectional (θ_o), monodisperse (ϕ_o) flow with an arbitrary speed distribution.

Fig. 19 Dependence of the mean temporal filter probability on the polar approach angle for a single speed (v_o), monodisperse (ϕ_o) flow with a uniform distribution over some angular interval $\Delta\theta$. Two center angles are shown, $\theta_o = \pi/4$ and $\pi/2$. The angular intervals about the center angles have been chosen as $\pi/100$, $\pi/4$, $\pi/2$ and π .

Fig. 20 Dependence of the mean temporal filter probability on particle size for a single speed (v_o), unidirectional (θ_o), flow with a uniform size distribution of width $\Delta\phi$, centered about $\phi_o = 26$ μm . Although distribution widths of 2, 10, 20 and 48 μm are shown, only the curves for 20 and 48 μm are distinguishable. The curves for 2 and 10 μm overlay the 20 μm curve.

Fig. 21 Joint temporal distribution function representative of experimentally observed data for a kerosene spray flame. The speed distribution at any given particle size, $m_t(v; \phi)$, is a Gaussian clipped at ± 3 standard deviations (3 m/s). The mean speed is equal to the atomizing-air speed (10 m/s) up to a particle size of 10 μm , then increases linearly with size up to 40 μm . Above 40 μm the mean speed is constant, the particles still travelling with their original injection speed (20 m/s). The size distribution function is an inverted parabola which spans from 2 to 50 μm , and the flow is considered unidirectional at $\theta_0 = \pi/2$. The straight line segments in the $v\phi$ plane indicate the outermost boundaries of the joint distribution, while the interior curves denote four equispaced population contours from $f_t = .001$ to .004 $\mu\text{m}^{-1}\text{m}^{-1}\text{s}$ moving radially inward. The dashed lines indicate the mean speed, mean speed $\pm$ the standard deviation in speed, and mean particle size.

Fig. 22. Joint distribution efficiency for the flow characterized by Fig. 21, with a particle flux rate of $10^{11} / \text{m}^2 / \text{s}$. The joint efficiency values are in the range indicated by the labels in the regions bounded by solid lines. The dashed lines indicate the ± 1 standard deviation band from Fig. 21 for reference.

Fig. 23. Joint distribution efficiency for the flow characterized by Fig. 21, with a particle flux rate of $10^{12} / \text{m}^2 / \text{s}$. The joint efficiency values are in the range indicated by the labels in the regions bounded by solid lines. The dashed lines indicate the ± 1 standard deviation band from Fig. 21 for reference.

Fig. 24 Size measurement efficiency for the flow characterized by Fig. 21. Little size bias is introduced by the filters for this flow.

Fig.25 True and measured (ideal) size distribution functions for the flow characterized by Fig. 21. Even though large numbers of particles may be rejected by the filters, little distortion of the distribution function occurs for this flow.

Fig. 26 Joint temporal distribution function for the reduced carrier speed case. This case is similar to that of Fig. 21, but with the carrier gas speed reduced to 1 m/s to simulate a spray decelerating into near-stagnant conditions. The width of the speed distribution has also been scaled back (0.3 m/s) below 10 μm , but increases linearly to the uninfluenced value (3 m/s) at 40 μm . The contours of population density shown correspond to equal intervals from .002 to .008 $\mu\text{m}^{-1}\text{m}^{-1}\text{s}$ in f_t .

Fig. 27. Joint distribution efficiency for the reduced carrier speed flow with a particle flux rate of $10^{11} / \text{m}^2 / \text{s}$. The joint efficiency values are in the range indicated by the labels in the regions bounded by solid lines. The dashed lines indicate the ± 1 standard deviation band from Fig. 26 for reference.

Fig. 28. Joint distribution efficiency for the reduced carrier speed flow with a particle flux rate of $10^{12} / \text{m}^2 / \text{s}$. The joint efficiency values are in the range indicated by the labels within the regions bounded by solid lines. The dashed lines indicate the ± 1 standard deviation band from Fig. 26 for reference.

Fig. 29 Size measurement efficiency for the reduced carrier speed flow. The size bias introduced by the filters for this flow is much more pronounced than the previous case.

Fig. 30 True and measured (ideal) size distribution functions for the reduced carrier speed flow. Significant distortion of the distribution function occurs for this flow.

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