Neoclassical Transport in Helically Symmetric Plasmas

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ABSTRACT

The neoclassical rate of diffusion is calculated in such a way that the result is applicable to both helically symmetric and toroidally symmetric plasmas.

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I. Introduction

Neoclassical diffusion in a toroidal stellarator has been discussed by several authors. For a recent review, and for references to this work, the reader is referred to the paper by Miyamoto.¹ Surprisingly, the corresponding calculations for a straight stellarator, or for a toroidal stellarator in which the helical effects dominate the toroidal, appear not to have been reported in the literature. It is the purpose of this paper to present the results of such a calculation. The theory will be developed in such a manner that, at every step along the way, it applies equally well to a helically symmetric or a toroidally symmetric plasma.

Two cases are considered. First, the collision operator is left completely arbitrary, but the magnitude of the magnetic field is assumed to vary only slightly over a magnetic surface. In this case, the neoclassical transport calculations for the two different symmetries become identical, and the diffusion coefficients calculated for the small aspect ratio, symmetric tokamak can be carried over to the helically symmetric plasma unchanged, except for the substitution of the appropriate helical parameters for the corresponding toroidal ones.

Secondly, a detailed calculation is carried through with the Lorentz collision operator, but with the variation of the magnetic field and the shape of the magnetic surface left arbitrary, except for the requirements of symmetry, helical or toroidal. The general result, for both symmetries, is presented in Eq. (77), and a special case in Eq. (83).

II. Helically Symmetric Magnetic Field

A. General Results

Starting with cylindrical coordinates, r, ϕ, z and the corresponding unit vectors, $\underline{e}_r, \underline{e}_\phi, \underline{e}_z$, we define the helical variable θ , and the vector $\underline{\zeta}$ in the direction of symmetry,

$$\theta = \ell(\phi - \alpha z) \quad , \quad (1)$$

$$\underline{\zeta} = (\underline{e}_z + \alpha \underline{e}_\phi) / (1 + \alpha^2 r^2) \quad , \quad (2)$$

where ℓ is an integer and α is an arbitrary constant. Helical symmetry is imposed by requiring that all physical quantities be functions of only two spatial variables, r and θ . The gradient in the $\underline{\zeta}$ direction of any such quantity will vanish, since $\underline{e}_r, \nabla\theta$, and $\underline{\zeta}$ are mutually perpendicular.

With the vector potential $\underline{A}$ a function of r and θ , it is easy to show that the magnetic field takes the following form,

$$\underline{B} = \nabla \times \underline{A} = g \underline{\zeta} + \underline{\zeta} \times \nabla \psi \quad . \quad (3)$$

Here, ψ is the usual flux coordinate which labels a magnetic surface,

$$-\psi = A_z + \alpha r A_\phi \quad , \quad (4)$$

and

$$g = \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) - \alpha r \frac{\partial A_z}{\partial r} - \frac{l}{r} (1 + \alpha^2 r^2) \frac{\partial A_r}{\partial \theta} . \quad (5)$$

Taking the curl of Eq. (3), we obtain an equation for $\nabla \times \underline{B}$ which, in terms of $\underline{\zeta}$, is structurally similar to that for $\underline{B}$,

$$\nabla \times \underline{B} = \Gamma \underline{\zeta} - \underline{\zeta} \times \nabla g , \quad (6)$$

$$\Gamma = \nabla^2 \psi + 2\alpha(g - \alpha r \frac{\partial \psi}{\partial r}) / (1 + \alpha^2 r^2) . \quad (7)$$

Assuming that we are in MHD equilibrium, $\nabla P = \frac{1}{c} \underline{J} \times \underline{B}$, we can readily deduce that g depends on space only through the function ψ . First, the equation $\underline{B} \cdot \nabla P = 0$, or $\underline{\zeta} \times \nabla \psi \cdot \nabla P = 0$, requires that P depend on r and θ only through ψ , $P = P(\psi)$. Second, the equation $\underline{J} \cdot \nabla P = 0$, or $\underline{\zeta} \times \nabla g \cdot \nabla P = 0$, requires that P depend on space only through g , $P = P(g)$, unless g is a constant. Therefore, g , if it has any spatial dependence at all, depends on r and θ only through ψ , $g = g(\psi)$. In other words, g is constant over a magnetic surface.

B. Irrotational Magnetic Field

In a low beta stellarator, the plasma currents are small compared to the external currents, and the magnetic field is approximately curl free,

$$\nabla \times \underline{B} = 0 . \quad (8)$$

It follows from Eq. (6) that, in this case,

$$g = B_0 = \text{constant} , \quad (9)$$

and

$$\Gamma = 0 . \quad (10)$$

The appropriate solution to Eq. (10) is

$$\psi = -\frac{1}{2}B_0 a r^2 + \sum_{\ell=1}^{\infty} b_{\ell} r I'_{\ell}(\ell a r) \cos \theta , \quad (11)$$

where b_{ℓ} is an arbitrary constant and I'_{ℓ} denotes the derivative of the modified Bessel function I_{ℓ} . If we substitute this expression for ψ in Eq. (3), we recover the well-known helically symmetric vacuum field.^{1,2}

