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UCID- 21103

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FIELDS II: A FREE ELECTRON AND A HARMONICALLY
BOUND ELECTRON

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July 1987

Lawrence
Livermore
National
Laboratory

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

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THEORY OF ATOMS IN STRONG, PULSED ELECTROMAGNETIC FIELDS II:
A FREE ELECTRON AND A HARMONICALLY BOUND ELECTRON*

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ABSTRACT

Multiple scale methods are used to treat several ingredients of the theory of atoms in strong electromagnetic fields. First, a focussed electromagnetic field is expanded. Second, a classical electron is described giving the ponderomotive potential. Third, the quantum electron is treated in the WKB approximation. Fourth, the one dimensional quantum harmonic oscillator is solved; and last the one dimensional square well, showing the low intensity limit of the theory.

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INTRODUCTION

In a previous paper¹ (referred to as I) a theory of atoms in strong, pulsed, narrow band radiation fields was outlined. In this paper we discuss some of the basic ingredients that enter into the theory, and make some order-of-magnitude estimates of their importance.

A major motivation in writing this paper is to show the power of the multiple scale method that was also used in I. It provides not only a simple and elementary derivation of known results, but it gives a procedure to assess their accuracy, and to extend them to higher order. In the first section we derive a convenient and very physical representation of a focused laser pulse with a slowly varying envelope: it is that of a pulse following a "path" determined by the optical system. In the second section we describe the motion of a non-relativistic classical electron in a laser pulse and solve it using the multiple scale method. This is a very direct way to derive the ponderomotive potential, and a time-averaged equivalent Hamiltonian. Also we show the basic asymmetry between the electron moving into (or out of) an essentially static electromagnetic field and the field overtaking a slow-moving electron. In the same section we derive the semiclassical wave function of a quantum mechanical electron using the action function calculated classically. In the third section we review the wave function of a one-dimensional harmonic oscillator in an external field. This is a good model for the linear response of an atom; in the strong-field regime it can provide a guideline only when non-linear resonances are not important.¹ Four dimensionless parameters enter into the theory. The smallness of the relativistic parameter $\mu^2 \ll 1$ (Eq. 2.1) limits the validity of the whole paper. The free electron parameter, n_f (Eq. 2.37) counts the number

of photons "dressing" the free electron. The parameter, n_{osc} (Eq. 2.39) counts the number of "missing" electron energy channels due to the "quiver energy" of the outgoing electrons. Finally the bound electron parameter, n_b (Eq. 3.5) counts the number of excited levels of the "atom." In the final section we make connections to experiments.

I. ELECTROMAGNETIC FIELDS WITH SLOWLY VARYING ENVELOPE

A strong optical pulse with a narrow spectrum always has a slowly varying envelope in time. We will assume that it has a slowly varying envelope in both space and time, and discuss the validity of this assumption at the end of the section.

The wave equation in vacuo, in radiation gauge for the vector potential $\bar{A}(x, y, z, t)$ is^{2,3}

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \bar{A} = 0, \quad (1.1)$$

together with the gauge conditions,

$$\bar{\nabla} \cdot \bar{A} = 0, \quad \varphi = 0. \quad (1.2)$$

The slowly varying envelope approximation (SVEA) can be expressed mathematically as

$$\bar{A} = \frac{1}{2} \left[\bar{a}(x, y, z, t) e^{-i \int \omega(\tau') d\tau'} + c.c. \right] \quad (1.3)$$

where

$$\tau = t - z/c \quad (1.4)$$

and

$$|\bar{\nabla} \bar{a}| \ll \frac{\omega}{c} |\bar{a}| \quad (1.5)$$

$$|\frac{\partial \bar{a}}{\partial \tau}| \ll \omega |\bar{a}|$$

$$|\frac{d\omega}{d\tau}| \ll \omega^2.$$

Thus, $\bar{a}$ (the complex analytic signal) and ω (the instantaneous frequency) are both slowly varying with respect to the light cone moving with the pulse: the spatial variation is slow compared to $1/k$ and the time variation is slow compared to $1/\omega$. Substituting Eq. 1.3 into 1.1, a simple calculation gives

$$\nabla_T^2 \bar{a} + 2i k \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \bar{a} + \frac{\partial^2 \bar{a}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \bar{a}}{\partial t^2} = 0 \quad (1.6)$$

where

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad k = \frac{\omega}{c} \quad (1.7)$$

We transform now to new variables x', y', z', τ ,

$$x' = x, y' = y, z' = z, \tau = t - z/c \quad (1.8)$$

This gives, after neglecting the primes on x, y, z ,

$$\nabla_T^2 \bar{a} + 2i k \frac{\partial}{\partial z} \left[\bar{a} - \frac{i}{2k} \frac{\partial \bar{a}}{\partial z} + \frac{i}{\omega} \frac{\partial \bar{a}}{\partial \tau} \right] = 0 \quad (1.9)$$

