Stochasticity, superadiabaticity, and the theory of
adiabatic invariants and guiding-center motion

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The theory of adiabatic invariants is discussed within the modern framework of symplectic Hamiltonian dynamics. The distinctions between exact, adiabatic, and superadiabatic invariants are clarified. The intimate connection between adiabatic (as opposed to exact) invariance and resonant interactions between motions on disparate time scales is elucidated. For the important case of charged particle motion in a strong magnetic field, resonances between gyration, bounce motion, and an external sinusoidal perturbation are described explicitly by introducing a time-dependent symplectic formulation of the guiding center motion. Destruction of invariance is discussed for quite general situations of physical interest, including the case of a trapped particle in a tokamak.

26
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I. INTRODUCTION

The roots of the concept of adiabatic invariance extend far back into the history of physics. We will not attempt here to provide a complete historical introduction, which would be worthy of a treatise in itself. However, the idea was familiar to, and used by, the founding fathers of statistical and quantum mechanics. In the 1911 Solvay Conference, Einstein considered a simple pendulum and argued that¹ "If one alters the length of the pendulum infinitely slowly, the energy of oscillation will remain always the same if it was initially the same; the energy of oscillation changes only through v . The same holds for undamped electrical oscillations and for free radiation." According to Born,² Ehrenfest considered in 1914 general classical mechanical systems in which certain dynamical variables took on only integral values, and argued that if such systems were disturbed sufficiently slowly, the initial integral value would remain unchanged — "adiabatically invariant." This principle led to the method of quantizing classical systems in the old quantum mechanics.² It persists in modern quantum mechanics in the form of the so-called "Adiabatic Theorem," which states³⁻⁴ that, under certain technical restrictions, the eigenstates of a sufficiently slowly perturbed quantum mechanical system will be those which evolve continuously from the unperturbed system.

Though applications of adiabatic invariance could be traced through the development of many modern physics specialties, we will concentrate here on the application to the motion of charged particles and, thus, to plasma physics. Both practical and theoretical reasons foster an intense interest in the dynamics of charged particles trapped in the terrestrial magnetic field; extensive reviews have been given.⁵ In the laboratory, the quest for controlled fusion has provided important examples of the utility of adiabatic invariants. The theory of one of the major confinement geometries, the magnetic mirror,⁶ depends fundamentally on the adiabatic invariance of the magnetic moment of gyration⁷ $\bar{\mu}$, when the gyroradius of the particle is much smaller than the macroscopic scale of variation of the magnetic field seen by the particle in the course of its motion, to reflect particles from the open ends of the container. Laboratory experiments⁸ have verified adiabatic confinement of trapped particles for more than 10^{10} gyroperiods. Toroidal geometry, as in the case of the tokamak,⁹ eliminates the need for such reflection; however, it also leads to an important component of trapped particles. The conventional single particle picture of confinement in a tokamak¹⁰ relies critically on the theory of the magnetic and electric drifts¹¹ which exist in the limit of small gyroradius, the same limit in which $\bar{\mu}$ is (adiabatically) conserved. Furthermore, free energy arguments¹² suggest that the dominant instabilities of confined tokamak plasmas will be low frequency, $\omega \ll \Omega = eB/mc$, again the limit in which $\bar{\mu}$ should be a good invariant. This idea is used extensively in the analyses of microfluctuations in tokamaks. Space precludes a discussion of the important role of the second, or longitudinal invariant^{13,14} $\bar{J}$, the adiabatic conservation of which was apparently first suggested by Rosenbluth on the basis of quantum mechanical analogies.

In the early historical discussions of adiabatic invariance, the concept of "sufficiently slowly" remained quantitatively undefined. Study of special models^{11,15} led to the conclusion that the exact relative change of an adiabatic invariant after a slow perturbation was often of the form $\exp(-\alpha/\epsilon)$, where α is a constant and ϵ is the small ratio of slow perturbation frequen-

cy to fast unperturbed oscillation frequency. Because the asymptotic expansion in powers of ϵ of $\exp(-a/\epsilon)$ vanishes, it is often said, and sometimes taken as a definition, that adiabatic invariants are constant to all orders^{5,16} in ϵ . In fundamental work, Kruskal¹⁷ gave an elegant description of the construction of the asymptotic series which represent the adiabatic invariants. Chirikov^{18,19} emphasized that the basic concept was not $\epsilon \ll 1$, but rather the presence or absence of resonances²⁰ of the perturbation with the periodic motion of the unperturbed system. If resonances existed in a particular region of phase space, the invariant was destroyed in that region; otherwise, it was preserved.²¹ The inequality $\epsilon \ll 1$ does not preclude resonances, since nonlinearity can create high harmonics from a slow fundamental. We will return to this point.

We now understand that in the region where the invariant is destroyed stochastic motion²² ensues. Loosely, in that region trajectories depend exponentially sensitively on initial conditions. The theory of stochastic regions have been pursued with various degrees of sophistication.²³⁻²⁵ In work of considerable intuitive and practical importance, Chirikov²⁵ has developed the theory of resonances to the point where one now understands how to estimate the dimensions of the stochastic regions for a wide variety of conservative systems, including certain models of particle motion in a magnetic mirror.