To obtain a simple expression for the field, it is customary to assume that the helix is loosely wound, or $\ell a r \ll 1$, and that only one ℓ component is present. For this case one finds,

$$\psi \approx -\frac{1}{2}B_0 a r^2 + \ell^{-1} r B_H \cos \theta , \quad (12)$$

$$\vec{B} = B_0 \vec{e}_z + \vec{B}_H , \quad (13)$$

$$\vec{B}_H \approx B_H (\vec{e}_r \sin \theta + \vec{e}_{\phi} \cos \theta - \vec{e}_z a r \cos \theta) , \quad (14)$$

where the magnitude of the helical field, B_H , is proportional to $r^{\ell-1}$ and otherwise arbitrary. From Eq. (12) it is clear that the surfaces of constant ψ , or constant pressure, will be nearly cylindrical if $B_H/B_0 \ll \ell a r/2$. To aid us in obtaining a physical feeling for the helically symmetric geometry, we will

assume, in the remainder of this section, that

$$B_H/B_O \ll \ell \alpha r/2 \ll 1 ,$$

which is the usual weak, loosely wound helix limit.

In evaluating the radial particle transport, one finds two functions to be of primary importance. These are the magnitude of the magnetic field B , and the quantity $B \cdot \nabla \theta$,

$$\begin{aligned} B \cdot \nabla \theta &= \nabla \theta \cdot \zeta \times \nabla \psi \\ &= \frac{\ell}{r} \frac{\partial \psi}{\partial r} . \end{aligned} \quad (15)$$

If we define the small parameter ε by

$$\varepsilon = \alpha r B_H/B_O , \quad (16)$$

we obtain, in the weak, loosely wound helix case,

$$B \cdot \nabla \theta \approx -\ell \alpha B_O \left(1 - \frac{\varepsilon}{2} \cos \theta \right) , \quad (17)$$

and $B \approx B_z \approx B_O (1 - \varepsilon \cos \theta)$, or

$$B \approx B_O \left(1 + 2\varepsilon \sin^2 \left(\frac{\theta}{2} \right) \right) . \quad (18)$$

In toroidal symmetry, the ratio of the poloidal field to the total field, $\theta = r B \cdot \nabla \theta_T / B$, where θ_T is the poloidal angle, often appears in transport formulas. The quantity which plays a similar role in the helically symmetric case is $\theta_H = r B \cdot \nabla \theta / \ell B \approx -\alpha r$. The

factor of $1/\ell$ in the expression for Θ_H can be understood in terms of the ℓ dependence of the radius of curvature and of the connection length of the magnetic field, as we will now show.

The curvature $\tilde{K}$ of a magnetic field $\tilde{B}$ is given by $\tilde{K} = (\tilde{B}/B) \cdot \nabla(B/B)$, which, for a curl-free field, can be written $\tilde{K} = (\nabla_{\perp} B)/B$. The radius of curvature, by definition, is equal to the inverse of the magnitude of $\tilde{K}$. With B given by Eq. (18), and with $\epsilon \propto r^{\ell}$, we find the radius of curvature for the helically symmetric field to be, $R_H \approx r/\ell\epsilon$. Then, writing down the well-known expression for the drift velocity in a curl-free field, we have

$$\tilde{v} \approx - \frac{v_{\perp}^2 + 2v_{\parallel}^2}{2\Omega R_H} \frac{\tilde{B}_H}{B_H}, \quad R_H \approx \frac{r}{\ell\epsilon}. \quad (19)$$

The connection length, L_H , is defined by

$$L_H = \frac{1}{2\pi} \int_0^{2\pi} \frac{B d\theta}{|\tilde{B} \cdot \nabla \theta|} \approx \frac{1}{\ell\alpha},$$

where we have taken note of Eq. (17). Clearly, $2\pi L_H$ is the distance along the field line from a given point to the nearest identical (by symmetry) point. In analogy with the toroidal case, we now write

$$|\Theta_H| \equiv \epsilon R_H / L_H \approx \alpha r, \quad (20)$$

in agreement with the earlier statement concerning Θ_H .

The expressions for the connection length and the radius of curvature allow us to estimate the rate of particle transport in a helix. In the long mean-free-path case, the transport is primarily determined by the $\sqrt{\epsilon}$ fraction of particles which are trapped in

the magnetic field [see Eq. (18)]. The typical drift time for such particles is, $\tau \approx L_H/v_{||} \approx L_H/\sqrt{\epsilon} v$, where v is the thermal velocity. Since the drift velocity is, $V \approx v\rho/R_H$ [see Eq. (19)], where ρ is the gyroradius, the orbit size is given by

$$\Delta \approx V\tau \approx \rho L_H/R_H \sqrt{\epsilon} \approx \sqrt{\epsilon} \rho / |\theta_H| .$$

This result for Δ is identical to the width of a banana orbit in a large aspect ratio, axisymmetric torus, if we replace θ_H by θ . Therefore, we expect the expression for neoclassical transport in a weak, loosely wound helix to be essentially identical to that in a large aspect ratio torus. A formal proof that this is so is the major point of this paper.