A similar substitution into the gauge condition, Eq. 1.2, gives

$$\bar{\nabla}_T \cdot \bar{a}_T + i k a_z \left[1 - \frac{i}{k a_z} \frac{\partial a_z}{\partial z} \right] = 0 \quad (1.10)$$

where $\bar{a}_T = \hat{x} a_x + \hat{y} a_y$ is the transverse part of $\bar{a}$. The SYEA condition,

Eq. 1.5, allows us to neglect the second term in the brackets in Eq. 1.10,

giving to first order in smallness,

$$a_z = \frac{i}{k} \bar{\nabla}_T \cdot \bar{a}_T, \quad (1.11)$$

thus $|a_z| \ll |\bar{a}_T|$. By a similar argument, the two last terms in the brackets in Eq. 1.9 can also be neglected. For the transverse component this results in the paraxial wave equation,

$$\nabla_T^2 \bar{a}_T + 2ik \frac{\partial \bar{a}_T}{\partial z} = 0. \quad (1.12)$$

As this equation has no τ -derivatives, it has solutions of the form

$$\bar{a}_T(\bar{x}, \tau) = \bar{g}_T(\bar{x}) f(\tau) \quad (1.13)$$

where $f(\tau)$ can be any slowly varying function: it is the pulse shape. Also, $\bar{g}_T$ is a stationary solution of the paraxial wave equation,

$$\nabla_T^2 \bar{g}_T + 2ik \frac{\partial \bar{g}_T}{\partial z} = 0, \quad (1.14)$$

determined by the optical system. The longitudinal component, a_z can be calculated from Eq. 1.11, it also satisfies the paraxial wave equation.

This result agrees with our intuition that a short laser pulse propagates in an optical system the same way as a long pulse does. There are several limitations to this result. The first is that the source (the laser) itself has to emit a pulse of the form, Eq. 1.13, i.e., the transverse distribution of its (complex) field amplitude should stay constant during the pulse. Second, the optical system should have a numerical aperture (f-number) substantially larger than unity, so the paraxial approximation hold. Third, we have to postulate the absence of discontinuities in the wavefront,⁴ or of nasty caustics.^{4,5} This latter condition can easily be violated in quite innocent-looking optical systems, e.g., in the presence of aberrations, or even slight non-linear effects.

II. FREE ELECTRON IN A RADIATION FIELD

In this section we treat the free electron in a radiation field. We will restrict ourselves to the non-relativistic case, when the parameter μ^2 is small:

$$\mu^2 = \left(\frac{e|\bar{A}|}{mc^2} \right)^2 \ll 1. \quad (2.1)$$

The Hamiltonian is

$$H = \frac{1}{2m} \left(\bar{P} - \frac{e}{c} \bar{A} \right)^2 \quad (2.2)$$

where $\bar{P}$ is the canonical momentum. We use the notation of Jackson,⁶ and radiation gauge. We first review the classical theory then we will construct the quasi-classical wave function for a quantum mechanical electron.

II.-1 CLASSICAL THEORY

The motion of the free electron in a finite radiation pulse was solved quite completely by Kibble,⁷ reviewed by Eberly,⁸ and recently by Mulser⁹ and by Ehlötzky.¹⁰ We will treat it using a multiple scale method. Hamilton's canonical equations, in Cartesian coordinates, are

$$\frac{d\bar{P}}{dt} = - \frac{\partial H}{\partial \bar{x}} \quad \frac{d\bar{x}}{dt} = \frac{\partial H}{\partial \bar{P}} \quad (2.3)$$

when the (non-conservative) Hamiltonian is given by Eq. 2.2; we will assume a slowly varying envelope of the field, Eqs. 1.3, 1.13. In order to simplify notation, we write

$$\bar{A} = \bar{a}(x, y, z, \tau) \cos \omega \tau \quad (2.4)$$

but all our calculations are equally valid for the more general expression,
 1.3. Sorting of orders in the equations gets simple by the introduction of the "scaled" variables,

$$\bar{\xi} = \epsilon \bar{x}; \text{ i.e. } (\xi, \eta, \zeta) = \epsilon(x, y, z). \quad (2.5)$$

The magnitude of ϵ is determined by the (rather vague) condition that

$$\frac{1}{\epsilon} \left| \frac{\partial \bar{a}}{\partial \bar{x}} \right| = \left| \frac{\partial \bar{a}}{\partial \bar{\xi}} \right| = \left| \bar{a} \right|. \quad (2.6)$$

(Alternatively, after the calculation we may put $\epsilon = 1$.) First we note, that from Eq. 1.11,

$$a_z \equiv \epsilon a_\zeta = \epsilon \frac{1}{k} \left(\frac{\partial a_x}{\partial \xi} + \frac{\partial a_y}{\partial \eta} \right). \quad (2.7)$$

The equations of motion are obtained by using Eqs. 2.4, 2.7, 2.2 in Eq. 2.3.

