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Computational Results

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Abstract

In this paper we examine two lattice Boltzmann methods (that are a derivative of lattice gas methods) for computing solutions to two two-dimensional nonlinear diffusion equations of the form

$$\frac{\partial}{\partial t} u = \nu \left(\frac{\partial}{\partial x} D(u) \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} D(u) \frac{\partial}{\partial y} u \right), \quad (1)$$

where $u = u(\vec{x}, t)$, $\vec{x} \in \mathcal{R}^2$, ν is a constant, and $D(u)$ is a nonlinear term that arises from a Chapman-Enskog asymptotic expansion. In particular, we provide computational evidence supporting recent results showing that the methods are second order convergent (in the L_1 -norm), conservative, conditionally monotone finite difference methods. Solutions computed via the lattice Boltzmann methods are compared with those computed by other explicit, second order, conservative, monotone finite difference methods. Results are reported for both the L_1 - and L_∞ -norms.

1 Introduction

There has been growing recent interest in lattice gas methods for modeling fluid dynamics, e.g., [24, 25]. The methods involve particles traversing the links of a lattice and interacting at the nodes. The particles advance in unison to neighboring nodes with unit speed according to a discrete clock. The system can be viewed as a discretization in time, space, and velocity of the Boltzmann equation, e.g., [24] (cf. [12, 35]). Of special note is that velocity space is coarsely discretized, and does not refine in the limit as the unit spatial and temporal lengths (Δx and Δt respectively) tend to zero. The fixed, coarse discretization of the velocity

makes the approach amenable to a mathematical description (of the microdynamics) over a Boolean field,¹ with an implementation using only logical operations, i.e., no floating point arithmetic; this point, in addition to the methods' vast implicit parallelism, has been argued in their favor. Although the microdynamics of these models is non-physical, several researchers have recovered macrodynamical descriptions that agree² with those based on physical microdynamics, e.g., [24, 25]. Most models regard the macroscopic description of fluid dynamics, i.e., the Navier-Stokes equations. The models have been known as, among other terms, *lattice gas automata*, *cellular automata*, and *cellular fluids*. Obtaining the macrodynamical description involves some kind averaging and an assumption regarding molecular chaos. Another related class of methods does not, however; they are lattice Boltzmann methods and have been the subject of much growing interest [2, 3, 23, 29, 32]. These methods originated from lattice gas methods, but differ in that particle distributions move along the links rather than particles themselves. In this way the molecular chaos assumption becomes an inherent part of the method, and the methods involve floating point arithmetic calculations. They are a form of finite difference methods. Lattice gas and lattice Boltzmann methods (herein collectively termed "lattice methods") have been used in a number of computational arenas: incompressible fluid flow (Navier-Stokes) [24, 25, 26, 27, 28], magnetohydrodynamics [13, 14, 34], the Poisson equation [15], mixed flows [1, 6, 9, 10, 11, 13], turbulence [3, 29], diffusion [5, 8, 20], reaction-diffusion [17], advection-diffusion [4, 31], and flow in porous media [7, 16].³ As the subject gains understanding, undoubtedly, its scope and spectrum will increase.

However, lattice methods have lacked a formal numerical theory, which includes proving convergence through consistency and stability arguments. Until recently [21] (and forthcoming in [22]), with but one exception [4], numerical analysis of lattice methods has only included deriving partial differential equations that describe their macroscopic behavior; this is the first part of a formal consistency argument, leaving stability and convergence untouched. Consequently, many questions regarding their applicability, effectiveness, efficientness, competitiveness, and utility have remained questions. Due mostly to limited knowledge about lattice methods, little progress toward answering the questions is being made. We address this issue in this paper by numerically investigating the order of convergence for two lattice Boltzmann methods for solving some two-dimensional nonlinear diffusion equations.

We present computational evidence supporting that the lattice methods are second order, conservative, conditionally monotone finite difference methods. We compare results computed by the lattice Boltzmann methods on coarser and coarser lattices and by standard second order, conservative, monotone finite difference methods on a fine lattice. The fine lattice finite difference-computed solutions are used as exact solutions for comparison purposes.

Section 2 describes lattice gas methods. Section 3 introduces the associated lattice Boltzmann methods. The section also outlines the Chapman-Enskog analysis for deriving the partial differential equations consistent with the lattice methods, and it discusses the conditions of monotonicity, under which a statement regarding convergence has and will

¹The microdynamical description can also be made in terms of cellular automata, e.g., [37].