In the collisional limit (mean free path short compared to L), one also obtains identical formulas for the Pfirsch-Schlüter transport in a large aspect ratio torus and a weak, loosely wound helix. However, the helical quantity, q_H , which is analogous to the toroidal "safety factor," q , is much less than one, so that the Pfirsch-Schlüter correction in a weak, loosely wound helix is of no great interest. To see this, note that the typical effective drift time for a particle in the collisional regime is $\tau \approx (L/\lambda)^2 \tau_C$, with τ_C the collision time and $\lambda = v\tau_C$ the mean free path. Again taking $\Delta \approx V\tau$ and $V \approx v\rho/R_H$, and noting that the diffusion coefficient is $D \approx \Delta^2/\tau \approx \Delta V$, we obtain

$$D \approx \rho^2 L_H^2 / R_H^2 \tau_C \approx \rho^2 q_H^2 / \tau_C ,$$

where we have defined q_H by $q_H \equiv L_H/R_H \approx \epsilon/|\theta_H|$. This expression

is identical to the well-known toroidal result, with q in a torus equal to ϵ/θ . However, q_H is small compared to unity, $q_H \approx \epsilon/|\theta_H| \approx B_H/B_0 \ll 1$.

In a torus, q is equal to the inverse of the rotational transform. However, in a helix, q_H , as defined here, is not so simply related to the rotational transform. By the rotational transform per period, i_p , we mean the angle through which a field line rotates in the ϕ direction during one helical period, i.e., while θ increases by 2π . In order to calculate this quantity, we need to know the radius of a magnetic surface as a function of θ . Setting ψ equal to a constant, $\psi = -B_0 \alpha r_0^2/2$, we find, to lowest order in ϵ ,

$$r = r_0(1 + \delta \cos \theta) \quad , \quad \delta = B_H/B_0 \ell \alpha r_0 = \epsilon/\ell \alpha^2 r_0^2 \quad . \quad (21)$$

With the differentials to be taken along the field line, we obtain

$$\frac{d\phi}{d\theta} = \frac{\tilde{B} \cdot \tilde{\nabla} \phi}{\tilde{B} \cdot \tilde{\nabla} \theta} \approx \frac{B_H \cos \theta}{-\ell \alpha B_0 (1 - \epsilon \cos \theta / \alpha^2 r^2)} \approx -\delta \cos \theta [1 + 2(\ell-1)\delta \cos \theta] \quad ,$$

where we have used Eq. (21) to expand B_H/r , which is proportional to $r^{\ell-2}$, in $r-r_0$. Averaging $-d\phi/d\theta$ over θ , we find immediately,

$$\frac{i_p}{2\pi} \approx (\ell-1)\delta^2 \approx (\ell-1)\epsilon^2/\ell^2(\alpha r)^4 \quad , \quad (22)$$

which, to this order, agrees with the result first calculated by Spitzer.³

Another difference between the helix and the torus, which is worthy of note, is evident in the expression for the ballooning mode

beta limit, β_L . In the simplest models, one finds $\beta_L = rR/L^2$, with r the pressure scale length, R the radius of curvature, and L the connection length. In a torus, $\beta_L \approx \epsilon/q^2$, which is quite small, while for weak, loosely wound helix, $\beta_L \approx 1/\delta \approx \alpha^2 r^2/\epsilon$, which is large compared to one. Unfortunately, in the weak, loosely wound helix we have maximum average and not minimum average B . (It is possible, however, to have a minimum average B in a straight, symmetric helix.)

III. Comparison of Helical and Toroidal Symmetry

First, we consider a general axisymmetric plasma. It is, again, convenient to use cylindrical coordinates, which we now denote by R, ϕ, Z . Physical quantities are assumed to be independent of the angle ϕ . With the vector potential A a function of R and Z only, we can write B in the form given by Eq. (3), once again, provided that

$$\zeta = e_\phi/R \quad , \quad (23)$$

$$\psi = -RA_\phi \quad , \quad (24)$$

$$g = RB_\phi \quad . \quad (25)$$

Also, the curl of B is given by Eq. (6), with the additional proviso that

$$\Gamma = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} \quad . \quad (26)$$

Next, we restrict ourselves to an axisymmetric torus with nested magnetic surfaces. It is natural, then, to switch from R and Z to circular coordinates in the plane of constant ϕ , namely r , the radial distance from the magnetic axis, and θ , the poloidal angle, chosen to be zero on the outside of the torus, and such that $\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi$. With $R B_\phi$ constant on a magnetic surface, and assuming the inverse aspect ratio to be small,

$$\epsilon = r/R_0 \ll 1, \quad (27)$$

we obtain for B the expression given by Eq. (18). Formally, then, B is the same function of θ as in the helical case, Eq. (18), but θ is now the poloidal angle, and ϵ the inverse aspect ratio.

IV. The Drift Kinetic Equation

We choose as velocity coordinates E , the total energy, μ , the magnetic moment, and $\sigma = v_{||}/|v_{||}|$, the sign of the velocity component in the B direction,

$$E = \frac{1}{2} m v^2 + e\phi, \quad (28)$$

$$\mu = m v_\perp^2 / 2B. \quad (29)$$

The phase of the perpendicular velocity has been averaged away. The parallel velocity

$$v_{||} = \pm \left[\frac{2}{m} (E - \mu B - e\phi) \right]^{1/2}, \quad (30)$$

is a function of the spatial coordinates r and θ (through B and ϕ) as well as the velocity coordinates.