They can be written, explicitly as

$$\begin{aligned} \dot{p}_x = - \frac{\partial H}{\partial x} = \frac{1}{m} & \left[\left(p_x - \frac{e}{c} a_x \cos \omega \tau \right) \left(\epsilon \frac{e}{c} \frac{\partial a_x}{\partial \xi} \cos \omega \tau \right) \right. \\ & + \left(p_y - \frac{e}{c} a_y \cos \omega \tau \right) \left(\epsilon \frac{e}{c} \frac{\partial a_y}{\partial \xi} \cos \omega \tau \right) \\ & \left. + \left(p_z - \epsilon \frac{e}{c} a_\zeta \cos \omega \tau \right) \left(\epsilon^2 \frac{e}{c} \frac{\partial a_\zeta}{\partial \xi} \cos \omega \tau \right) \right] \end{aligned} \quad (2.8x)$$

$$\dot{p}_y = \text{similar} \quad (2.8y)$$

$$\begin{aligned} \dot{p}_z = - \frac{\partial H}{\partial z} = \frac{1}{m} & \left[\left(p_x - \frac{e}{c} a_x \cos \omega \tau \right) \left(\epsilon \frac{e}{c} \frac{\partial a_x}{\partial \zeta} \cos \omega \tau + k \frac{e}{c} a_x \sin \omega \tau \right) \right. \\ & + \left(p_y - \frac{e}{c} a_y \cos \omega \tau \right) \left(\epsilon \frac{e}{c} \frac{\partial a_y}{\partial \zeta} \cos \omega \tau + k \frac{e}{c} a_y \sin \omega \tau \right) \\ & \left. + \left(p_z - \epsilon \frac{e}{c} a_\zeta \cos \omega \tau \right) \left(\epsilon^2 \frac{e}{c} \frac{\partial a_\zeta}{\partial \zeta} \cos \omega \tau + \epsilon k \frac{e}{c} a_\zeta \sin \omega \tau \right) \right] \end{aligned} \quad (2.8z)$$

$$\dot{\bar{x}} = \frac{\partial H}{\partial \bar{p}} = \frac{1}{m} \left(\bar{p} - \frac{e}{c} \bar{a} \cos \omega \tau \right) \quad (2.9)$$

We expand $\bar{p}$ and $\bar{x}$ in orders of ϵ ,

$$\bar{p} = \bar{p}^0 + \epsilon \bar{p}^1 + \dots, \quad \bar{x} = \bar{x}^0 + \epsilon \bar{x}^1, \quad (2.10)$$

substitute into the equations, and equate the various orders of ϵ . To zero order,

$$\dot{\bar{p}}_x^0 = \dot{\bar{p}}_y^0 = 0 \quad (2.11_{x,y})$$

$$\begin{aligned} \dot{\bar{p}}_z^0 &= \frac{1}{m} \left[\left(p_x^0 - \frac{e}{c} a_x \cos \omega \tau \right) \left(-k \frac{e}{c} a_x \sin \omega \tau \right) \right. \\ &\quad \left. + \left(p_y^0 - \frac{e}{c} a_y \cos \omega \tau \right) \left(k \frac{e}{c} a_y \sin \omega \tau \right) \right] \\ &= \frac{ke}{mc} \left(p_x^0 a_x + p_y^0 a_y \right) \sin \omega \tau - \frac{e^2 (a_x^2 + a_y^2)}{2 mc^2} \sin 2 \omega \tau \end{aligned} \quad (2.11z)$$

$$\dot{\bar{x}}^0 = \frac{1}{m} \left(\bar{p}^0 - \frac{e}{c} \bar{a} \cos \omega \tau \right) \quad (2.12)$$

It is interesting to note, that the "figure eight" motion due to the magnetic field comes out correctly; also the "jitter" motion in the x, y plane does not appear in the canonical momentum $\bar{p}$, but in the kinetic momentum $m \dot{\bar{x}}$.

To first order, we get

$$\begin{aligned} \dot{\bar{p}}_x^1 &= \frac{1}{m} \left[\left(p_x^0 - \frac{e}{c} a_x \cos \omega \tau \right) \left(\frac{e}{c} \frac{\partial a_x}{\partial \xi} \cos \omega \tau \right) \right. \\ &\quad \left. + \left(p_y^0 - \frac{e}{c} a_y \cos \omega \tau \right) \left(\frac{e}{c} \frac{\partial a_y}{\partial \xi} \cos \omega \tau \right) \right] \\ &= \frac{e}{mc} \left(p_x^0 \frac{\partial a_x}{\partial \xi} + p_y^0 \frac{\partial a_y}{\partial \xi} \right) \cos \omega \tau \\ &\quad - \frac{e^2}{2 mc^2} \left(a_x \frac{\partial a_x}{\partial \xi} + a_y \frac{\partial a_y}{\partial \xi} \right) (1 + \cos 2 \omega \tau) \end{aligned} \quad (2.13x)$$