²A more precise term will be defined subsequently.

³Note that the references listed provide examples of the application areas and are by no means meant to be exhaustive listings. References [18, 19, 30, 33] are collections of works and proceedings of conferences and workshops that encompass lattice methods (and related topics). Several of the articles therein contain thorough reference lists, e.g., [24].

be made [21, 22]. Section 4 discusses the computational results supporting the purported second order convergence of the lattice Boltzmann methods. Section 5 is the conclusion.

2 The Lattice Gas Methods

The lattice gas/Boltzmann methods of this paper are one-speed models. In one-speed models, particles move about (in unison according to the ticks of a discrete clock) along the links of a regular lattice with unit speed and interacting at the nodes of the lattice, a set of collision rules determining the nature of the interactions. The interactions change the velocity of the particles, which are indistinguishable, and the outcomes may be deterministic or nondeterministic. In our case, the methods are deterministic. Lattice gas methods adhere to the Fermi Exclusion Principle, which allows at any given time, at most one particle per direction per node. Adhering to this principle ensures either absence or presence of a particle moving in a particular direction at a node. It is in this way that a finite number of bits may be used to describe the state at a node—one bit for each direction. Supposing that each node has d links, a node state may be described as an element of a Boolean field, $\mathcal{B}^d$ ($\mathcal{B} = \{0, 1\}$). In our case there are $d = 4$ links.

Toward building notation for quantifying a lattice gas method's behavior, the lattice gas methods discussed have the following characteristics: a doubly periodic (toroidal) regular, square lattice, $\mathcal{L}$, suspended in two-dimensional space, $\Omega = [0, L]^2$, with four links per node, each link having unit length $\Delta x = L\delta$; time advances in units of $\Delta t = T\delta^2$, where L and T are the spatial and temporal scale lengths, respectively.

2.1 Occupation Numbers

The occupation number denotes the number of particles (0 or 1) with a particular velocity at a node. We denote this quantity $b_k(\vec{\tau}, m) \in \mathcal{B}$, for the number of particles in direction k at lattice node $\vec{\tau} \in \mathcal{L}$ and time step m . For a particular node, we denote the vector of occupation numbers $\mathbf{b} = [b_k]_{k=0}^{d-1} \in \mathcal{B}^d$, where $d = 4$ is the number of directions in the lattice gas method. (Throughout the paper, boldface lowercase letters denote vectors either in $\mathcal{B}^4$ or the four-dimensional hypercube, $[0, 1]^4$.)

2.2 Microdynamical Equation

The behavior of a lattice gas method is governed directly by its rules: rules for advecting particles to new locations and rules for determining the new directions of the particles. These are characterized in the microdynamical evolution equation for a lattice gas method:

$$\mathcal{A}\mathbf{b}(\vec{\tau}, m) = \mathbf{b}(\vec{\tau}, m) + \mathcal{C}(\mathbf{b}(\vec{\tau}, m)), \quad (2)$$

where $\mathcal{A}$ and $\mathcal{C}$ are the advection and collision operators, respectively. Let $\vec{e}_k$ denote the *unit direction vector* for direction k . (For our square lattice, $\mathcal{L}$, the unit direction vectors are $\vec{e}_0 = (1, 0)$, $\vec{e}_1 = (0, 1)$, $\vec{e}_2 = (-1, 0)$, and $\vec{e}_3 = (0, -1)$.) Then the microdynamical equation, Eq. (2), states that the new occupation numbers (on the left) at the new locations, $(\vec{\tau} + \vec{e}_k, m + 1)$, are the same as the occupations numbers at the current location, $(\vec{\tau}, m)$, plus some correction for particles changing directions. More specifically, the term, $\mathbf{b}(\vec{\tau}, m)$, is the vector of occupation numbers at $(\vec{\tau}, m)$; the *collision operator*, $\mathcal{C}$, changes the direction of particles at $(\vec{\tau}, m)$ before the advection; the sum, $\mathbf{b}(\vec{\tau}, m) + \mathcal{C}(\mathbf{b}(\vec{\tau}, m))$, then, describes

the new occupation numbers that are to be advected; the *advection operator*, $\mathcal{A}$, moves the particles to their new locations, i.e., advects the new occupation numbers:

$$\mathcal{A}\mathbf{b}(\vec{i}, m) = \left[b_k(\vec{i} + \vec{e}_k, m + 1) \right]_{k=0}^3 = \begin{bmatrix} b_0(i + 1, j, m + 1) \\ b_1(i, j + 1, m + 1) \\ b_2(i - 1, j, m + 1) \\ b_3(i, j - 1, m + 1) \end{bmatrix}.$$