A particularly convenient expression for the guiding center drift velocity is given by

$$\tilde{\mathbf{v}} = (v_{||}/B) [\tilde{\mathbf{B}} + \tilde{\mathbf{v}} \times (\rho_{||}\tilde{\mathbf{B}})] \quad , \quad (31)$$

where

$$\rho_{||} = v_{||}/\Omega \quad (32)$$

is the parallel velocity divided by the gyrofrequency. This form of the drift velocity was developed by Morozov and Solov'ev⁴ for the special case of no parallel current, $\tilde{\mathbf{B}} \cdot \tilde{\mathbf{v}} \times \tilde{\mathbf{B}} = 0$, and for the general case by Boozer.⁵ The component of $\tilde{\mathbf{v}}$ perpendicular to $\tilde{\mathbf{B}}$, obtained from Eq. (31) is completely equivalent to the conventional expression for $V_{\perp}$. Our $V_{||}$, however, differs from the conventional $V_{||}$ through the presence of the second term in Eq. (31), which represents a correction of first order in the gyroradius to the dominant first term. This difference is immaterial for the calculation presented here. In any case, as shown by Boozer,⁵ the expression given by Eq. (31) is the more correct one, since it leads to a particle-conserving drift kinetic equation.

If we insert into Eq. (31) the expressions for $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}$ given by Eqs. (3) and (6), valid for both helically symmetric and axially symmetric plasmas, we obtain

$$\tilde{\mathbf{v}} = (v_{||}/B) [\zeta(\mathbf{g} + \rho_{||}\mathbf{r} + \tilde{\mathbf{v}}\rho_{||} \cdot \nabla\psi) + \zeta \times \nabla(\psi - \rho_{||}g)] \quad . \quad (33)$$

With a symmetric distribution function, $f(r, \theta, \psi)$, and recalling that $\tilde{\zeta}$ is perpendicular to $\tilde{e}_r$ and $\nabla\theta$, we arrive at the following time-independent drift kinetic equation,

$$\tilde{v} \cdot \nabla f = (v_{||}/B) \tilde{\zeta} \times \nabla \psi \cdot \nabla f = C(f) \quad , \quad (34)$$

where $C(f)$ is a collision term, and where

$$\tilde{\psi} = \psi - \rho_{||} g \quad , \quad (35)$$

is a constant of the motion in the absence of collisions.

Under the assumption of weak collisions, i.e., a collision frequency low compared to the bounce frequency of trapped particles, we may employ an iterative procedure:

$$f = f_v + f_c \quad , \quad (36)$$

$$\tilde{v} \cdot \nabla f_v = 0 \quad , \quad (37)$$

$$\tilde{v} \cdot \nabla f_c = C(f_v) \quad . \quad (38)$$

Clearly, the collisionless equation, Eq. (37), or

$$\tilde{\zeta} \times \nabla \tilde{\psi} \cdot \nabla f_v = 0 \quad , \quad (39)$$

is satisfied by an arbitrary function of $\tilde{\psi}$. We will assume that f_v is close to a local Maxwellian, $f_M(E, \tilde{\psi})$, of energy E , whose density n and temperature T are functions of $\tilde{\psi}$,

$$f_v = f_M(E, \tilde{\psi}) + h(E, \mu, \sigma, \tilde{\psi}) \quad . \quad (40)$$

Here, h , which represents the deviation from a Maxwellian, is assumed to be of first order in the gyroradius. Since $\tilde{\psi}$ differs from ψ by a term of order Ω^{-1} , it is convenient to expand Eq. (40) to first order,

$$f_v = f_M(E, \psi) + f_1 \quad , \quad (41)$$

$$f_1 = -\frac{\partial f_M}{\partial \psi} \rho_{||} g + h(E, \mu, \sigma, \psi) \quad . \quad (42)$$

We turn now to f_c . To the lowest significant order in the gyroradius, Eq. (38) takes the form

$$(v_{||}/B) \tilde{z} \times \nabla \psi \cdot \nabla f_c = C(f_v) \quad . \quad (43)$$

If we regard f_c as a function of ψ and θ , rather than r and θ , only the θ -derivative of f_c enters,

$$\frac{\partial f_c}{\partial \theta} = C(f_v) B / v_{||} \tilde{z} \cdot \nabla \theta \quad . \quad (44)$$

By integrating this equation in θ , keeping ψ constant, and demanding that f_c be single-valued, and periodic in θ , we arrive at

$$0 = \oint d\theta C(f_v) / \rho_{||} \tilde{z} \cdot \nabla \theta \quad , \quad (45)$$

where the integral extends from $-\pi$ to π for untrapped particles,

and over a full cycle of the motion for trapped particles.

Equation (45) is valid for an arbitrary collision term. It simplifies considerably if we consider a linearized collision operator for which $C(f_M) = 0$. For untrapped particles, Eq. (45) then reduces to

$$\langle C(h)/\rho_{||} \rangle = g \frac{\partial f_M}{\partial \psi} \langle C(v_{||})/v_{||} \rangle, \quad (46)$$

where the bracket is defined as the following average over ∂ ,

$$\langle f \rangle = \frac{\int_{-\pi}^{\pi} d\theta f / B \cdot \nabla \theta}{\int_{-\pi}^{\pi} d\theta / B \cdot \nabla \theta}. \quad (47)$$

If we assume, further, that the collision operator is even in $v_{||}$, then, for trapped particles, the contribution to Eq. (45) from the first term in f_1 , as given by Eq. (42), vanishes. We are left with the $C(h)$ term, and Eq. (45) is satisfied by

$$h = 0, \quad \mu \geq \mu_0, \quad (48)$$

where μ_0 is the value of μ at the boundary between trapped and untrapped particles.