As is clear from the work of Chirikov and many others, the appropriate framework in which to discuss questions of invariance and stochasticity is Hamiltonian dynamics. However, in the application to adiabatic invariance the usual Hamiltonian formulation meets with certain difficulties, and historically the theory of adiabatic invariants was generally developed from some version of the method of averaging^{17,26} applied directly to the equations of motion. Some of these approaches are reviewed in the monograph by Northrop.²⁷

To illustrate the difficulty with the Hamiltonian procedures, let us consider the theory of motion of a charged particle.²⁸ The Hamiltonian thereof is well-known²⁹:

$$H(p, q; t) = |p - eA(q, t)/c|^2/2m + e\phi(q, t). \quad (1)$$

Here A and ϕ are the vector and scalar potentials, q is the spatial coordinate $\tilde{x}$ and p is the canonical momentum conjugate to q . The problem is concerned with writing (1) in the standard form, appropriate for perturbation theory,

$$H = H_0 + \epsilon H_1 + \dots, \quad (2)$$

where ϵ is a small parameter and H_0 is soluble, and becomes apparent when (1) is written in dimensionless variables. Normalize velocities to a typical velocity v_0 , introduce a typical magnetic field B_0 and a macroscopic scale length L , and define a gyroradius ρ_0 by $\rho_0 \equiv v_0/\Omega_0$, where $\Omega_0 \equiv eB_0/mc$. If we then introduce a dimensionless momentum $\tilde{p} \equiv p/mv_0$, a dimensionless vector potential $\tilde{A} \equiv A/LB_0$, and a dimensionless scalar potential $\tilde{\phi} \equiv e\phi/(1/2 mv_0^2)$, we find

$$H/(1/2 mv_0^2) = 1/2 |\tilde{p} - \epsilon^{-1} \tilde{A}|^2 + \tilde{\phi}, \quad (3)$$

where $\epsilon \equiv p_0/L$ is assumed to be small. Then, if the dimensionless velocity $\tilde{v} \equiv p/\epsilon^{-1}A$ is to be of order unity, we see that $\tilde{p}$ must be of order ϵ^{-1} . It is not excluded, and, in fact, must be true that $\tilde{p}$ also contains terms of higher order in ϵ , but the forms of these terms are by no means apparent. Thus, while it is sometimes justifiable to set $\tilde{\phi}=0(\epsilon)$, it is not obvious how to extract the $O(\epsilon)$ part related to the magnetic field, and we have not succeeded in writing the Hamiltonian in the form (2).

In an attempt to deal with this difficulty, Gardner³⁰ introduced an ingenious canonical transformation, and indicated how one could apply canonical perturbation theory to his transformed Hamiltonian. Stern³¹ studied the same method in much greater detail. However, the variables of Gardner and Stern were neither completely intuitive nor systematic, and interpretation of the resulting perturbation theory was somewhat opaque. Dragt and Finn³² studied Hamiltonians of a special form appropriate to mirror machines with the aid of Lie transforms. Mynick³³ studied more general Hamiltonians, employing a novel combination of mixed variable generating functions and Lie transforms to give a systematic procedure for doing perturbation theory to all orders. All of these authors worked exclusively with canonical variables.

Recently Littlejohn³⁴ reconsidered the possibility of finding an efficient Hamiltonian theory of guiding center motion and, thereby, of the magnetic moment. By introducing relatively unfamiliar yet strikingly simple techniques — in particular, the use of noncanonical phase space coordinates — he succeeded in providing a beautiful, mathematically elegant, and physically clear Hamiltonian formulation of charged particle motion which in ease of use and interpretation far superseded all previous work and made accessible to the study of adiabatic invariance a number of important techniques and theorems from the general theory of stochasticity. Part of this article is devoted to a review, brief of necessity, of Littlejohn's procedure, on which our work depends heavily. We wish to emphasize that Littlejohn deserves the fullest credit for his insight and fundamental contributions and calculations. Our modest contributions will be primarily to discuss certain physics points which Littlejohn did not completely interpret, and to touch on questions of resonances and stochasticity, which Littlejohn did not address (though he emphasized that his method afforded a powerful tool in this direction). Whereas Littlejohn's emphasis was on the guiding center equations of motion, we will be concerned more with the adiabatic invariants and the conditions under which they are destroyed.

