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COHERENT-STATE DESCRIPTION OF FREE-ELECTRON LASERS

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I. INTRODUCTION

Free-electron lasers (FEL) have great advantages over traditional lasers because they have proved to be by far the most efficient way to generate coherent radiations which are also tunable over the widest range of frequencies. This is made possible by eliminating the “middle-man”, *i.e.*, the atoms, molecules, or crystals in the traditional laser systems. Using quantum mechanical analysis, Maydey [1] first explained the gain mechanism in propagating relativistic electrons through a static periodic magnetic field, the wiggler, of an FEL system. Then the first operative FEL amplifier [2] indicated that the most essential features of FEL can be understood in terms of classical mechanics [3]. However, the problem of photon statistics of FEL and, consequently, the very question of whether or not an FEL is a laser, in the sense that it produces coherent-state radiation, can only be studied quantum mechanically; and coherent properties of a laser beam are very important, especially if it is intended to be transported through a long distance. Furthermore, as the experiments are expanded to new frontiers, it is highly probable that questions will arise which can only be answered by quantum-mechanical description. This research project has been concerned mainly about the quantum aspects of FEL.

II. BAMBINI-RENIERI HAMILTONIAN

We consider a nonlinear quantum model of an FEL working in the Compton regime. The FEL is reduced to its very bare essentials; it consists of a single electron moving along an untapered wiggler at a relativistic speed and interacting with a single mode of radiation propagating along the same direction.

It is very convenient to carry out quantum-mechanical analysis of an FEL starting from the Bambini-Renieri Hamiltonian [4] which describes the system in a frame moving along with the electron at a speed close to that of light so that the frequency of the wiggler field coincides with that of the laser and that the wiggler field appears almost as a plane-wave radiation consisting of quasi-photons (Weizsäcker-William approximation).

The Bambini-Renieri Hamiltonian has been derived by Bambini and Stenholm [5] as

$$H = p^2/2m_e + \hbar\omega \left(a_L^\dagger a_L + a_W^\dagger a_W \right) + \hbar\Lambda \left(a_L^\dagger a_W e^{-2ikz} + a_W^\dagger a_L e^{2ikz} \right), \quad (1)$$

where p is the operator representing the momentum of the electron, m_e is the mass of the electron, $a_L^\dagger$ ($a_W^\dagger$) is the creation operator of the laser (wiggler) field, a_L (a_W) is the corresponding annihilation operator, $\Lambda \equiv e^2/2m_e\omega\epsilon_0 V_W$ is the coupling constant with V_W being the wiggler volume, and $\hbar k = \hbar\omega/c$ is the momentum of a photon. This Hamiltonian implies the conservation of photon number and linear momentum.

III. RESEARCH OBJECTIVE

Even in its simplest form, the problem described in the preceding section is still too difficult to solve completely. Therefore, most of the previous works on this subject have been done by neglecting the quantum recoils completely. So the main objective of this research project is to include the effects of the quantum recoils in the analysis. Our approach is to use coherent-state representations to describe the system. Coherent-state representation have proven to be extremely powerful mathematical tools to solve problems in other areas of quantum optics. In this project we have shown once again how powerful these tools can be in solving the quantum-mechanical problem of FEL.

IV. RESEARCH ACHIEVEMENTS

We have considered two different initial conditions for the evolution of the FEL system: 1. Fock states or definite "photon" number states for the initial wiggler field and the initial laser field. 2. Coherent states for the initial fields. Depending on these different initial states, we have introduced different coherent-state descriptions, *i.e.*, atomic-coherent-state description and double-coherent-state description.