As an example of a collision term which obeys the restrictions of the previous paragraph, we consider the Lorentz operator

$$C(f) = \frac{v_M}{B} v_{||} \frac{\partial}{\partial \mu} (v_{||} \mu \frac{\partial f}{\partial \mu}). \quad (49)$$

With this collision term, we have $C(v_{||})/v_{||} = -v$, and Eq. (46) becomes

$$\frac{\partial}{\partial \mu} \langle v_{||} \rangle \mu \frac{\partial h}{\partial \mu} = -\frac{c}{e} g \frac{\partial f_M}{\partial \psi} \quad (50)$$

It is easy to verify that Eq. (50) is satisfied by

$$h = \frac{c}{e} g \frac{\partial f_M}{\partial \psi} \int_{\mu}^{\mu_0} \frac{d\mu'}{\langle v_{||}(\mu') \rangle}, \quad 0 \leq \mu \leq \mu_0, \quad (51)$$

where the upper limit of integration has been chosen such that the two expressions for h , given by Eqs. (48) and (51), agree for $\mu = \mu_0$. With the Lorentz collision model, then, the integral condition on f_v , given by Eq. (45), is sufficient to determine f_v completely. It is given explicitly through Eqs. (41), (42), (48) and (51).

It should be noted that every equation in this section applies equally well to helically symmetric and axisymmetric plasmas. Also, the magnetic field, except for the symmetry, is left completely general. In particular, we have made no use of the simplifying assumptions on ψ and B , which were introduced in Sec. IIB.

V. Transport Across Magnetic Surfaces

Assuming that we have a helically symmetric or toroidally symmetric plasma with nested magnetic surfaces, we wish to calculate the particle flux across magnetic surfaces, i.e. in the $\nabla\psi$

direction,

$$F = \int d^3v v_{\psi} f = \sum_{\sigma} \int_0^{\infty} dE \int_0^{\frac{E/B}{m}} d\mu \frac{2\pi B}{m^2 |v_{||}|} v_{\psi} f, \quad (52)$$

where the sum is over the sign of $v_{||}$, and where v_{ψ} is the component of the drift velocity in the $\nabla\psi$ direction. For simplicity, we will assume that there is no electric field, $\phi=0$. Taking $\rho_{||}$ to be a function of ψ and θ , we obtain from Eq. (33),

$$v_{\psi} = \frac{v_{||}}{B} g \frac{\partial \rho_{||}}{\partial \theta} \nabla\theta \times \zeta \cdot \frac{\nabla\psi}{|\nabla\psi|}, \quad (53)$$

which is to be substituted in Eq. (52).

Our main interest is the particle flux averaged over the surface of constant ψ . Since there is no need to average in the direction of symmetry, the surface average reduces to an average over θ with the appropriate weighting factor,

$$\bar{F} = \int_{-\pi}^{\pi} \frac{d\theta |\nabla\psi| F}{B \cdot \nabla\theta} \bigg/ \int_{-\pi}^{\pi} \frac{d\theta |\nabla\psi|}{B \cdot \nabla\theta}. \quad (54)$$

Pulling together Eqs. (52) - (54), we obtain

$$\bar{F} = \frac{2\pi g}{m^2} \int_{-\pi}^{\pi} d\theta \sum_{\sigma} \int_0^{\infty} dE \int_0^{\frac{E/B}{m}} d\mu \frac{\partial \rho_{||}}{\partial \theta} f \bigg/ \int_{-\pi}^{\pi} \frac{d\theta |\nabla\psi|}{B \cdot \nabla\theta}. \quad (55)$$

It is convenient to consider the integral over θ first. In interchanging the integrals in θ and μ , however, we must take

account of the θ dependence of E/B , the limit of integration in μ . If we examine the area of integration in μ, θ space, we find that

$$\int_{-\pi}^{\pi} d\theta \int_0^{E/B} d\mu = \int_0^{E/B_{\min}} d\mu \int_{-\gamma}^{\gamma} d\theta, \quad (56)$$

where $B_{\min}$ is the minimum value of B on the magnetic surface in question, and where $\pm \gamma$ is $\pm \pi$ for untrapped particles, but equal to $\pm \theta_T$, the values of θ at the turning points, for trapped particles. Integrating by parts in θ , we obtain

$$\int_{-\gamma}^{\gamma} d\theta \frac{\partial \rho_{||}}{\partial \theta} f = - \int_{-\gamma}^{\gamma} d\theta \rho_{||} \frac{\partial f}{\partial \theta}, \quad (57)$$

where we have noted that $\rho_{||}$ vanishes at $\pm \theta_T$ for trapped particles, and that $\rho_{||} f$ takes on the same value at the two limits, $\pm \pi$, for untrapped particles. The f_v part of f does not contribute to this integral, since

$$\frac{\partial f_v}{\partial \theta} = - \frac{\partial f_M}{\partial \psi} g \frac{\partial \rho_{||}}{\partial \theta}, \quad (58)$$

and

$$\int_{-\gamma}^{\gamma} d\theta \rho_{||} \frac{\partial \rho_{||}}{\partial \theta} = 0.$$