2.3 The State Transition Function and Collision Rules

Each lattice location has a *state* which describes its configuration. We have denoted this state with the vector of occupation numbers, $\mathbf{b}(\vec{i}, m)$. Because $\mathbf{b}(\vec{i}, m) \in \mathcal{B}^4$, there are exactly 2^4 possible configurations or *states* that a particular lattice location can have. Given that a location is in state $\alpha = \mathbf{b}(\vec{i}, m) \in \mathcal{B}^4$, then there are 2^4 possible new states, $\beta \in \mathcal{B}^4$. Depending on the lattice gas, there will be a certain probability that a location in state α will change to state β before the advection occurs. This is determined by the collision operator. We can define a state transition function, $\mathcal{S} : \mathcal{B}^4 \times \mathcal{B}^4 \rightarrow [0, 1]$, where $\mathcal{S}(\alpha \rightarrow \beta)$ is the probability of changing from state α to state β . The state transition function determines the probability of a lattice node changing state prior to advection. Put another way, let α be the state at a particular lattice node; then it describes configuration of particles at that node. Let β be a new state at the same lattice node prior to advection, so that β describes particles and their directions. Note that α and β may, of course, be different. Given that a node is in state α , the probability that it will change to state β prior to advection, is $\mathcal{S}(\alpha \rightarrow \beta)$. If the number of particles in α and β are the same, then the state transition function describes the process of particles obtaining new directions. Note that since $\mathcal{S}$ is a probability, it obeys conservation of probability, i.e., $\sum_{\beta \in \mathcal{B}^4} \mathcal{S}(\alpha \rightarrow \beta) = 1$, $\forall \alpha \in \mathcal{B}^4$. A lattice gas is deterministic if $\mathcal{S} \in \mathcal{B}$ and non-deterministic otherwise. The *collision operator* for a lattice gas is defined as

$$\mathcal{C}(\mathbf{b}) = \sum_{\alpha, \beta \in \mathcal{B}^4} \mathcal{S}(\alpha \rightarrow \beta)(\beta - \alpha) \prod_{k=0}^3 (b_k)^{\alpha_k} (1 - b_k)^{(1 - \alpha_k)},$$

in which $\mathcal{S}(\alpha \rightarrow \beta)$ is the probability of a transition from state α to β , $(\beta_k - \alpha_k) \in \{-1, 0, +1\}$ is the correction for direction k , and $\prod_{k=0}^3 (b_k)^{\alpha_k} (1 - b_k)^{(1 - \alpha_k)}$ is 1 if $\mathbf{b} = \alpha$ and is 0 otherwise. Here and in the sequel we assume that $\mathbf{b} = \mathbf{b}(\vec{i}, m)$.

2.4 Statement of the Rules

We now introduce our lattice gas methods, denoted LB_1 and LB_2 . As previously indicated they are on a two-dimensional toroidal lattice, in which each node has four directions. The collision rules for the LB_1 are described as follows: if a state has exactly two particles, then the new state (prior to advection) is determined by sending the particles into the two remaining unoccupied directions; otherwise, particles do not interact (passing through each other). Then in the two-particle cases, particles make a 90° phase transition in head-on collisions; particles “bounce back” in orthogonal collisions. The collision rules for LB_2 are a variation of the LB_1 rules. The LB_2 rules are the LB_1 rules with the addition of single-particle collisions, i.e., in one-particle states, the particle returns—bouncing back—whence

it came.⁴ With $\bar{b} \equiv 1 - b$, the k th element of the collision operator, $\mathcal{C}^{(1)}$, for LB₁ follows:

$$\mathcal{C}_k^{(1)}(\mathbf{b}) = \frac{+\bar{b}_k \bar{b}_{k+1} b_{k+2} b_{k+3} + \bar{b}_k b_{k+1} \bar{b}_{k+2} b_{k+3} + \bar{b}_k b_{k+1} b_{k+2} \bar{b}_{k+3}}{-b_k \bar{b}_{k+1} \bar{b}_{k+2} b_{k+3} - b_k \bar{b}_{k+1} b_{k+2} \bar{b}_{k+3} - b_k b_{k+1} \bar{b}_{k+2} \bar{b}_{k+3}}. \quad (3)$$

Here (and in the sequel), the k indices are evaluated modulo four. The k th element of the collision operator, $\mathcal{C}^{(2)}$, for LB₂ is written

$$\mathcal{C}_k^{(2)}(\mathbf{b}) = \mathcal{C}_k^{(1)}(\mathbf{b}) + b_k b_{k+1} \bar{b}_{k+2} b_{k+3} - b_k \bar{b}_{k+1} \bar{b}_{k+2} \bar{b}_{k+3}, \quad (4)$$

Note that the collision operators can also be expressed in terms of Boolean operations, e.g., [4, 5].