A. Atomic-Coherent-State Description

Let the initial quantum state be writtens as

$$|\psi_o\rangle \equiv K_o e^{ip_o z/\hbar} |n_o\rangle_L |N - n_o\rangle_W, \quad (2)$$

where K_o is the normalization constant, p_o is the initial momentum of the electron, n_o is the initial photon number in the laser field, and N is the fixed total number of photons in both the wiggler field and the laser field. Because of the conservation of total photon number, the quantum state evolving from the initial state defined by Eq. (2) can always be written as linear combinations of the $N + 1$ basic states, each being sufficiently identified by a single quantum number n , the photon number in the laser field, denoted by

$$|n\rangle \equiv K_n \exp \{i [p_o/\hbar - 2(n - n_o)k] z\} |n\rangle_L |N - n\rangle_W, \quad (3)$$

where K_n is again a normalization constant. In terms of these basic states, the time-dependent quantum state of the system can be expressed as

$$|\psi(t)\rangle = \exp \left\{ -i \left[(p_o + 2n_o \hbar k)^2 / 2m_e \hbar + N\omega \right] t \right\} \sum_{n=0}^N C_n(t) |n\rangle. \quad (4)$$

Substituting this expression into the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \quad (5)$$

with the Hamiltonian given by (1), we obtain the following difference-differential equation:

$$i \frac{d}{dt} C_n(t) = (-2n\Delta + n^2) C_n(t) + \Lambda \sqrt{(N-n)(n+1)} C_{n+1}(t) \\ + \Lambda \sqrt{(N-n+1)n} C_{n-1}(t), \quad (6)$$

where $C_n(t)$ is the probability amplitude that n photons are in the laser field at time t , and

$$\Delta \equiv k(p_o + 2n_o \hbar k) / m_e, \quad E \equiv 2\hbar k^2 / m_e \quad (7)$$

are two constants, the latter being related to the quantum recoils of the electron. Equation (6) is the so-called spherical Raman-Nath equation [6,7], which is difficult to solve because of the existence of the nonlinear term $n^2 E$.

To solve Eq. (6) with the initial condition $C_n(0) = \delta_{n,n_o}$ as implied by Eq. (2), we will introduce the atomic coherent-state description. The atomic coherents are defined in terms of $|n\rangle$ as

$$|\theta, \phi\rangle \equiv \sum_{n=0}^N |n\rangle \binom{N}{n}^{1/2} \cos^{N-n}(\theta/2) \sin^n(\theta/2) e^{-in\phi}. \quad (8)$$

The Q-function, which is the probability density over the spherical surface, is defined as

$$Q(\theta, \phi) \equiv \langle \theta, \phi | \rho(t) | \theta, \phi \rangle \equiv P^*(\theta, \phi, t) P(\theta, \phi, t), \quad (9)$$

where $\rho(t)$ is the density operator which is to be constructed from the solution to Eq. (6) as

$$\rho(t) \equiv \sum_{n=0}^N \sum_{m=0}^N C_m^*(t) C_n(t) |n\rangle \langle m| \quad (10)$$

and

$$P(\theta, \phi, t) = \sum_{n=0}^N \binom{N}{n}^{1/2} C_n(t) \cos^{N-n}(\theta/2) \sin^n(\theta/2) e^{in\phi} \quad (11)$$

is the probability amplitude.

It will be convenient to introduce the following dimensionless parameters:

$$\delta \equiv \Delta/\Omega, \quad \lambda \equiv \Lambda/\Omega, \quad \epsilon \equiv NE/\Omega, \quad \tau \equiv 2\Omega t, \quad (12)$$

where

$$\Omega \equiv \sqrt{\Delta^2 + \Lambda^2}, \quad (13)$$

which implies

$$\delta^2 + \lambda^2 = 1. \quad (14)$$

Using Eqs. (6) and (11), we can obtain the partial differential equation for $P(\theta, \phi, \tau)$ as

$$\left\{ i \frac{\partial}{\partial \tau} + i [\lambda \cot(\theta/2) \cos \phi - \delta] \frac{\partial}{\partial \phi} + \lambda e^{i\phi} \frac{\partial}{\partial \theta} + \frac{\epsilon}{2N} \frac{\partial^2}{\partial \phi^2} \right\} P(\theta, \phi, \tau) = 0 \quad (15)$$

with the initial condition

$$P(\theta, \phi, 0) = \binom{N}{n_o}^{1/2} \cos^{N-n_o}(\theta/2) \sin^{n_o}(\theta/2) e^{in_o \phi}. \quad (16)$$