We are left with the contribution from f_c ,

$$\int_{-\gamma}^{\gamma} d\theta \frac{\partial \rho_{||}}{\partial \theta} f = - \int_{-\gamma}^{\gamma} d\theta \rho_{||} \frac{\partial f_c}{\partial \theta} . \quad (59)$$

If we now change the order of integration in θ and μ back again, and substitute from Eq. (44) in Eq. (59), Eq. (55) becomes

$$\bar{F} = - \frac{2\pi c g}{em \langle |\nabla \psi| \rangle} \sum_{\sigma} \sigma \int_0^{\infty} dE \left\langle \int_0^{E/B} d\mu C(f_v) \right\rangle , \quad (60)$$

where the bracket indicates the average over θ defined by Eq. (47). This result is quite general. It holds for every collision operator, and for both helically symmetric and axially symmetric plasmas. No particular assumption has been made about the θ dependence of the magnetic field or the shape of the surface of constant ψ .

We will carry Eq. (60) further in two respects. First, we will leave the collision term completely arbitrary, but assume that the magnetic field is given by Eq. (18). Secondly, we will choose the Lorentz collision model, but leave the θ dependence of the magnetic field unspecified.

A. General Collision Term

Without specifying the collision term, the particle flux $\bar{F}$, given by Eq. (60), cannot be calculated in detail. However, we can relate the result for the helically symmetric plasma to that for a toroidally symmetric one, if we assume that Eq. (18) holds

for both symmetries. In that case, not only B , but also $v_{||}$, $\rho_{||}$, and f_v , which depend on θ only through B , will have exactly the same θ dependence for the two different symmetries.

In evaluating the integral in Eq. (45), we find that the θ dependence of $v_{||}$ is crucial. Indeed, for trapped particles, $v_{||}$ vanishes for some values of θ , $\theta = \pm \theta_T$. In contrast, for $\epsilon \ll 1$, the θ dependence of $\underline{B} \cdot \underline{\nabla} \theta$ is slight, and will be ignored. (The term $\underline{B} \cdot \underline{\nabla} \theta$ is equal to $(l/r)(\partial\psi/\partial r)$ for the helical case, and $(\partial\psi/\partial r)/rR$, or B_θ/r , for the toroidal case.) Therefore, Eq. (45) becomes

$$0 = \oint d\theta C(f_v)/\rho_{||} \quad (61)$$

The θ dependence of the integrand in Eq. (61) is identical for the two different symmetries. For every f_v that satisfies Eq. (61) in the toroidally symmetric case, a perfectly equivalent f_v , with the same dependence on θ and v , exists for the helically symmetric case. Similarly, if we ignore the θ dependence of $|\underline{\nabla}\psi|$ and $\underline{B} \cdot \underline{\nabla} \theta$, the integrand in Eq. (60) will have exactly the same θ , E , and μ dependence for the two different symmetries. We end up with identical expressions for $\tilde{F}$, whether the plasma is helical or toroidal. This conclusion in no way depends on our ability to find an analytical expression for f_v from Eq. (61). We have shown, therefore, that all the calculations of neoclassical transport for the large aspect ratio, axisymmetric tokamak can be carried over to the helically symmetric case without change, no matter what collision operator is employed.

It must be remembered, of course, that the symbols g , ψ and ϵ mean somewhat different things in the two cases. For example, the term $g/\langle |\nabla\psi| \rangle$, which appears in Eq. (60), is approximately equal to $(\alpha r)^{-1}$ in the helical case, and B_ϕ/B_θ in the toroidal case.

B. Lorentz Collision Term

From now on we will use the Lorentz collision operator, given by Eq. (49), in which case we find that

$$\int_0^{E/B} d\mu C(f_1) = -v \int_0^{E/B} d\mu f_1, \quad (62)$$

where we have integrated by parts twice, and noted that μ vanishes at one limit and f_1 at the other. Furthermore, the function f_1 , given by Eqs. (42), (48), and (51), is linear in σ , the sign of $v_{||}$, so that $\sum_{\sigma} \sigma f_1 = 2f_1^+$, where f_1^+ is equal to f_1 with $\sigma = +1$. Therefore, Eq. (60) becomes

$$\bar{F} = \frac{4\pi v c g}{\epsilon m \langle |\nabla\psi| \rangle} \int_0^{\infty} dE \left\langle \int_0^{E/B} d\mu f_1^+ \right\rangle. \quad (63)$$

The function h vanishes for $\mu \geq \mu_0$, where $\mu_0 = E/B_{\max}$, and $B_{\max}$ is the maximum value of B on our surface of constant ψ . For the h part of f_1 , therefore, the integral over μ in Eq. (63) may be taken from 0 to μ_0 . With the aid of Eqs. (42) and (51), we write

$$\left\langle \int_0^{E/B} d\mu f_1^+ \right\rangle = -\frac{c}{e} g \frac{\partial f_M}{\partial \psi} (S_1 - S_2) \quad , \quad (64)$$

$$S_1 = \left\langle \frac{m}{B} \int_0^{E/B} d\mu |v_{||}| \right\rangle \quad , \quad (65)$$

$$S_2 = \int_0^{\mu_0} d\mu \int_{\mu}^{\mu_0} \frac{d\mu'}{\langle |v_{||}(\mu')| \rangle} \quad . \quad (66)$$