The picture which emerges of charged particle motion in a strong magnetic field is essentially Chirikov's^{18,19} coupled to the modern symplectic techniques which Littlejohn introduced. Consider, for definiteness, the case of a single magnetically-trapped particle in time-independent magnetic and electric fields. The phase space of the particle can then be described in terms of a pair of field line coordinates $\{a, \theta\}$ and two action-angle pairs $\{p, \phi\}$ and $\{J, \psi\}$. The latter pairs (which are not unique) describe, respectively, gyration about, and bouncing along a field line. These variables are not canonical, but are rather semi-canonical in a technical sense to be described. In general, there will be resonances between low harmonics of the gyration and high harmonics of the bounce motion. In regions of phase space where the resonance condition is not satisfied, p and J will be connected, by a transformation which differs from the identity by an amount of $O(\epsilon)$, to quantities $\bar{p}$ and $\bar{J}$ which label invariant tori²² — KAM surfaces — embedded in the phase

space. Thus, away from resonance $\bar{\mu}$ and $\bar{J}$ are exactly (not merely to all orders) conserved. Near resonances, the KAM tori do not exist — colloquially, they "are destroyed." They are replaced by stochastic regions in which μ and J diffuse. For analytic Hamiltonians and small ϵ it can be shown that the amplitudes of the resonant harmonics, and thus the width of the stochastic regions and the rates of change of μ and J , are exponentially small [of the form $\exp(-\alpha/\epsilon)$], in agreement with the historical picture based on special cases and with Kruskal's asymptotic theory. The quantities μ and J are called adiabatic invariants, the adjective "adiabatic" reminding us that the barred quantities are not defined in the stochastic region.

More precisely, the resonant and nonresonant regions are not isolated. Rather, there are stochastic regions near any point of the phase space, densely intermixed with the KAM tori. However, for small ϵ , the fraction of the space which is stochastic is very small, and the intuition presented in the last paragraph is useful for many purposes. As another caveat, when a third periodicity is present Arnol'd diffusion²⁵ can occur. Lack of space precludes us from discussing this interesting possibility.

If one now adds a sinusoidal perturbation — e.g., $\phi \sim \exp(-i\omega t)$ — one of two things can happen depending on the point in phase space: the KAM tori can shift and distort relative to fixed phase space axes, but remain topologically intact; or the invariants may be destroyed because of new resonances between the perturbation and the unperturbed motion. In the first case, the invariants change their functional form — e.g., $\mu \rightarrow \mu(\omega)$. The ω -dependent functions have been called superadiabatic invariants.³⁵ However, it is too restrictive to reserve the term superadiabatic for invariants topologically preserved under temporal perturbation, since we will show that time dependence can be placed on a footing equal to that of the other phase space coordinates. To us, the term superadiabatic invariant is synonymous with a KAM torus — i.e., with an exact invariant in a certain part of phase space.

We must comment on some confusing imprecision in common usage. The invariants which we find will emerge as power series in the small parameter ϵ : away from resonance, the series converge. Although sometimes the lowest order term — e.g., $\mu_0 \approx v_1^2/2\Omega$ — is loosely called the adiabatic invariant, to us the adiabatic invariant is the full series, whose value does not change in the nonresonant region where it is defined, even though μ_0 does change. We call the gyration-related quantity μ the "first adiabatic invariant" or the "magnetic moment"; we call the bounce-related quantity J the "second adiabatic invariant," the "longitudinal invariant," or the "bounce action." The quantity μ_0 would be called the "lowest order magnetic moment."

Along with the adiabatic invariants, the various magnetic and electric drifts emerge naturally from the Hamiltonian formalism. The magnetic drifts are relatively straightforward and have been adequately discussed by Littlejohn.^{34,36} However, the way in which the $\mathbf{E} \times \mathbf{B}$ and polarization drifts enter the theory are somewhat subtle, particularly when E_1 is ordered small. Careful, though straightforward, applications of Lie transformations are needed to describe these effects. We will describe this part of the theory in some detail, since it demonstrates a beautiful link between fundamental, intuitive physics and the somewhat formal mathematical apparatus of Lie transformations.

The paper, then, is organized as follows. In Sec. II we briefly review relevant notions of symplectic Hamiltonian dynamics and of Lie transforms; we give many references to the literature. In Sec. III we formulate and, to some extent, discuss a time-dependent symplectic Hamiltonian theory of guiding center motion and the magnetic moment for a particular ordering of the electric and magnetic fields. In this section we ignore possible interactions with the longitudinal motion. In Sec. IV we treat gyration and bouncing on equal footing, so that the resonant interaction between them emerges naturally; we discuss with the aid of an example how one estimates the extent of the nonadiabatic regions. We devote Sec. V to a brief discussion. Intermediate algebraic results are collected in several appendices.

II. SYMPLECTIC HAMILTONIAN DYNAMICS AND TRANSFORMATIONS

We review here the modern, coordinate-free representation of Hamiltonian dynamics, and sketch Littlejohn's procedure for applying this formalism to (1). The key ingredients of his method are the preparatory use of non-canonical variables to place the Hamiltonian into the form (2), the construction of semi-canonical variables to isolate the (nearly)-periodic motion, then the use of near-identity symplectic (Lie) transformations to remove angle dependence order by order.