$$\dot{p}_y^1 = \text{similar} \quad (2.13y)$$

$$\begin{aligned} \dot{p}_z^1 = \frac{1}{m} \left[\left(p_x^0 - \frac{e}{c} a_x \cos \omega \tau \right) \left(\frac{e}{c} \frac{\partial a_x}{\partial \zeta} \cos \omega \tau \right) \right. \\ + \left(p_y^0 - \frac{e}{c} a_y \cos \omega \tau \right) \left(\frac{e}{c} \frac{\partial a_y}{\partial \xi} \cos \omega \tau \right) \\ + \left(p_x^1 - \frac{e}{c} a_x \cos \omega \tau \right) \left(k \frac{e}{c} a_y \sin \omega \tau \right) \\ + \left(p_y^1 - \frac{e}{c} a_y \cos \omega \tau \right) \left(k \frac{e}{c} a_x \sin \omega \tau \right) \\ \left. + p_z^0 \frac{e}{c} a_\zeta k \sin \omega \tau \right] \quad (2.13z) \end{aligned}$$

$$\dot{\bar{x}}^1 = \frac{1}{m} \dot{\bar{p}}^1 \quad (2.14)$$

It is clear, that Eqs. 2.13 contain a secular (non-oscillating) term,

$$\dot{\bar{p}}_{\text{secular}} = - \frac{e^2}{4 m c^2} \nabla |\bar{a}|^2 \quad (2.15)$$

It has the form of a gradient of a potential, the so-called ponderomotive potential:

$$E_{\text{osc}} = \frac{e^2 |\bar{a}|^2}{4 m c^2} \quad (2.16)$$

Such secular terms should be added to the equations of a lower order. The same term can be obtained by the method of averaging.¹⁰

For completeness, we present the resulting classical equations, correct to order ε^0 for the Hamiltonian with a potential

$$H = \frac{1}{2m} \left(\bar{p} - \frac{e}{c} \bar{a} \cos \int^\tau \omega(\tau') d\tau' \right)^2 + V(\bar{x}) \quad (2.17)$$

where $\bar{a}$, ω are slowly varying (Eq. 1.5). They are, in Cartesian coordinates,

$$\dot{p}_x^o = - \frac{e^2}{4 mc^2} \frac{\partial |\bar{a}|^2}{\partial x} - \frac{\partial V}{\partial x} \quad (2.18x)$$

$$\dot{p}_y^o = - \frac{e^2}{4 mc^2} \frac{\partial |\bar{a}|^2}{\partial y} - \frac{\partial V}{\partial y} \quad (2.18y)$$

$$\begin{aligned} \dot{p}_z^o &= - \frac{e^2}{4 mc^2} \frac{\partial |\bar{a}|^2}{\partial z} - \frac{\partial V}{\partial z} + \frac{ke}{mc} (p_x^o a_x + p_y^o a_y) \sin \int^\tau \omega(\tau') d\tau' \\ &\quad + \frac{e^2 (a_x^2 + a_y^2)}{2 mc^2} \sin \int^\tau 2 \omega(\tau') d\tau' \\ &\approx - \frac{e^2}{4 mc^2} \frac{\partial |\bar{a}|^2}{\partial z} - \frac{\partial V}{\partial z} + \frac{ke}{mc} (\bar{p}^o \cdot \bar{a}) \sin \int^\tau \omega(\tau') d\tau' \\ &\quad + \frac{e^2 |\bar{a}|^2}{2 mc^2} \sin \int^\tau 2 \omega(\tau') d\tau' \end{aligned} \quad (2.18z)$$

$$\dot{\bar{x}}^o = \frac{1}{m} (\bar{p}^o - \frac{e}{c} \bar{a} \cos \int^\tau \omega(\tau') d\tau') \quad (2.19)$$

Time averaging these equations of motion yield

$$\langle \dot{\bar{p}}^o \rangle = - \nabla \left(\frac{e^2 |\bar{a}|^2}{4 mc^2} + V \right) \quad (2.20)$$

$$\langle \dot{\bar{x}}^o \rangle = \frac{1}{m} \langle \bar{p}^o \rangle \quad (2.21)$$

These are derivable from an equivalent Hamiltonian,

$$H_{eq} = \frac{1}{2m} \langle \bar{p}^o \rangle^2 + \frac{e^2 |\bar{a}|^2}{4 mc^2} + V \equiv E_k + E_{osc} + V \quad (2.22)$$

Note that the ponderomotive potential depends on both space and time. In general, the equations don't conserve energy or canonical momentum. Also when an electron enters or exits the electromagnetic field, the canonical momentum $\langle \bar{p}^o \rangle$ changes its direction as well as its magnitude.