2.5 Conservation Laws

Each of LB₁ and LB₂ possess one *local* conservation law. The conserved quantity is $b_0 + b_1 + b_2 + b_3$. Because of the assumed periodic nature of the lattice, the lattice methods each have *global* conservation laws. The conserved quantity is the sum of the occupation

numbers over the entire lattice, i.e., $\sum_{\vec{r} \in \mathcal{L}} \sum_{k=0}^3 b_k(\vec{r}, m)$.

3 Analysis

3.1 The Lattice Boltzmann Methods

The lattice Boltzmann methods that we will discuss are related to the dynamics of the mean occupation numbers in the associated lattice gas methods. Let $\langle \cdot \rangle$ denote the expected value. Then we define the mean occupation numbers,

$$\mathbf{n}(\vec{r}, m) = \langle \mathbf{b}(\vec{r}, m) \rangle.$$

Note that while $\mathbf{b} \in \mathcal{B}^4$, we have $\mathbf{n} \in [0, 1]^4$. Neglecting the correlations,⁵ i.e., assuming that the expected value passes through the nonlinear terms, we obtain the lattice Boltzmann equations,⁶

$$\mathcal{A}n_k = n_k + \mathcal{C}_k^{(1)}(\mathbf{n}), \quad \text{and} \quad (5)$$

$$\mathcal{A}n_k = n_k + \mathcal{C}_k^{(2)}(\mathbf{n}), \quad (6)$$

for LB₁ and LB₂, respectively. Here, the arguments of the advection and collision operators are in $[0, 1]^4$ instead of in $\mathcal{B}^4$ as for the lattice gas. Eq. (5) and Eq. (6) are the lattice Boltzmann methods we mean to study.

Respectively examining $\mathcal{C}^{(1)}(\mathbf{n})$ and $\mathcal{C}^{(2)}(\mathbf{n})$, we find that⁷ the homogeneous equilibria to the lattice Boltzmann equations (above) have the form $n_0 = n_1 = n_2 = n_3 \equiv u$ so that a zeroth order approximation to $\mathbf{n}$ is $\mathbf{h}^{(0)} = [u, u, u, u]^T$ (T for transpose).

⁴Note that another, equivalent lattice method from LB₂ may be obtained by symmetry considerations under the transformation of exchanging particles and holes.

⁵This is the molecular chaos assumption for lattice gases. See, e.g., [36, p. 523] for a discussion of the molecular chaos assumption in statistical physics.

⁶In the present context this is also known as the lattice Boltzmann approximation to lattice gases, e.g., [24].

⁷It is a simple exercise to verify this by setting the respective collision operators, given in Eq. (3) and Eq. (4) to zero and looking for solutions such that $n_k \in (0, 1)^4$.

3.2 Discrete Chapman-Enskog Asymptotic Analysis

We look for approximate solutions to the lattice Boltzmann equation,

$$\mathcal{A}\mathbf{n} = \mathbf{n} + \mathcal{C}(\mathbf{n}), \quad (7)$$

of the form $\mathbf{n}(\vec{i}, m) \approx \mathbf{h}(\vec{i}\Delta x, m\Delta t)$, for smooth function $\mathbf{h}(\vec{x}, t)$, where $\Delta x = L\delta$ and $\Delta t = T\delta^2$. (Recall that Δx is the lattice spacing and Δt is the temporal spacing.) We expand $\mathbf{h}$ about equilibrium, $\mathbf{h}^{(0)}$, as

$$\mathbf{h} = \mathbf{h}^{(0)} + \delta\mathbf{h}^{(1)} + \delta^2\mathbf{h}^{(2)} + \dots. \quad (8)$$