It is, perhaps, not widely enough appreciated that Hamiltonian dynamics does not require the use of canonical coordinates, but can be couched in a coordinate-free representation. Once understood, such a formalism is both more transparent and often more useful than the original representation in terms of p 's and q 's, just as the concept of a vector v as an abstract entity transcends in importance and utility the idea of the collection of numbers $\{v^i\}$, the components of the vector with respect to a particular basis. Now for the purpose of this review, though not for most of the rest of the paper, we must be prepared to deal with simple ideas about p -forms, exterior calculus, and the like. These concepts, perhaps unfamiliar, are beautifully introduced at a level appropriate to our needs in Ref. 37, a pedagogical masterpiece. A more compressed and rigorous treatment can be found in Ref. 38. Also summarized in Ex. 4.11 of Ref. 37 is the germ of the application to Hamiltonian dynamics. Namely, define³⁷ the action integral

$$I \equiv \int_A^B [\bar{p}^i \underline{d}q^i - H(\bar{p}^i, q^i; t) \underline{d}t] \equiv \int_A^B \underline{\Omega}.$$

(We denote p -forms, functions of p arguments, by an under-bar, vectors by an under-tilde; we use the summation convention.) The Principle of Least Action^{29,37-39} states that the variation of I between two adjacent paths in the $2n+1$ -dimensional space $\{\bar{p}^i, q^i, t\}$ vanishes for the physical path $\underline{p}$. An application of Stokes' Theorem^{37,38} then leads to

$$\underline{d}\Omega(\dots, \underline{d}\underline{p}/\underline{d}t) = 0, \quad (4)$$

where $\underline{d}$ signifies exterior differentiation and the dots signify³⁷ that the first argument of $\underline{\Omega}$ remain unspecified. Thus, by definition, $\underline{\Omega}$ is a closed differential 2-form.^{37,38} The component of (4) in the phase space is

$$\omega(\dots, \underline{d}\underline{p}/\underline{d}t) = \underline{d}H, \quad (5)$$

where

$$\underline{\omega} \equiv dp^i dq^i = d(p^i dq^i) \quad (6)$$

is called the fundamental or distinguished (differential) 2-form, or the Lagrange tensor.³⁴ It is also closed ($d\underline{\omega} = 0$) and nondegenerate.³⁸ This important result means, by definition,³⁸ that the phase space together with the fundamental 2-form is a symplectic manifold M^{2n} .

Equation (5) is one form of the desired abstract expression of Hamiltonian dynamics. It establishes an isomorphism³⁸ $\tilde{\sigma}$ between the phase space velocity (tangent vector) dP/dt and the differential 1-form dH :

$$dP/dt = \tilde{\sigma}(dH), \quad (7)$$

which is another useful abstract expression of the Hamiltonian equations of motion.

Loosely, the isomorphism $\tilde{\sigma}$ can be thought of as the inverse of the fundamental 2-form $\underline{\omega}$. More precisely, expand $\underline{\omega}$ (covariantly) with respect to an arbitrary basis dz^i : $\underline{\omega} = 1/2 \omega_{ij} dz^i dz^j$. Similarly, expand (contravariantly) $dP/dt = \dot{z}^i dz_i$ and note that one has $dH = (\partial H / \partial z^i) dz^i$. The covariant components of Eq. (5) are then readily shown to be $\omega_{ij} \dot{z}^j = \partial H / \partial z^i$. Defining the contravariant tensor σ^{ij} by $\omega_{ij} \sigma^{jk} = \delta_i^k$, one also has

$$\frac{dz^i}{dt} = \sigma^{ij} \frac{\partial H}{\partial z^j}, \quad \text{or} \quad \frac{dz}{dt} = \tilde{\sigma} \frac{\partial H}{\partial z}, \quad (8a, b)$$

the desired covariant expression of Hamiltonian dynamics.

Following Arnol'd, define the Poisson bracket $[F, H]$ of two phase functions F and H as the derivative of F in the direction of the flow $U^t: M^{2n} \rightarrow M^{2n}$ generated by H :

$$[F, H] \equiv \frac{d}{dt} F(U^t z) \Big|_{t=0}. \quad (9)$$

A corollary³⁸ of this definition is that

$$[F, H] = \underline{\omega}[\tilde{\sigma}(dH), \tilde{\sigma}(dF)], \quad (10)$$

from which follow the important results that

$$[F, H] = \frac{\partial F}{\partial z^i} \sigma^{ij} \frac{\partial H}{\partial z^j}, \quad \text{or} \quad \sigma^{ij} = [z^i, z^j] \equiv [z, z]^{ij}. \quad (11, 12)$$

$\tilde{\sigma}$ is called the Poisson tensor. With (12), Eq. (8) becomes

$$\frac{dz^i}{dt} = [z^i, z^j] \frac{\partial H}{\partial z^j}, \quad (13)$$

which is the form we shall actually use.