The integral in S_1 can be completed without difficulty,

$$S_1 = \frac{1}{3} (2E)^{3/2} m^{1/2} \langle B^{-2} \rangle \quad , \quad (67)$$

and S_2 , after an integration by parts, becomes

$$S_2 = \int_0^{\mu_0} \frac{d\mu \mu}{\langle |v_{||}| \rangle} \quad . \quad (68)$$

In S_2 , before integrating over μ , we must average $|v_{||}|$ over θ , in the sense of Eq. (47). This requires knowledge of the θ dependence of B . In general, on a surface of constant ψ , B varies, periodically in θ , between the limits of $B_{\min}$ and $B_{\max}$. If we define ε by,

$$\varepsilon = (B_{\max} - B_{\min}) / 2B_{\min} \quad , \quad (69)$$

we can always express the B field by

$$B = B_{\min} (1 + 2\varepsilon \tau(\theta)) \quad , \quad (70)$$

where $B_{\min}$ and ϵ depend only on ψ , and where τ is some periodic function of θ , with period 2π , varying between 0 and 1. Then, $|v_{||}|$ may be written,

$$|v_{||}| = [2(E - \mu B_{\min})/m]^{1/2} (1 - p^2 \tau)^{1/2}, \quad (71)$$

where

$$p = [2\epsilon \mu B_{\min}/(E - \mu B_{\min})]^{1/2}. \quad (72)$$

If we now change the variable of integration from μ to p , S_2 takes the form

$$S_2 = 2E^{3/2} m^{1/2} B_{\min}^{-2} \epsilon^{1/2} \int_0^1 \frac{dp p^3}{(p^2 + 2\epsilon)^{5/2} \langle (1 - p^2 \tau)^{1/2} \rangle}. \quad (73)$$

Combining Eqs. (67) and (73), we obtain

$$S_1 - S_2 = 2K m^{1/2} E^{3/2} B_{\min}^{-2}, \quad (74)$$

where K is the dimensionless, energy independent factor,

$$K = \frac{\sqrt{2}}{3} \left\langle (1 + 2\epsilon \tau)^{-2} \right\rangle - \sqrt{\epsilon} \int_0^1 \frac{dp p^3}{(p^2 + 2\epsilon)^{5/2} \langle (1 - p^2 \tau)^{1/2} \rangle}. \quad (75)$$

With this expression for $S_1 - S_2$, and with

$$\frac{\partial f_M}{\partial \psi} = f_M \left[\frac{1}{n} \frac{dn}{d\psi} + \frac{1}{T} \frac{dT}{d\psi} \left(\frac{E}{T} - \frac{3}{2} \right) \right], \quad (76)$$

the E dependence in Eq. (64) is determined, and the integral over E in Eq. (63) may be carried out. The result is

$$\bar{F} = -\frac{3K}{\sqrt{2}} \frac{v m c^2 g^2}{e^2 B_{\min}^2 \langle |\nabla \psi| \rangle} \frac{dp}{d\psi} \quad (77)$$

where

$$P = nT \quad (78)$$

is the pressure of the particle species under consideration. This expression for the particle flux across magnetic surfaces is valid for both helically symmetric and toroidally symmetric plasmas. Aside from the symmetry, no other conditions have been imposed on ψ or B . For instance, the parameter ϵ , which is defined by Eq. (69), and which enters in the expression for K , is not assumed to be small. It is clear, of course, that the numerical factor K cannot be fully determined, unless the function $\tau(\theta)$, defined by Eq. (70), is specified.

In practice, the parameter ϵ often is quite small, $\epsilon \ll 1$. In that case, we can expand Eq. (75) in a power series in $\sqrt{\epsilon}$, with the result, to lowest order

$$K \approx \sqrt{\epsilon} \left[1 - \int_0^1 \frac{dp}{p^2} \left(\frac{1}{\langle (1 - p^2 \tau)^{1/2} \rangle} - 1 \right) \right] \quad (79)$$

The terms of order ϵ cancel, so that the correction to Eq. (79) is of order $\epsilon^{3/2}$. In general, the integral over p in Eq. (79) must be evaluated numerically. However, we can get a reasonable

estimate by expanding the integrand in p^2 ,

$$K \approx \sqrt{\epsilon} \left[1 - \frac{1}{2} \langle \tau \rangle - \frac{1}{12} \langle \tau \rangle^2 - \frac{1}{24} \langle \tau^2 \rangle \dots \right] \quad (80)$$

Furthermore, to the lowest order in ϵ , the operation indicated by the bracket $\langle f \rangle$, defined in Eq. (47), reduces to $(2\pi)^{-1} \int_{-\pi}^{\pi} d\theta f$.