In general, we can solve for the parameters of this expansion by substituting Eq. (8) into the lattice Boltzmann equation, Eq. (7). Applying this to the lattice Boltzmann equation for LB₁, i.e., Eq. (5), we find $h_k^{(0)}(\vec{x}, t) = u(\vec{x}, t)$ for $k \in \{0, 1, 2, 3\}$, where $u(\vec{x}, t)$ satisfies Eq. (1), in which

$$D(u) = \frac{1}{4u(1-u)} - \frac{1}{2}, \quad (9)$$

and $\nu = \frac{L^2}{2T}$. The same analysis applies to LB₂, in which case

$$D(u) = \frac{1}{2(1-u)(1+u)} - \frac{1}{2}. \quad (10)$$

3.3 Conditions of Monotonicity

Consider the Jacobian matrix of the right-hand side of the lattice Boltzmann equation, Eq. (7). The condition of monotonicity is that the elements of the Jacobian matrix be nonnegative. It can be shown, [21], that LB₁ and LB₂ meet this condition if

$$M_- \leq n_k(\vec{i}, 0) \leq M_+ \quad (11)$$

for all lattice directions k and nodes $\vec{i}$, where for LB₁, $M_{\pm} = (1 \pm 1/\sqrt{5})/2$ and for LB₂, $M_{\pm} = 3/4 \pm 1/12$. Under the conditions of monotonicity, it can be shown further, [21], that LB₁ and LB₂ converge with second order accuracy in the L_1 -norm to the solutions of Eq. (1) for $D(u)$ defined in Eq. (9) and Eq. (10), respectively.

4 Computational Studies

Our numerical investigation regards examining the second order convergence of lattice Boltzmann methods, LB₁ and LB₂. We compare coarser solutions computed by the lattice Boltzmann methods with those computed by a conservative, monotone finite difference method on a fine grid.

Toward developing notation for comparison of the finite difference- and lattice Boltzmann-computed solutions, consider solving the partial differential equation, Eq. (1), with periodic boundary conditions for $\vec{x} = (x, y) \in [0, L]^2$ and $\nu = 1/2$. Let V_i^m and U_i^m denote the finite difference and lattice Boltzmann approximations to $u(\vec{i}\Delta x, m\Delta t)$, respectively, for a square grid sized $N \times N$, where $u(\vec{x}, t)$ is an exact solution to the Eq. (1). Note that $U_i^m = \left(\sum_{k=0}^3 n_k(\vec{i}, m)\right)/4$. Then let

$$E^{(N)}(\vec{i}, m) = U_i^m - V_i^m,$$

and $E^{(N)}(m)$ be a vector of the $E^{(N)}(\vec{r}, m)$ over all spatial grid points $\vec{r} = (i, j)$.

Quantitative results are reported in each of two norms, L_1 and L_∞ , with the norms defined respectively for vector E by

$$\begin{aligned}\|E\|_{\ell_1} &= \frac{\Delta x \Delta y}{L^2} \sum_{i,j} |E_{i,j}|, \\ \|E\|_{\ell_\infty} &= \sup_{i,j} |E_{i,j}|.\end{aligned}$$

We pay particular attention to the ratios,

$$\|E^{(N)}\|_{\ell_1} / \|E^{(2N)}\|_{\ell_1} \quad \text{and} \quad \|E^{(N)}\|_{\ell_\infty} / \|E^{(2N)}\|_{\ell_\infty},$$

which for second order convergence should be near four; for superconvergence, greater than four.

4.1 Finite Difference Methods

To generate solutions with which to compare the lattice Boltzmann-computed solutions, we employ a conservative, monotone finite difference method. The difference scheme follows, in which $D(u)$ is defined in Eq. (9) and Eq. (10) for LB₁ and LB₂, respectively:

$$\begin{aligned}U_{i,j}^{m+1} &= U_{i,j}^m \\ &+ \frac{\nu \Delta t}{2(\Delta x)^2} \left[\left(D(U_{i+1,j}^m) + D(U_{i,j}^m) \right) (U_{i+1,j}^m - U_{i,j}^m) \right. \\ &\quad \left. - \left(D(U_{i,j}^m) + D(U_{i-1,j}^m) \right) (U_{i,j}^m - U_{i-1,j}^m) \right] \\ &+ \frac{\nu \Delta t}{2(\Delta y)^2} \left[\left(D(U_{i,j+1}^m) + D(U_{i,j}^m) \right) (U_{i,j+1}^m - U_{i,j}^m) \right. \\ &\quad \left. - \left(D(U_{i,j}^m) + D(U_{i,j-1}^m) \right) (U_{i,j}^m - U_{i,j-1}^m) \right].\end{aligned}\tag{12}$$