There is a shorter, if less intuitive, route to Eq. (13). It can be readily verified that the components of $\underline{\omega}$ with respect to an arbitrary basis dz^i are Lagrange brackets²⁹:

$$\omega_{ij} = \{z^i, z^j\} \equiv \{z, z\}_{ij} = -\sum_l \left(\frac{\partial q^l}{\partial z^i} \frac{\partial p^l}{\partial z^j} - \frac{\partial q^l}{\partial z^j} \frac{\partial p^l}{\partial z^i} \right). \quad (14)$$

Since it is well-known²⁹ that the Lagrange and Poisson brackets are inverses,

$$\{z, z\}_{ij} \{z, z\}^{jk} = \delta_i^k, \quad (15)$$

Eq. (13) follows immediately from Eq. (5). Indeed, it is interesting to note that Goldstein remarks that "[Eq. (15)] is invariant under all transformations of the coordinates, without restriction even to canonical coordinates." (Last emphasis added.) Unfortunately, Goldstein did not pursue the implications of this observation.

It is assumed that the Hamiltonian describes rapid nearly-periodic motion in an angle θ . Then, with the appropriate choice of non-canonical coordinates, it is possible to cast the Hamiltonian into the standard form (2), and various averaging techniques are available for effecting the time-honored procedure of removing angle dependence order by order. However, unlike the very special case of canonical coordinates, such averaging does not, in general, remove angle dependence from the equations of motion for any of the variables, because in noncanonical coordinates the Poisson tensor may depend on θ . To isolate the rapid motion, an intermediate set of ("semi-canonical") variables must be constructed for which the fast angle and its conjugate action are decoupled from — have zero Poisson brackets with — the remaining variables. That such a transformation can be found is a consequence of the Darboux Theorem,³⁸ which states that, given a closed nondegenerate differential 2-form ω in a certain neighborhood in R^{2n} , a local coordinate system $\{p^i, q^i\}$ can be chosen such that ω has the standard form (6). Littlejohn³⁴ has discussed the Darboux Theorem in detail. The proof of the theorem is constructive, so that explicit equations for the desired transformations are available.

The Hamiltonian written in Darboux variables can now be subjected to a near-identity symplectic transformation³⁸ which removes angle dependence to any desired order. The functional form of the Poisson tensor is preserved under such a transformation.³⁴ The most efficient technique for this purpose is Lie transformation, the theory and application of which has been thoroughly discussed.^{34,40-42} A Lie transformation T to new variables $\tilde{z}$ and to a new Hamiltonian K is effected by $\tilde{z} = Tz$ and $K = T^{-1}H$; if $L_n \equiv \{w_n, \dots\} = \sum_{\ell=0}^{\infty} \epsilon^\ell L_n^{(\ell)}$, then^{34,36}

$$T = 1 - \epsilon L_1^{(0)} + 1/2 \epsilon^2 [-L_2^{(0)} + (L_1^{(0)})^2 - 2L_1^{(1)}] + O(\epsilon^3). \quad (16)$$

The expression $K = T^{-1}H$ is expanded order by order, and the generating functions w_n are chosen to make K_n independent of angle.

The function $\bar{\mu} = T\mu$, the Lie transform of the Darboux action variable μ conjugate to the angle θ , is the adiabatic invariant associated with the nearly-periodic motion described by θ . If the Lie transformation, constructed as a power series in ϵ , converges, $\bar{\mu}$ exists and is an exact invariant. The Lie series will not converge everywhere if the system supports one or more additional periodicities which can resonate with the first. This effect can be isolated by first introducing Darboux action-angle variables for

all periodicities, and by designing T to remove dependence on all angles simultaneously. Potential loss of convergence will be signified by the appearance of resonant denominators in the formal expressions for the w_n 's, and can be analyzed with the aid of the KAM and related theorems. See Sec. IV.

III. A NON-RELATIVISTIC, TIME-DEPENDENT, GUIDING CENTER HAMILTONIAN

Littlejohn's original work³⁴ on the guiding center problem ignored time dependence and electric fields. Here we indicate a possible generalization which includes these effects.⁴³ The results of the formalism will be needed in the next section where we study resonances. We use the well-known technique²⁹ of extending the phase space by introducing the energy and the reversed time as a new canonical momentum-coordinate pair. Thus, defining $s \equiv -t$, we introduce a variable w such that $[s, w] = 1$, while $[s, s] = [w, w] = [s, z^i] = [w, z^i] = 0$. Then, we define an extended Hamiltonian Λ by

$$\Lambda(q, p, s, w; \tau) \equiv H(q, p; t) - w, \quad (17)$$

where τ now plays the role of the time and does not appear on the right-hand side of (17). We have $\partial\Lambda/\partial w = -1 = ds/d\tau$, so $\tau = -s = t$ (choosing the origin of time to vanish). Then

$$\frac{\partial\Lambda}{\partial p} = \frac{\partial H}{\partial p} = \frac{dq}{dt} = \frac{dq}{d\tau},$$

and similarly $\partial\Lambda/\partial q = -dp/d\tau$; thus, the original canonical dynamics are preserved in the extended phase space. Finally, from

$$\frac{dw}{d\tau} = -\frac{\partial\Lambda}{\partial s} = \frac{\partial H}{\partial t} = \frac{dH}{dt} = \frac{dH}{d\tau},$$

we conclude that we can choose $w = H$, consistent with $\partial\Lambda/\partial\tau = d\Lambda/d\tau = 0$. The fact that the value of Λ is identically zero with this choice does not, of course, affect the canonical apparatus.