We have used the perturbation method to obtain an approximate solution to Eq. (15) as follows:

$$P(\theta, \phi, \tau) = e^{iN\delta\tau/2} \binom{N}{n_o}^{1/2} \left\{ [F_1(\theta, \phi, \tau)]^2 + \epsilon G_1(\theta, \phi, \tau) \right\}^{(N-n_o)/2} \times \left\{ [F_2(\theta, \phi, \tau)]^2 + \epsilon G_2(\theta, \phi, \tau) \right\}^{n_o/2}, \quad (17)$$

where

$$F_1(\theta, \phi, \tau) \equiv [\cos(\tau/2) - i\delta \sin(\tau/2)] \cos(\theta/2) - i\lambda \sin(\tau/2) \sin(\theta/2) e^{i\phi}, \quad (18)$$

$$F_2(\theta, \phi, \tau) \equiv -i\lambda \sin(\tau/2) \cos(\theta/2) + [\cos(\tau/2) + i\delta \sin(\tau/2)] \sin(\theta/2) e^{i\phi}, \quad (19)$$

$$\begin{aligned}
G_1(\theta, \phi, \tau) \equiv & -\frac{i}{8}\lambda^4(3\sin\tau - \tau\cos\tau - 2\tau)\cos^2(\theta/2) \\
& + \frac{\lambda^3}{4}[(2\cos\tau + \tau\sin\tau - 2) + i\delta(3\sin\tau - \tau\cos\tau - 2\tau)] \\
& \times \sin(\theta/2)\cos(\theta/2)e^{i\phi} \\
& - \frac{\lambda^2}{8}\left\{2\delta(3\sin\tau - \tau\cos\tau - 2\tau) \right. \\
& \quad \left. - i[(3\lambda^2 - 2)\sin\tau - (\lambda^2 - 2)\tau\cos\tau - 2\lambda^2\tau]\right\} \\
& \times \sin^2(\theta/2)e^{2i\phi},
\end{aligned} \tag{20}$$

$$\begin{aligned}
G_2(\theta, \phi, \tau) \equiv & -\frac{\lambda^2}{8}\left\{2\delta(2\cos\tau + \tau\sin\tau - 2) \right. \\
& \quad \left. + i[(3\lambda^2 - 2)\sin\tau - (\lambda^2 - 2)\tau\cos\tau - 2\lambda^2\tau]\right\}\cos^2(\theta/2) \\
& + \frac{\lambda}{4}\left\{[(4 - 6\lambda^2)(\cos\tau - 1) + (4 - 3\lambda^2)\tau\sin\tau] \right. \\
& \quad \left. + i\delta[(4 - 3\lambda^2)\sin\tau - (4 - \lambda^2)\tau\cos\tau + 2\lambda^2\tau]\right\} \\
& \times \cos(\theta/2)\sin(\theta/2)e^{i\phi} \\
& + \left\{\delta[\lambda^2(\cos\tau - 1) - (1 - \lambda^2/2)\tau\sin\tau] \right. \\
& \quad \left. + i[\lambda^2(1 - 3\lambda^2/8)\sin\tau + (1 - \lambda^2 + \lambda^4/8)\tau\cos\tau + \lambda^4\tau/4]\right\} \\
& \times \sin^2(\theta/2)e^{2i\phi}.
\end{aligned} \tag{21}$$

We have considered a special case of this general solution when $n_o = 0$ and have evaluated the wiggler-length dependence of the laser output measured in terms of average photon number in the laser field. The result is presented as Fig. 1.