With $\Delta x = \Delta y$, it can be shown that the scheme is a $\mathcal{O}((\Delta x)^2) + \mathcal{O}(\Delta t)$ monotone finite difference scheme for $0 < U_{i,j}^m < 1$ with a stability criterion of

$$\Delta t \leq \frac{(\Delta x)^2}{4\nu}.$$

4.2 Computations

We compute solutions to Eq. (1) for $\vec{x} = (x, y) \in [0, 1] \times [0, 1]$, $t > 0$, periodic boundary conditions, and initial condition

$$u(x, y; 0) = u_I(x, y) = A \sin(2\pi x) \sin(2\pi y) + B,$$

for constants A and B such that $0 < u_I(x, y) < 1$. We vary A and B to generate initial conditions satisfying or not satisfying the conditions of monotonicity for the respective methods, LB₁ and LB₂. The initial condition for the finite difference method, Eq. (12), is $V_i^0 = u_I(\vec{r}\Delta x)$, while for the lattice Boltzmann methods the initial occupation numbers are $n_k(\vec{r}, 0) = u_I(\vec{r}\Delta x)$. Note that $U_i^0 = u_I(\vec{r}\Delta x)$.

While the lattice Boltzmann calculations are computed on a grid of size $N \times N$, where $N < 256$, the finite difference ones are computed originally on a grid of 256×256 points

and rendered (for comparison purposes) on the coarser grids via pointwise projection. To avoid conflict with the limit of stability in the finite difference calculations, we use a Δt that is half that limit.

For each lattice Boltzmann method, we computed solutions for two different initial conditions. Figure 1 and Figure 2 exhibit the solution at time $t = 0$ (the initial conditions) and time $t = 1/32$ for LB_1 and LB_2 , respectively, for initial conditions satisfying the respective conditions of monotonicity. Figure 3 and Figure 4 exhibit the solution at time $t = 0$ (the initial conditions) and time $t = 1/32$ for LB_1 and LB_2 , respectively, for initial conditions not everywhere satisfying the respective conditions of monotonicity, e.g., the monotonicity conditions are not satisfied in either case at the locations $(x, y) \in \{(1/4, 1/4), (3/4, 1/4), (1/4, 3/4), (3/4, 3/4)\}$. At these points the initial occupation numbers exceed the respective conditions of monotonicity. The graphs at $t = 1/32$ are the result of the 256×256 finite difference computations.

4.2.1 Quantitative Analysis of LB_1

The first case regards Table 1 and Table 2. Here, the initial condition satisfies the conditions of monotonicity. We look at the ratio between the norm of the difference between the finite difference- and lattice Boltzmann-computed solutions for grid sizes a factor of two apart, expecting the ratio to be near four. The L_1 -norm results support the theoretical second order convergence in that norm of LB_1 , and even suggest a slight superconvergence. The L_∞ results indicate that LB_1 may also be second order convergent in the L_∞ -norm.

The second case regards Table 3 and Table 4. Here, the initial condition does not everywhere satisfy the conditions of monotonicity for LB_1 . Again, we look at the ratio between the norm of difference between the finite difference- and lattice Boltzmann-computed solutions for grid sizes a factor of two apart, but not necessarily expecting the ratio to be four (because of a lack of a statement of convergence for initial conditions not satisfying the monotonicity conditions). The L_1 -norm results show that the theoretical second order convergence LB_1 may still hold for some initial conditions not everywhere satisfying the conditions of monotonicity. One might argue in favor of a slight superconvergence here, too. The L_∞ -norm results indicate similar findings, i.e., LB_1 may be second order accurate in the L_∞ -norm for not necessarily everywhere monotonicity preserving initial conditions.

4.2.2 Quantitative Analysis of LB_2

The first case regards Table 5 and Table 6. Here, the initial condition satisfies the conditions of monotonicity for LB_2 . We look at the ratio between the norm of the difference between the finite difference- and lattice Boltzmann-computed solutions for grid sizes a factor of two apart, expecting the ratio to be near four. The L_1 -norm results support the theoretical second order convergence of LB_2 , though not as strongly as in the corresponding (i.e., both initial conditions satisfying the monotonicity conditions) LB_1 case. The L_∞ -norm results suggest that LB_2 is second order convergent in that norm, and, again, not as strongly as in the LB_1 case.