As an important example, we now consider a helically or toroidally symmetric plasma, for which the B field is given by Eq. (18). With τ equal to $\sin^2(\theta/2)$, we have

$$\langle (1 - p^2 \tau)^{1/2} \rangle = \frac{2}{\pi} E(p) \quad , \quad (81)$$

where $E(p)$ is the elliptic integral. A numerical integration over p yields

$$K \approx 0.69 \sqrt{\epsilon} \quad , \quad (82)$$

and, to the lowest order in $\sqrt{\epsilon}$, we may rewrite Eq. (77) as follows,

$$\bar{F} \approx -1.46 \sqrt{\epsilon} \, v m c^2 g^2 \frac{dP}{dr} / e^2 B_o^2 \left(\frac{d\psi}{dr} \right)^2 \quad , \quad (83)$$

where we have ignored the slight θ -dependence of ψ . For an axisymmetric torus, $g^2/B_o^2 \left(\frac{d\psi}{dr} \right)^2$ is equal to B_θ^{-2} and ϵ is the inverse aspect ratio; Eq. (83) then gives, essentially, the result first obtained by Galeev and Sagdeev.^{5,7} For a helically symmetric plasma, $g^2/B_o^2 \left(\frac{d\psi}{dr} \right)^2$ is equal to $(B_o \alpha r)^{-2}$, where α is defined through Eq. (1), and ϵ is the parameter given by Eq. (16) or (22).

Finally, we will end this section with a remark about the so-called bootstrap current. It is clear from the form of the distribution function, that there will exist a current in the direction of the B field,

$$\begin{aligned} j_{||} &= e \int d^3v v_{||} f \\ &= \frac{4\pi eB}{m^2} \int_0^\infty dE \int_0^{E/B} d\mu f_1^+ \end{aligned} \quad (84)$$

It follows from Eq. (63), that there is a simple relationship between the outward particle flux and the bootstrap current,

$$\bar{F} = \frac{\nu m c g}{e^2 \langle |\nabla \psi| \rangle} \langle \frac{j_{||}}{B} \rangle \quad (85)$$

In calculating $\bar{F}$, we have determined $\langle j_{||}/B \rangle$, and vice versa. Thus, combining Eqs. (77) and (85), we arrive at

$$\langle \frac{j_{||}}{B} \rangle = - \frac{3K}{\sqrt{2}} \frac{cg}{B_{\min}^2} \frac{dP}{d\psi} \quad (86)$$

When the conditions appropriate to Eq. (83) apply, Eq. (86) reduces to

$$j_{||} \approx 1.46 \sqrt{\epsilon} \frac{c}{B_0 \alpha r} \frac{dP}{dr} \quad (87)$$

for the helically symmetric case. With toroidal symmetry, B_0 or in Eq. (87) should be replaced by $-B_\theta$.

VI. Conclusion

The neoclassical particle flux across magnetic surfaces has been calculated in such a manner that, at every step along the way, it applies equally well to helically symmetric and toroidally symmetric plasmas.

When the variation in the magnetic field strength over a magnetic surface is small, we find that, for a completely arbitrary collision operator, the transport calculations for the two different symmetries become identical in every way. The results of the many neoclassical transport calculations⁷ for the large aspect ratio, axisymmetric tokamak, therefore, can be trivially carried over to the helically symmetric case. For example, the transport coefficients calculated by Rosenbluth, Hazeltine, and Hinton,⁸ will apply to the helically symmetric plasma, if we replace the poloidal magnetic field, B_θ , by $B \alpha r$, where B is the magnitude of the magnetic field, and α is a constant which determines the pitch of the helix. (More specifically, in terms of α and the cylindrical coordinates, r , ϕ , and z , we define the helical variable, $\theta = \ell(\phi - \alpha z)$, which, together with r , is one of the two non-ignorable spatial coordinates in the helically symmetric case.) Also, the parameter ϵ , which is equal to the inverse aspect ratio, r/R , in the toroidally symmetric case, should be written in its alternate form, $\epsilon = (B_{\max} - B_{\min})/2B_{\min}$ where $B_{\max}$ and $B_{\min}$ are the extreme values of B on the magnetic surface. When this form of ϵ is used, no further change is necessary in going from toroidal to helical symmetry. More

explicit expressions for ϵ , valid in the case of helical symmetry, are given in Eqs. (16) and (22).

A detailed calculation of the particle flux is carried through with a Lorentz collision operator, but with the variation of the magnetic field and the shape of the magnetic surface left arbitrary, except for the requirements of symmetry. The general result, valid for both symmetries, is given by Eq. (77). As a special case, we consider the magnetic field to be given by Eq. (18), with $\epsilon \ll 1$. In that case, the radial particle flux is given by Eq. (83). The bootstrap current is also calculated.

This work was supported by the U. S. Department of Energy Contract No. EY-76-C-02-3073.

References

- ¹K. Miyamoto, Nucl. Fusion 18, 243 (1978).
- ²A. I. Morozov and L. S. Solov'ev, Reviews of Plasma Physics, edited by M. A. Leontovich (Consultants Bureau, New York, 1966) Vol. 2, p. 42.
- ³L. Spitzer Jr., Phys. Fluids 1, 253 (1958).
- ⁴A. I. Morozov and L. S. Solov'ev, Reviews of Plasma Physics, edited by M. A. Leontovich (Consultants Bureau, New York, 1966) Vol. 2, p. 229.
- ⁵A. H. Boozer, to be published.
- ⁶A. A. Galeev and R. Z. Sagdeev, Sov. Phys. - JETP 26, 233 (1968).
- ⁷F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys., 48, 241 (1976).
- ⁸M. N. Rosenbluth, R. D. Hazeltine, and F. L. Hinton, Phys. Fluids 15, 116 (1972).