B. Double-Coherent-State Description

To deal with the situation when the initial state for the wiggler field is a coherent state with average “photon” number N and that for the laser field is a vacuum, we introduce the double-coherent-state description. In this description, the initial quantum state of the system

can be expressed as

$$|\psi_o\rangle = K_o e^{ip_o z/\hbar - N/2} \sum_{m=0}^{\infty} (N^m/m!)^{1/2} |m\rangle_W |0\rangle_L \quad (22)$$

and the general time-dependent quantum state can be expressed as

$$|\psi(t)\rangle = K_o \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}(t) e^{i(p_o/\hbar - 2nk)z} |m\rangle_W |n\rangle_L, \quad (23)$$

where $C_{m,n}(t)$ is the probability amplitude that n photons are in the laser field and m “photons” are in the wiggler field at time t . The dependence on z appears in a phase factor which can be absorbed by the replacement $e^{-2ink}|n\rangle_L \rightarrow |n\rangle_{EL}$.

The double-coherent states are defined as

$$|u, v\rangle \equiv \exp\left(-|u|^2/2 - |v|^2/2\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(u^m v^n / \sqrt{m!n!}\right) |m\rangle_W |n\rangle_{EL}, \quad (24)$$

where u and v are two complex variables. The distribution function in the Q-representation (corresponding to antinormal ordering of operators) [8] is defined as

$$Q(u, u^*, v, v^*, t) \equiv \langle u, v | \rho(t) | u, v \rangle \equiv \exp\left(-|u|^2 - |v|^2\right) A^*(u, v, t) A(u^*, v^*, t), \quad (25)$$

where

$$A^*(u, v, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}^*(t) u^m v^n / \sqrt{m!n!}. \quad (26)$$

In this description, the Schrödinger equation is transformed into the following partial differential equation for $A^*(u, v, t)$ as

$$\left\{ i \frac{\partial}{\partial \tau} - 2\delta v \frac{\partial}{\partial v} + \lambda \left[u \frac{\partial}{\partial v} + v \frac{\partial}{\partial u} \right] + \frac{\epsilon}{N} v^2 \frac{\partial^2}{\partial v^2} \right\} A^*(u, v, \tau) = 0 \quad (27)$$

with the initial condition

$$A^*(u, v, 0) = \exp\left(\sqrt{N}u - N/2\right). \quad (28)$$

Without going into the details, we simply present the probability density resulting from the approximate perturbative solution for Eq. (27) as follows:

$$Q(u, u^*, v, v^*, \tau) = \exp \left\{ - \left(|u|^2 + |v|^2 + N \right) / 2 + \sqrt{N} e^{-i\delta\tau} [(\cos \tau + i\delta \sin \tau) u + (i\lambda \sin \tau) v] \right\} \quad (29)$$

where

$$g_1(\tau) \equiv i\lambda^4 h(\tau), \quad (30)$$

$$g_2(\tau) \equiv \lambda^3 h'(\tau) - 2i\lambda^3 \delta h(\tau), \quad (31)$$

$$g_3(\tau) \equiv -\lambda^2 \delta h'(\tau) - i\frac{\lambda^2}{2} [2\lambda^2 h(\tau) + h''(\tau)] \quad (32)$$

with

$$h(\tau) \equiv [2\tau(2 + \cos 2\tau) - 3 \sin 2\tau] / 16 \quad (33)$$

and $h'(\tau)$ and $h''(\tau)$ being the first and the second derivatives of $h(\tau)$, respectively.