We must now decide how to order the electric field. It is conventional to order $E_{\parallel} = O(\epsilon)$; we will make the same choice for $E_{\perp}$. In the (drift-kinetic) ordering we shall enforce, where gradients and time derivatives are taken to be $O(\epsilon)$, this implies that the potentials A and ϕ which generate the electric field are $O(1)$ [while the potential which generates the equilibrium magnetic field is $O(\epsilon^{-1})$]. In fact, it is often true that $\phi = O(\epsilon)$. For example, in the theory of drift-type fluctuations in tokamaks, various arguments⁴⁴ predict saturation at fluctuation levels $e\langle\delta\phi^2\rangle^{1/2}/T \lesssim (k_{\perp}L_n)^{-1}$, where $k_{\perp}$ is a typical wavenumber of the fluctuations and L_n is the (macroscopic) density scale length. We take $(k_{\perp}L_n)^{-1} = O(\epsilon)$, thus precluding discussion of the very longest wavelengths. In the laboratory, measured fluctuation levels are on the order of several percent. Littlejohn⁴³ has recently treated the MHD ordering $E_{\perp} = O(1)$, $E_{\parallel} = O(\epsilon)$.

Following Littlejohn^{34,36} closely, we now effect a succession of variable changes designed to bring the Hamiltonian into the standard form (2). We first pass to the set $\{z^i\} = \{x, y, s, w\}$, where $y \equiv p - eA(x, s)/c$, and find

$$\Lambda(\underline{x}, \underline{v}, s, w; \tau) = 1/2 m v^2 + e\phi(\underline{x}, s) - w. \quad (18)$$

In these variables, the Poisson bracket of two arbitrary functions F and G , which defines the Poisson tensor $\bar{\sigma}$, is given in App. A. Next one introduces a spatial, orthonormal basis $\{\hat{b}, \hat{\tau}_1, \hat{\tau}_2\}$, where $\hat{b}(\underline{x}, s) \equiv \underline{B}/B$ and $\hat{\tau}_1$ and $\hat{\tau}_2$ are space-time fields arbitrary except for the constraint of orthonormality. One then introduces the coordinates $\{x, v_{\parallel}, \theta, v_{\perp}, s, w\}$, where $v_{\parallel} \equiv \hat{b} \cdot \underline{v}$, $v_{\perp} \equiv |\hat{b} \times (\underline{v} \times \hat{b})|$, and $\theta \equiv \tan^{-1}(\underline{v} \cdot \hat{\tau}_1 / \underline{v} \cdot \hat{\tau}_2)$. The Hamiltonian becomes

$$\Lambda(\underline{x}, v_{\parallel}, \theta, v_{\perp}, s, w; \tau) = 1/2 m (v_{\parallel}^2 + v_{\perp}^2) + e\phi(\underline{x}, s) - w. \quad (19)$$

The non-vanishing Poisson brackets in these coordinates are also recorded in App. A.

To isolate the fast θ dependence, we design a Darboux transformation $\{x, v_{\parallel}, \theta, v_{\perp}, s, w\} \rightarrow \{X, U, \theta, \mu, T, W\}$ by requiring

$$\{\theta, \mu\} = (m\epsilon)^{-1}, \quad (20a)$$

$$\{X, \theta\} = \{U, \theta\} = \{W, \theta\} = \{T, \theta\} = 0, \quad (20b)$$

$$\{X, \mu\} = \{U, \mu\} = \{W, \mu\} = \{T, \mu\} = 0. \quad (20c)$$

That is, we determine the variable μ semi-canonically conjugate to θ . The solution of these equations subject to the conditions $\mu = 0$, $X = x$, $U = v_{\parallel}$, $W = w$, and $T = s$ on the $v_{\perp} = 0$ surface is recorded in App. B. The Hamiltonian which results can be written in the form

$$\begin{aligned} \Lambda(\underline{x}, U, \theta, \mu, T, W) = & m(1/2 U^2 + \mu\Omega) + e\phi_0 - W + \epsilon \{ e\phi_1 + \mu \rho [(1/3)(F_3 + \hat{a} \cdot \underline{\nabla} \ln \Omega) v_{\perp}^2 \\ & + 1/2 (1/2 Z_0 + Z_2 + S_0) v_{\perp} U + (U^2 F_0 - e \underline{E} \cdot \hat{a} / m)] \} + O(\epsilon^2), \end{aligned} \quad (21)$$

where we have redefined $v_{\perp}$ as $v_{\perp} \equiv (2\Omega\mu)^{1/2}$; also, $\rho \equiv v_{\perp}/\Omega$, $\underline{E}(\underline{x}, T) \equiv -\partial\phi/\partial\underline{x} + c^{-1}\partial A/\partial T$ is the electric field, and the remaining symbols are defined in App. A. Expression (21) will be used in Sec. IV where we introduce bounce motion explicitly.