There are two cases of physical interest that are easy to discuss. The first is the "long pulse" case - the electron transits the field in a time that is short with respect to its pulse length. Note that this is a statement about the relative magnitude of the pulse duration, τ_p , its transverse dimension, $x_T = |\bar{a}_T| / |\nabla_T \bar{a}_T|$ and the electron's transverse speed, $v_T = |\bar{p}_T| / m$:

$$v_T \gg \frac{\omega_T}{\tau_p} \quad (2.23)$$

In this case, the time dependence of $|\bar{a}|^2$ in Eq. 2.22 can be neglected, thus

$$E_0 = \frac{p_0^2}{2m} = \frac{|\langle \bar{p}_0 \rangle|^2}{2m} + \frac{e^2 |\bar{a}|^2}{4 mc^2} + V \quad (2.24)$$

is a conserved quantity, where $E_0 = p_0^2/2m$ is the (initial, and final) energy of the electron, as measured outside the field. Note that the time averaged canonical momentum inside the field is diminished by the ponderomotive potential (or "quiver energy").

The opposite limit is when the (poor, innocent) electron is overtaken by the laser pulse. Note that no electron can escape the pulse if the latter's transverse dimension is large enough, $x_T > c\tau_p$. For simplicity, let us take a plane wave pulse, of the form, Eq. 1.13, with a constant $\bar{g}_T$,

$$\bar{a}_T(\vec{x}, \tau) = \bar{g}_T f(\tau); \quad (2.25)$$

also, $V = 0$ and $\bar{p}_0 = p_z \hat{z} = m v_z \hat{z}$. The time averaged equations thus reduce to

$$\langle p_z^0 \rangle = - \frac{e^2}{4 mc^2} \frac{\partial |\bar{a}|^2}{\partial z} \quad (2.26)$$

$$\langle \dot{z}^0 \rangle = \frac{1}{m} \langle p_z^0 \rangle \quad (2.27)$$

using Eqs. 1.4, 2.25, we get

$$\frac{\partial |\bar{a}|^2}{\partial z} = |\bar{g}_T|^2 \frac{df^2(\tau)}{d\tau} \frac{\partial \tau}{\partial z} = -\frac{1}{c} |\bar{g}_T|^2 \frac{df^2(\tau)}{d\tau} \quad (2.28)$$

Integrating Eq. 2.26 from $\tau = -\infty$ to the peak of the pulse, $\tau = 0$, $f^2(0) = 1$, gives

$$\langle p_z^0 \rangle = p_z + \frac{1}{c} \frac{e^2 |\bar{g}_T|^2}{4 mc^2} \quad (2.29)$$

The "canonical kinetic energy," to first order, is given by

$$\frac{\langle p_z^0 \rangle^2}{2m} = \frac{p_z^2}{2m} + \frac{v_z}{c} \frac{e^2 |\bar{g}_T|^2}{4 mc^2} \quad (2.30)$$

the translational energy is conserved, the pondermotive potential energy changes it only by v_z/c of its magnitude. This effect was coined "surfing" by P. Bucksbaum.

Very recently, similar conclusions were obtained by Jönsson,¹² Becker et al.¹³ and have been dramatically demonstrated experimentally by Bucksbaum et al.¹⁴

II.2 QUANTUM THEORY

From the arguments of Kibble⁷ we expect that the quantum description of the electron (in a C-number, external field) gives the same results as the classical one. (See also Refs. 9, 10.) In this section we derive the semiclassical wave function; the reasons to do so are to define the parameters, to show the power of the multiple scale method, and the need to have the wave functions for actual calculations.

The semiclassical (or quasi-classical, or WKB) wave function is given to first order by¹⁵

$$\Psi = \Phi \exp\left[\frac{i}{\hbar} S(\bar{x}, t)\right] \quad (2.31)$$

where $S(\bar{x}, t)$ is Hamilton's first principal function (the action integral calculated over the classical path), and the amplitude is

$$\Phi = \Phi_0 |\bar{p}|^{-1/2}, \quad (2.32)$$

where $\bar{p}$ is the canonical momentum. The action satisfies the Hamilton-Jacobi equations, (see e.g., Ref. 11, p. 255),

$$\begin{aligned} \frac{\partial S}{\partial t} &= -H(\bar{p} = \bar{\nabla} S, \bar{x}, t) \\ \bar{p} &= \frac{\partial S}{\partial \bar{x}} \equiv \bar{\nabla} S \end{aligned} \quad (2.33)$$

The advantage of this formulation is that if the classical equations can be solved, the quantum mechanical wave function can easily be written down.