The second case regards Table 7 and Table 8. Here, the initial condition does not everywhere satisfy the conditions of monotonicity for LB_2 . Again, we look at the ratio between the norm of difference between the finite difference- and lattice Boltzmann-computed solutions for grid sizes a factor of two apart, but not necessarily expecting the ratio to be near four. The L_1 -norm results show that the theoretical L_1 -norm second order convergence of

LB₂ indeed has trouble for initial conditions not everywhere satisfying the monotonicity conditions. Of particular note in this regard are the $N = 64$ entries in Table 7 and Table 8. It should be noted that $N = 128$ entries do uphold the ratio of sixteen for errors on grid sizes a factor of four apart, e.g., at $t = 1/32$ the ratio, $\|E^{(16)}\|_{\ell_1} / \|E^{(128)}\|_{\ell_1} = 16.042$ based on Table 7. The apparent discrepancy may be restricted to smaller grid sizes.

5 Conclusions

We have introduced two lattice Boltzmann methods for solving some two-dimensional non-linear diffusion equations, outlined their derivation, given their respective conditions of monotonicity, and presented supporting computational results that the methods are second order accurate finite difference methods. A theory substantiating these conclusions is in [21] and will appear in [22].

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Figure 1: Evolution of $u(x,y;t)$: (a) Initial condition, $u(x,y;0) = u_I(x,y) = A \sin(2\pi x) \sin(2\pi y) + B$, where $A = 1/\sqrt{20}$ and $B = 1/2$; (b) $u(x,y;t)$ at $t = 1/32$, using LB_1 .

Figure 2: Evolution of $u(x,y;t)$: (a) Initial condition, $u(x,y;0) = u_I(x,y) = A \sin(2\pi x) \sin(2\pi y) + B$, where $A = 1/12$ and $B = 3/4$; (b) $u(x,y;t)$ at $t = 1/32$, using LB_2 .

Figure 3: Evolution of $u(x,y;t)$: (a) Initial condition, $u(x,y;0) = u_I(x,y) = A \sin(2\pi x) \sin(2\pi y) + B$, where $A = 0.45$ and $B = 1/2$; (b) $u(x,y;t)$ at $t = 1/32$, using LB_1 .

Figure 4: Evolution of $u(x,y;t)$: (a) Initial condition, $u(x,y;0) = u_I(x,y) = A \sin(2\pi x) \sin(2\pi y) + B$, where $A = 0.45$ and $B = 1/2$; (b) $u(x,y;t)$ at $t = 1/32$, using LB_2 .

Table 1: $\|\cdot\|_{\ell_1}$ comparison of LB₁ and finite difference computed solutions with initial condition parameters, $A = 1/\sqrt{20}$ and $B = 1/2$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_1}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.00241	0.00301	0.00241	0.00170
16	1/256	0.000605	0.000759	0.000624	0.000449
32	1/1024	0.000151	0.000190	0.000157	0.000114
64	1/4096	0.0000366	0.0000465	0.0000387	0.0000279
128	1/16384	0.00000804	0.0000107	0.00000902	0.00000662

N	$\ E^{(N)}\ _{\ell_1} / \ E^{(2N)}\ _{\ell_1}$
8	3.991
16	4.010
32	4.124
64	4.550

Table 2: $\|\cdot\|_{\ell_\infty}$ comparison of LB₁ and finite difference computed solutions with initial condition parameters, $A = 1/\sqrt{20}$ and $B = 1/2$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_\infty}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.0110	0.00885	0.00683	0.00476
16	1/256	0.00191	0.00194	0.00157	0.00113
32	1/1024	0.000455	0.000473	0.000387	0.000280
64	1/4096	0.000110	0.000115	0.0000950	0.0000675
128	1/16384	0.0000250	0.0000275	0.0000225	0.0000175

N	$\ E^{(N)}\ _{\ell_\infty} / \ E^{(2N)}\ _{\ell_\infty}$
8	5.755
16	4.198
32	4.137
64	4.394

Table 3: $\|\cdot\|_{\ell_1}$ comparison of LB_1 and finite difference computed solutions with initial condition parameters, $A = 0.45$ and $B = 1/2$.