The next step is to (attempt to) construct a Lie transformation to a set of barred variables such that $\bar{\theta}$ dependence is removed from the Hamiltonian to any desired order. The barred variables can be reasonably called guiding center variables; $\bar{\mu}$ will be the magnetic moment associated with the guiding center. In general, carrying the Lie transformation through second order is complicated, though straightforward, and is not of particular interest here, since Littlejohn has discussed the magnetic effects in detail.^{34,36} It is, however, instructive to consider the special case where the magnetic field is independent of space and time, while the electric field is left arbitrary. Here, the relations (B1-4) simplify considerably:

$$\underline{\bar{x}} = \underline{x} - \epsilon \rho \hat{a}, \quad U = v_{\parallel}, \quad (22a, b)$$

$$\mu = v_{\perp}^2 / 2\Omega, \quad W = w - \epsilon \left(\frac{e}{c} \right) \rho \hat{a} \cdot \frac{\partial A}{\partial s} + 1/2 \epsilon^2 \left(\frac{e}{c} \right)^2 \rho^2 \hat{a} \cdot \underline{\nabla} \left(\hat{a} \cdot \frac{\partial A}{\partial s} \right). \quad (22c, d)$$

The Lie operations described in Sec. II are also straightforward; one finds, for the low- β approximation where $\underline{E}_\perp = -\nabla_\perp \phi + ik_\perp \phi$,

$$\Lambda(\bar{X}, \bar{U}, \bar{\theta}, \bar{\mu}, T, \bar{W}) = m(1/2 \bar{U}^2 + \bar{\mu} \Omega) + [1 - 1/4 (k_\perp \rho)^2] e \bar{\phi} - (\bar{W} + 1/2 m \bar{u}_E^2) + O(\epsilon^3), \quad (23)$$

where $\underline{u}_E \equiv c \underline{E} \times \hat{b} / B$ is the familiar electric drift, and

$$\bar{X} = X + O(\epsilon^3), \quad \bar{\theta} = \theta - \epsilon (\Omega v_\perp)^{-1} e \hat{c} \cdot \underline{E} / m + O(\epsilon^2), \quad (24a, b)$$

$$\bar{U} = U + \epsilon^2 \left(\frac{c}{B} \right) \rho \hat{b} \cdot \nabla (\hat{c} \cdot \underline{E}) + O(\epsilon^3), \quad (24c)$$

$$\bar{\mu} = \mu - \epsilon \rho \frac{c}{B} \hat{a} \cdot \underline{E} + 1/2 \epsilon^2 \left\{ \frac{c}{B} \left(\frac{\rho}{\Omega} \right) [-1/2 v_\perp (\hat{a} \hat{a} - \hat{c} \hat{c}) : \nabla \underline{E} + 2 \frac{d}{dT} (\hat{c} \cdot \underline{E})] + u_E^2 / \Omega \right\}. \quad (24d)$$

where $d/dT \equiv \partial/\partial T - U \hat{b} \cdot \nabla$. The factor $1 - 1/4 (k_\perp \rho)^2$ in (23) is the long-wavelength limit of the well-known factor $J_0(k_\perp \rho)$ which describes the effective field felt by the guiding center. (A gyrokinetic ordering, in which terms of all orders in $k_\perp \rho$ are retained, is straightforward, but is not discussed here.)

The equations of motion which follow from (23) are

$$\frac{d\bar{X}}{d\tau} = \bar{U} \hat{b} + \epsilon \bar{u}_E + O(\epsilon^2), \quad \frac{d\bar{U}}{d\tau} = e \bar{E}_\parallel / m + O(\epsilon), \quad (25a, b)$$

$$\frac{d\bar{\theta}}{d\tau} = \bar{\Omega} + O(\epsilon^2), \quad \frac{d\bar{\mu}}{d\tau} = 0 \quad (\text{to all orders}); \quad (25c, d)$$

the physical interpretations are obvious. However, it is important to note that Eq. (25a) contains no hint of the polarization drift. There is no paradox; the polarization drift is an effect seen in laboratory, not guiding center, coordinates. Indeed, using $da/d\tau = cd\theta/d\tau$ and $\underline{x} = \bar{\underline{x}} + \epsilon \rho \hat{a}$, we have

$$\frac{d\underline{x}_\perp}{d\tau} = \frac{d\bar{\underline{x}}}{d\tau} + \epsilon \left(\frac{1}{v_\perp} \frac{d\mu}{d\tau} \hat{a} + \rho \frac{d\theta}{d\tau} \hat{c} \right).$$