First, we will solve Eq. 2.33 for a constant amplitude electromagnetic field ($\bar{a} = \text{const.}$ in Eq. 2.4). We note that the oscillating terms in Eq. 2.18z are of the order ϵ , thus to order ϵ^0 , we can write

$$\bar{p}^0 = \text{const}; \quad (2.34)$$

using the Hamiltonian, Eq. 2.2, we get immediately the plane wave solution,

$$\begin{aligned} S(\bar{x}, t) &= \bar{p}^0 \cdot \bar{x} - \left(\frac{|\bar{p}^0|^2}{2m} + \frac{e^2 |\bar{a}|^2}{4 mc^2} \right) t \\ &- \frac{e \bar{p}^0 \cdot \bar{a}}{mc \omega (1 - p_z^0/mc)} \sin \omega \tau + \frac{e^2 |\bar{a}|^2}{8 mc^2 \omega (1 - p_z^0/mc)} \sin 2 \omega \tau \end{aligned} \quad (2.35)$$

The wave function, $\Psi = \exp(i S/\hbar)$, is the "Volkov" wave function, see e.g., Rosenberg,¹⁶ Mittleman.¹⁷ It is of the Floquet form (see I, Eq. 2.10). The Floquet-exponent, or quasi-energy is

$$E_F = \frac{|\vec{p}^0|^2}{2m} + \frac{e^2 |\vec{a}|^2}{4 mc^2} \quad (2.36)$$

it is the same as the one appearing in the time averaged Hamiltonian, Eq.

2.22. The presence of the field thus alters the dispersion relation from $E = p^2/2m$, to Eq. 2.36. The time-dependent, periodic terms can be interpreted as the "dressing" of the electron by photons.¹⁶ Let us consider the $\sin \omega \tau$ term; its coefficient determines the free electron parameter, (with $p_z^0 = 0$)

$$n_f = \frac{e \vec{p}^0 \cdot \vec{a}}{\hbar mc \omega} = (8 n_k n_{osc})^{1/2} \cos \theta \quad (2.37)$$

where

$$n_k = E_k / \hbar \omega \quad , \quad (2.38)$$

$$n_{osc} = \frac{e^2 |\vec{a}|^2}{4 mc^2 \hbar \omega} \quad , \quad (2.39)$$

$$\cos \theta = \frac{\vec{p}^0 \cdot \vec{a}}{|\vec{p}^0| |\vec{a}|} \quad (2.40)$$

Using these parameters, we can write

$$S(\vec{x}, t) = \vec{p}^0 \cdot \vec{x} - E_F t - \hbar n_f \sin \omega \tau + \frac{1}{2} \hbar n_{osc} \sin 2 \omega \tau \quad (2.41)$$

Using the formula,

$$\exp(i n_f \sin \omega \tau) = \sum_{m=-\infty}^{\infty} J_m(n_f) \exp(im \omega \tau) \quad (2.42)$$

where J is the cylindrical Bessel function,¹⁸ we can see that n_f measures the number of photons that "dress" the electron. For $n_f \ll m$, $J_m(n_f) \approx n_f^m / 2m!$; thus, using Eqs. 3.23, 2.30 of I, we conclude that in the

"weak field" region, all m-photon photoionization behaves as I^m whether it is dominated by the "dressing" of the free- or the bound-electron.

In the more general case of a slowly varying field, it can be seen from Eq. 2.33 that Eq. 2.35 still gives an approximate solution, at least locally; the reason being that - to order ϵ^0 - $\bar{P}$ is constant in time and H is constant in space. Note, that in the general case of oblique incidence of the electron on a realistic laser pulse, it is difficult to calculate even the semiclassical wave function. Fortunately, as seen from the discussion following Eq. 3.15 of I, the wave function is needed only inside the electromagnetic field.¹⁶

In the two special cases discussed in Eqs. 2.24, 2.30 the semiclassical wavefunction can be written down in analogy with Eq. 2.35. In the "long pulse" case, Eq. 2.24 is valid. It can be solved for $\bar{P}^0$. Let us consider a high energy electron in a Coulomb potential $V = -e^2/r$, incident in the x direction, Then,

$$\frac{\langle P_x^0 \rangle^2}{2m} = \frac{p_0^2}{2m} - E_{osc}(x) + \frac{e^2}{x} \quad (2.44)$$

$$\langle P_x^0 \rangle \approx p_0 - \frac{m E_{osc}(x)}{p_0} + \frac{e^2 m}{p_0 x} \quad (2.45)$$

$$S = p_0 x - \frac{m}{p_0} \int^x E_{osc}(x') dx' + \frac{e^2 m}{p_0} (\ln x - \ln x_0) - \frac{p_0^2}{2m} t - \hbar \omega_f \sin \omega \tau + \frac{1}{2} \hbar \omega_{osc} \sin 2 \omega \tau \quad (2.46)$$

correct to order $1/r^2$. This exhibits the well known logarithmic dependence of the wavefunction in a Coulomb potential. The second case, $\bar{p}_0 = p_z \hat{z}$, can be written even more easily, using Eq. 2.30,

$$S = p_z z - \frac{p_z^2}{2m} t - \left(1 + \frac{p_z}{mc}\right) \frac{e^2}{4 mc^2} \int^\tau g_1^2(t') dt' \\ - \hbar n_f \sin \omega \tau + \frac{1}{2} \hbar n_{osc} \sin 2 \omega \tau. \quad (2.47)$$