N	$1/N^2$	$t =$	$\ E^{(N)}\ _{\ell_1}$			
			1/32	1/16	3/32	1/8
8	1/64		0.00576	0.00427	0.00386	0.00281
16	1/256		0.00118	0.00125	0.00108	0.000789
32	1/1024		0.000251	0.000325	0.000279	0.000203
64	1/4096		0.0000603	0.0000818	0.0000704	0.0000513
128	1/16384		0.0000142	0.0000199	0.0000175	0.0000130

N	$\ E^{(N)}\ _{\ell_1} / \ E^{(2N)}\ _{\ell_1}$			
8	4.875	3.409	3.557	3.560
16	4.715	3.846	3.884	3.884
32	4.161	3.981	3.964	3.959
64	4.231	4.116	4.025	3.943

Table 4: $\|\cdot\|_{\ell_\infty}$ comparison of LB_1 and finite difference computed solutions with initial condition parameters, $A = 0.45$ and $B = 1/2$.

N	$1/N^2$	$t =$	$\ E^{(N)}\ _{\ell_\infty}$			
			1/32	1/16	3/32	1/8
8	1/64		0.0438	0.0173	0.0123	0.00854
16	1/256		0.00545	0.00344	0.00272	0.00199
32	1/1024		0.00124	0.000860	0.000685	0.000502
64	1/4096		0.000300	0.0002125	0.000172	0.000127
128	1/16384		0.0000700	0.0000525	0.0000425	0.0000325

N	$\ E^{(N)}\ _{\ell_\infty} / \ E^{(2N)}\ _{\ell_\infty}$			
8	8.041	5.043	4.505	4.294
16	4.402	3.997	3.971	3.955
32	4.125	4.047	3.971	3.941
64	4.284	4.047	4.059	3.921

Table 5: $\|\cdot\|_{\ell_1}$ comparison of LB₂ and finite difference computed solutions with initial condition parameters, $A = 1/12$ and $B = 3/4$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_1}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.000692	0.00202	0.00144	0.000827
16	1/256	0.000276	0.000443	0.000321	0.000198
32	1/1024	0.0000722	0.000106	0.0000782	0.0000491
64	1/4096	0.0000180	0.0000264	0.0000197	0.0000123
128	1/16384	0.00000463	0.00000668	0.00000502	0.00000316

N	$\ E^{(N)}\ _{\ell_1} / \ E^{(2N)}\ _{\ell_1}$
8	2.504
16	3.827
32	4.009
64	3.893

Table 6: $\|\cdot\|_{\ell_\infty}$ comparison of LB₂ and finite difference computed solutions with initial condition parameters, $A = 1/12$ and $B = 3/4$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_\infty}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.00287	0.00504	0.00389	0.00226
16	1/256	0.00104	0.00118	0.000830	0.000502
32	1/1024	0.000263	0.000284	0.000200	0.000123
64	1/4096	0.0000650	0.0000700	0.0000500	0.0000300
128	1/16384	0.0000175	0.0000175	0.0000125	0.0000100

N	$\ E^{(N)}\ _{\ell_\infty} / \ E^{(2N)}\ _{\ell_\infty}$
8	2.762
16	3.952
32	4.037
64	3.711

Table 7: $\|\cdot\|_{\ell_1}$ comparison of LB₂ and finite difference computed solutions with initial condition parameters, $A = 0.45$ and $B = 1/2$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_1}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.0191	0.0131	0.0126	0.0137
16	1/256	0.00483	0.00337	0.00368	0.00259
32	1/1024	0.00116	0.000914	0.000738	0.000594
64	1/4096	0.00617	0.00830	0.00920	0.00969
128	1/16384	0.0000657	0.0000471	0.0000359	0.0000295

N	$\ E^{(N)}\ _{\ell_1} / \ E^{(2N)}\ _{\ell_1}$
8	3.949
16	4.158
32	0.188
64	93.855

Table 8: $\|\cdot\|_{\ell_\infty}$ comparison of LB₂ and finite difference computed solutions with initial condition parameters, $A = 0.45$ and $B = 1/2$.

N	$1/N^2$	$\ E^{(N)}\ _{\ell_\infty}$			
		$t = 1/32$	$1/16$	$3/32$	$1/8$
8	1/64	0.0561	0.0859	0.114	0.148
16	1/256	0.0141	0.0228	0.0320	0.0332
32	1/1024	0.00462	0.00598	0.00946	0.0131
64	1/4096	0.0130	0.0187	0.0262	0.0362
128	1/16384	0.000288	0.000338	0.000503	0.000777

N	$\ E^{(N)}\ _{\ell_\infty} / \ E^{(2N)}\ _{\ell_\infty}$
8	3.984
16	3.046
32	0.355
64	45.217

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