The statistical properties of the laser field are determined by the marginal distribution in the v complex plane which can be calculated from the above solution as follows:

$$P(v, v^*, \tau) \equiv \int \frac{d^2 u}{\pi} Q(u, u^*, v, v^*, \tau) \propto \exp \left[- (ax^2 + 2bxy + cy^2 + dx + ey) \right]. \quad (34)$$

In the above equation, we have let $v \equiv e^{i\delta\tau} (x + iy)$ and also have used

$$a = 1 - \epsilon \left[g_3(\tau) + g_3^*(\tau) + \epsilon |g_2(\tau)|^2 \right] \alpha(\tau), \quad (35)$$

$$b = i\epsilon \left[g_3^*(\tau) - g_3(\tau) + g_3^*(\tau) - g_3(\tau) + \epsilon^2 \{ g_1^*(\tau) g_2(\tau) [g_2(\tau) - 4g_1(\tau)] + c.c. \} \right] \alpha(\tau), \quad (36)$$

$$c = 1 - \epsilon \left[g_3(\tau) + g_3^*(\tau) - \epsilon |g_2(\tau)|^2 \right] \alpha(\tau), \quad (37)$$

$$d = -2\sqrt{N}\epsilon [\Re\beta(\tau)] \alpha(\tau), \quad (38)$$

$$e = 2\sqrt{N} \{ \lambda \sin \tau - \epsilon [\Im\beta(\tau)] \alpha(\tau) \}, \quad (39)$$

where

$$\alpha(\tau) \equiv \left[1 - 4\epsilon^2 |g_1(\tau)|^2 \right]^{-1}, \quad (40)$$

$$\beta(\tau) \equiv g_2^*(\tau) [(\cos \tau + i\delta \sin \tau) + 2\epsilon(\cos \tau - i\delta \sin \tau) g_1(\tau)]. \quad (41)$$

We have used this distribution function to calculate various physical quantities which are of great current interest concerning the nonclassical natures of a quantum state as follows:

(a) For the uncertainties in the two quadrature components measured by their standard deviations to determine the existence of squeezing, we have

$$\sigma_x = \sqrt{c/2(ac - b^2) - 1/4}, \quad (42)$$

$$\sigma_y = \sqrt{a/2(ac - b^2) - 1/4}. \quad (43)$$

The numerical evaluation of σ_x and σ_y is presented in Fig. 2.

(b) For the laser output in terms of photon number, we have

$$\langle n \rangle = \left[(ae - bd)^2 + (cd - be)^2 \right] / 4 (ac - b^2)^2. \quad (44)$$

The numerical evaluation of $\langle n \rangle$ is presented in Fig. 3.

(c) For Mandel's parameter defined to determine the existence of photon antibunching, we have

$$q = \frac{2(ae^2 + cd^2 - 2bde) \frac{(a+c)^2}{(ac-b^2)} + 4bde - 4(ae^2 + cd^2) - 2(ad^2 + ce^2)}{(ae - bd)^2 + (cd - be)^2} - 2. \quad (45)$$

A plot of q as a function of τ with ϵ as a parameter is presented in Fig. 4.

C. Gyrotron Radiation

We have considered a free electron moving in a uniform magnetic field. The system is governed by a relativistic wave equation. The relevant part of the eigenfunctions are of the type of generalized Laguerre polynomials; and the corresponding eigenenergies can be expressed as

$$E_j = \sqrt{1 + (2j + 1) \hbar\omega / m_o c^2}, \quad (46)$$

where m_o is the rest mass of the electron, c is the speed of light, and ω is the cyclotron frequency. It is obvious that we always have $\hbar\omega \ll m_o c^2$. So a good approximation of the eigenenergies are

$$E_j \approx E_o + j\hbar\omega - j^2 (\hbar^2 \omega^2 / 2m_o c^2), \quad (47)$$

where E_o is the energy of the ground state.

Then we have considered the interaction of the electron with a radiation field described by the interaction Hamiltonian

$$H_{int} = -e\vec{E} \cdot \vec{r}. \quad (48)$$

Let the quantum state of the whole system be expressed as

$$|\Psi(t)\rangle = \sum_{j=0}^N \sum_{n=0}^{N-j} C_{j,n}(t) |j\rangle |n\rangle, \quad (49)$$

where $|j\rangle$ is the eigenstate of the free electron with eigenenergy E_j , $|n\rangle$ is the Fock state with n photons, and N is the total energy in the system measured in terms of photon number.