III. ONE DIMENSIONAL HARMONIC OSCILLATOR

It is known that in the presence of a small perturbation every bound system behaves as a harmonic oscillator. As the one-dimensional harmonic oscillator in an external field can be solved by quadratures, we will use it as a paradigm. The Hamiltonian can be written as

$$H = \frac{1}{2m} (P - \frac{e}{c} A)^2 - V_0 + \frac{1}{2} m \omega_0^2 x^2, \quad P = -i \hbar \frac{\partial}{\partial x} \quad (3.1)$$

This model has an exact analytic solution for the adiabatic states, which is calculated in the Appendix, following the derivation by Kerner. The general solution of Eq. 3.1, for a vector potential, $A = a \cos \omega \tau$, $a = \text{const}$, is

$$\Psi_E(x, t) = N_{EF} \exp[-\frac{i}{\hbar} E_F(n) t] \exp[-\frac{i}{\hbar} \frac{e a_0 \omega_0^2}{c(\omega_0^2 - \omega^2)} x \cos \omega \tau] \\ \exp[-\frac{i}{\hbar} \frac{e^2 a^2 \omega(\omega_0^2 + \omega^2)}{8 mc^2 (\omega_0^2 - \omega^2)^2} \sin 2 \omega \tau] (C U(b, y) + D V(b, y)), \quad (3.2)$$

where

$$y = \sqrt{\frac{2m\omega_0}{\hbar}} (x - u(t)) \quad , \quad u(t) = -\frac{e a_0}{mc} \frac{\omega}{\omega_0^2 - \omega^2} \sin \omega \tau \quad , \\ = -u_0 \sin \omega \tau \quad (3.3)$$

$$b = -\frac{E + V_0}{\hbar\omega_0}, \quad E_F = E - \frac{e^2 a_0^2}{4mc^2} \frac{\omega^2}{\omega_0^2 - \omega^2}, \quad (3.4)$$

U, V are parabolic cylinder functions, and C, D are arbitrary constants. The stationary states have to vanish at $|x| \rightarrow \infty$, giving $C = 1, D = 0$, and $a_n = -(n + 1/2)$, $n = 0, 1, \dots$ in complete analogy to the unperturbed harmonic oscillator. The states n are the adiabatic Floquet states.¹ They are adiabatically connected to the corresponding unperturbed states; their quasi-energy is shifted by the AC Stark shift $-e^2 a_0^2 \omega^2 / 4mc^2 (\omega_0^2 - \omega^2)$. Comparing its value to the free electron's, Eq. 2.16, we see that the bound electron's Stark shift is reduced by a factor $\omega^2 / (\omega_0^2 - \omega^2)$; and it is of opposite sign for $\omega \ll \omega_0$.

In a manner similar to the definition of the free electron parameter (Eq. 2.37) we define the bound electron parameter,

$$n_b^2 = (u_0 \alpha)^2 = 4 n_{osc} \frac{\omega_0 \omega^3}{(\omega_0^2 - \omega^2)^2} \quad (3.5)$$

where α is given by Eq. A.23. Using obvious relations, we get,

$$\frac{n_b}{n_f} = \sqrt{\frac{\hbar\omega_0}{2E_k}} \frac{\omega^2}{|\omega_0^2 - \omega^2|} \quad (3.6)$$

It is clear that in the case of near resonance, $n_b > n_f$, thus the dressing of the bound electron is more important than that of the free electron. In the opposite case, unfortunately, we cannot draw firm conclusions because a harmonic oscillator, unlike an atom, does not have non-linear resonances. These latter can be important in atomic multiphoton ionization, as discussed at length in I.

The wave function, Eq. 3.2, can be written in the Floquet form, Eq. 2.10 of Ref. 1; the square integral of the individual Fourier components can be interpreted as the probabilities of "dressing" the oscillator by the appropriate number of photons. It is clear that this is not given by the naive formula $e^{-n_b} n_b^k / k!$ (see e.g., Messiah, Ref. , p. 461) the latter giving the probability of finding the oscillator in the k'th unperturbed state if the electromagnetic field is suddenly turned off.

APPENDIX

The one dimensional harmonic oscillator.

The Hamiltonian is (Eq. 3-1)

$$H = \frac{1}{2m} \left(P - \frac{e}{c} A \right)^2 - V_0 + \frac{1}{2} m \omega_0^2 x^2, \quad P = -i\hbar \frac{\partial}{\partial x} - i\hbar \frac{\partial}{\partial x} \quad (A1)$$

Schrödinger's equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi, \quad (A2)$$

can be solved by quadratures for an arbitrary external field $A(t)$. Our derivation follows closely that of Kerner, but we use a different gauge. We substitute,

$$\Psi(x,t) = \phi(\xi,t) \exp(i g(t)), \quad \xi = x - u(t). \quad (A3)$$

We get the equation

$$\begin{aligned} & -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial \xi^2} + \left[i\hbar \left(u - \frac{e}{mc} A \right) - \frac{\hbar^2}{m} g \right] \frac{\partial \phi}{\partial \xi} \\ & + \left(-V_0 + \frac{1}{2} m \omega_0^2 x^2 - i\hbar x g - \frac{\hbar^2}{2m} g^2 - \frac{i\hbar e}{mc} A g + \frac{e^2 A^2}{2mc^2} \right) \phi = i\hbar \frac{\partial \phi}{\partial t} \end{aligned} \quad (A4)$$

Setting the square brackets to zero,

$$i\hbar \left(u - \frac{e}{mc} A \right) - \frac{\hbar^2}{m} g = 0, \quad (A5)$$

and also

$$ku = i\hbar g = 0, \quad (A6)$$

we get

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial \xi^2} + \left(\frac{1}{2} k \xi^2 - V_0 - \frac{1}{2} k u^2 + \frac{1}{2} m \dot{u}^2 \right) \phi = i\hbar \frac{\partial \phi}{\partial t}. \quad (A7)$$

From Eqs. A5, 6, we get,

$$m\ddot{u} + m\omega_0^2 u = \frac{e}{c} \dot{A} \quad , \quad (A8)$$

which is recognized as the classical motion of the oscillator. Using a sinusoidal driving force, the adiabatic (or steady state) solution, in radiation gauge is

$$A = a_0 \cos \omega\tau \quad , \quad E = E_0 \sin \omega\tau = \frac{\omega}{c} a \sin \omega\tau \quad (A9)$$

$$u = - \frac{e E_0 \sin \omega\tau}{m(\omega_0^2 - \omega^2)} \quad (A10)$$

$$g(t) = \frac{i}{\hbar} \frac{e}{c} a \frac{\omega_0^2}{\omega_0^2 - \omega^2} \cos \omega\tau \quad (A11)$$

$$\delta(t) \equiv - \frac{1}{2} m \omega_0^2 u^2 + \frac{1}{2} m \dot{u}^2 = \frac{e^2 E_0^2}{\omega m (\omega_0^2 - \omega^2)^2} (\omega^2 \cos^2 \omega\tau - \omega_0^2 \sin^2 \omega\tau) \quad (A12)$$

substituting

$$\phi(\xi, t) = \chi(\xi) e^{iT(t)} \quad (A13)$$

we get

$$iT(t) = - \frac{i}{\hbar} \int^t (\delta(t') + E) dt' \quad (A14)$$

where E is the yet undefined separation constant and $\chi(\xi)$ satisfies the equation

$$- \frac{\hbar^2}{2m} \frac{d^2 \chi}{d\xi^2} + (-V_0 + \frac{1}{2} m \omega_0^2 \xi^2 - E) \chi = 0 \quad (A15)$$

This is the harmonic oscillator equation, it can be brought to canonical form, by the substitutions,

$$y = \sqrt{\frac{1m \omega_0}{\hbar}} \xi, \quad b = -\frac{E_F + V_0}{\hbar \omega_0}, \quad (A16)$$

$$E_F = E - \frac{e^2 \epsilon_0^2}{4m(\omega_0^2 - \omega^2)} \quad (A17)$$

$$\frac{d^2 x(y)}{dy^2} - \left(\frac{y^2}{4} + b \right) x = 0 \quad (A18)$$

The general solution of Eq. A18 is a linear combination of parabolic cylinder functions,¹⁸

$$x_E(\xi) = N_E [CU(b, y) + DV(b, y)] \quad (A19)$$

where N_E is a normalization constant. In order to make x vanish for $|x| \rightarrow \infty$, we have to choose

$$b = -\left(\frac{1}{2} + n\right), \quad n = 0, 1, 2, \dots \quad (A20)$$

giving the well known harmonic oscillator wave functions.

Collecting all terms, we get for the stationary states,

$$\begin{aligned} \Psi_n(x, t) = N_n \exp\left[-\frac{i}{\hbar} E_F(n)t\right] \exp\left[-\frac{i}{\hbar} \frac{e}{c} \frac{a \omega_0^2}{(\omega_0^2 - \omega^2)} x \cos \omega t\right] \\ \exp\left[-\frac{i}{\hbar} \frac{e^2 a^2 \omega(\omega_0^2 + \omega^2)}{8m c^2 (\omega_0^2 - \omega^2)^2} \sin 2 \omega t\right] \exp\left(-\frac{1}{2} a^2 \xi^2\right) H_n(a\xi) \end{aligned} \quad (A21)$$

where, H_n are Hermite polynomials

$$E_F(n) = -V_0 + \left(n + \frac{1}{2}\right) \hbar \omega_0 - \frac{e^2 \epsilon_0^2}{4m(\omega_0^2 - \omega^2)} \quad (A22)$$

$$\alpha = \sqrt{\frac{m \omega_0}{\hbar}} \quad (\text{A23})$$

$$N_n^2 = \frac{\alpha}{\sqrt{\pi} 2^n n!} \quad (\text{A24})$$

The state Ψ_n is adiabatically connected to the n 'th eigenstate of the unperturbed harmonic oscillator. These are the adiabatic Floquet solutions.

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