Using the properties of the generalized Laguerre polynomials, we have derived a difference-differential equation for the probability amplitude $C_{j,n}(t)$ as follows:

$$i \frac{d}{dt} C_{j,n}(t) = -j^2 \left(\frac{\hbar^2 \omega^2}{2m_o c^2} \right) C_{j,n}(t) + \Lambda \sqrt{j(n+1)} C_{j-1,n+1}(t) \\ + \Lambda \sqrt{(j+1)n} C_{j+1,n-1}(t), \quad (50)$$

where Λ is the coupling constant. It is obvious that the above equation is of the type of the spherical Raman-Nath equation about which we have learned a great deal in our previous studies of FEL.

V. CUMULATIVE LIST OF PUBLICATIONS

1. C. T. Lee, "Solution to the Spherical Raman-Nath Equation", *Journal of Physics A* **18**, L1139–L1142 (1985).
2. C. T. Lee, "Wiggler-Length Dependence of the Output of a Free-Electron Laser", *Physical Review A* **36**, 3245–3247 (1987).
3. C. T. Lee, "General Solution to the Spherical Raman-Nath Equation", *Journal of Physics A* **20**, 5473–5479 (1987).
4. C. T. Lee, "Double-Coherent-State Description of a One-Particle FEL", *Nuclear Instruments and Methods in Physics Research A* **272**, 330–333 (1988)
5. Qiying Fu and C. T. Lee, "Spherical Raman-Nath Equation for Gyrotron Radiation", accepted for publication in *Journal of Physics A*.

REFERENCES

- [1] J. M. Madey, J. Appl. Phys. **42**, 1906 (1971).
- [2] L. R. Elias, W. M. Fairbank, J. M. Madey, H. A. Schwettman, and T. I. Smith, Phys. Rev. Lett. **36**, 717 (1976).
- [3] See, for example, N. M. Kroll, P. L. Morton, and M. N. Rosenbluth, IEEE QE-**17**, 1436 (1981).
- [4] A. Bambini and A. Renieri, Lett. Nuovo Cimento **31**, 399 (1978); A. Bambini, A. Renieri, and S. Stenholm, Phys. Rev. A **19**, 2013 (1979).
- [5] A. Bambini and S. Stenholm, Opt. Commun. **30**, 391 (1979).
- [6] C. W. Raman and N. S. Nath, Proc. Ind. Acad. Sci. **2**, 406 (1937).
- [7] P. Bosco, J. Gallardo, and G. Dattoli, J. Phys. A **17**, 2739 (1984).
- [8] Y. Kano, J. Math. Phys. **6**, 1913 (1965).

FIGURE CAPTIONS

Fig. 1. FEL output vs wiggler length. The values of the parameter are $\lambda = \delta = 1/\sqrt{2}$ and $\epsilon = 0, 0.2, \text{ and } 0.5$.

Fig. 2. Uncertainties in the two quadrature components of the output radiation as functions of time. The lines correspond to $\epsilon = 0, 0.1, 0.2, \text{ and } 0.3$ while $\delta = \lambda = 1/\sqrt{2}$ are fixed. Values of the uncertainties less than 0.5 imply the existence of squeezing, a nonclassical property of radiation.

Fig. 3. Laser output as a function of time. The lines correspond to $\epsilon = 0, 0.2, \text{ and } 0.4$ while $\delta = \lambda = 1/\sqrt{2}$ are fixed.

Fig. 4. The q parameter as a function of time. The lines correspond to $\epsilon = 0, 0.1, 0.2, \text{ and } 0.3$ while $\delta = \lambda = 1/\sqrt{2}$ are fixed. Negative value of q imply the existence of photon antibunching, another nonclassical property of radiation.

$\epsilon = .5$
 $\epsilon = .2$
 $\epsilon = 0$

τ

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