Using Eqs. (24) and (25), we then find

$$\frac{d\underline{x}_\perp}{d\tau} = \underline{v}_\perp + \epsilon \underline{u}_E + \frac{\epsilon^2}{\Omega} \frac{d}{d\tau} \left(\frac{c}{B} \underline{E}_\perp \right) + O(\epsilon^3), \quad (26)$$

the last term of which is the well-known polarization drift.¹¹

Further insight into the role of the barred, guiding center variables and of the Lie transformation is gained by considering the form of the magnetic moment $\bar{\mu}$. For constant $\underline{E}_\perp$, this can be written correct to $O(\epsilon^2)$ as

$$\bar{\mu} = |\underline{v}_\perp - \underline{u}_E|^2 / 2\Omega, \quad (27)$$

the well-known result²⁷ for the adiabatic invariant in the presence of electric fields. Thus, among other things the Lie transformation effects a Galilean transformation to the drift frame. If this interpretation is to be consistent, the new variable $\bar{\theta}$ should describe gyration in the drift frame. That is, it should be true that $\bar{v}_\perp \sin \bar{\theta} = (\underline{v}_\perp - \underline{u}_E) \cdot \underline{\tau}_\perp$. Writing $\bar{\theta} = \theta + \delta\theta$ and noting that $\bar{v}_\perp = v_\perp - \underline{c} \cdot \underline{u}_E$, one finds that $\delta\theta = \underline{a} \cdot \underline{u}_E / v_\perp = -(c/B) \underline{c} \cdot \underline{E} / v_\perp$, in agreement with the formal result (24).

IV. THE LONGITUDINAL INVARIANT, RESONANCES, AND DESTRUCTION OF INVARIANCE

The procedure of Sec. III produces a magnetic moment $\bar{\mu}$ which is formally constant to all orders in ϵ . However, because the choice of coordinates took no explicit account of possible near-periodicities other than the gyration, it is not guaranteed that the Lie series converges. We will study this issue with the aid of a specific example, and consider the effects on the invariance of μ of the longitudinal periodicity (bouncing) of a particle trapped in a magnetic well, as well as the effects of temporal resonance.

In unpublished work of considerable importance, Littlejohn⁴⁵ has discussed the longitudinal invariant J (for the time-independent case). However, he transformed the gyration-averaged Hamiltonian into bounce coordinates, and thus was unable to examine the resonant interaction between bouncing and gyration. As he indicated, the appropriate technique is to introduce bounce coordinates before Lie transformation is applied, and this is what we shall do. We begin with the Hamiltonian (21), and assume that the magnetic structure is such that particles with sufficiently small $v_\perp$ are trapped in a magnetic well. We assume that the bounce frequency is $O(\epsilon)$ relative to the gyrofrequency. To lowest order, particles bounce along a field line. The natural spatial coordinates are therefore the field line coordinates $\{y, \ell\}$, where

$$\ell(\underline{X}, T) = \int_{\underline{X}}^{\underline{X}} d\underline{X}' \cdot \underline{b}(\underline{X}', T)$$

is the distance along a field line (computed with time frozen), $y^1 \equiv \beta(\underline{X}, T)$, $y^2 \equiv e\phi(\underline{X}, T)/mc$, where α and β are the Euler potentials⁴⁶ for the field: $\underline{B}(\underline{X}, T) = \nabla\alpha \times \nabla\beta$. One can choose a gauge such that $\underline{B} = \nabla \times \underline{A}$ with

$$\underline{A} = 1/2 (\alpha \nabla\beta - \beta \nabla\alpha), \text{ or } A_i = 1/2 \left(\frac{mc}{e} \right) \gamma_{ab} y^b \frac{\partial y^a}{\partial z^i},$$

where γ is a two-dimensional antisymmetric matrix with $\gamma_{12} = 1$. As an intermediate variable, it is convenient to introduce the zeroth order particle energy E_0 :

$$E_0 \equiv 1/2 \underline{U}^2 + \mu R + e\phi_0/m. \quad (28)$$

The zeroth order bounce angle of the particle is then

$$\psi = \omega_b \int_{\ell_0}^{\ell} d\ell' \chi(\underline{y}, \ell', E_0, \mu, T), \quad (29)$$

where

$$2\pi/\omega_b \chi(\underline{y}, E_0, \mu, T) \equiv \int_b d\ell' / U(\underline{y}, \ell', E_0, \mu, T) \quad (30)$$

is the period of the nonlinear bounce motion ($\int_b$ denotes the line integral over a complete period), and ℓ_0 is a turning point ($U = 0$) of the zeroth order motion. For later use, it is convenient to introduce the quantity (the zeroth order longitudinal action)

$$I(\underline{y}, E_0, \mu, T) \equiv \frac{1}{2\pi} \int_b d\xi' U,$$

in terms of which (30) can be written

$$\omega_b(\underline{y}, E_0, \mu, T) = (\partial I / \partial E_0)^{-1} \equiv \Omega_b(\underline{y}, I, \mu, T). \quad (31)$$

Rather than determine the Poisson brackets of the new variables directly, it is convenient to note⁴⁵ that since $\underline{w}$ is closed it can be written as the exterior derivative of a distinguished 1-form $\underline{p}$: $\underline{w} = d\underline{p}$. The 1-form $\underline{p}$, being isomorphic to a vector $\underline{p}$, can be computed simply, $\underline{w}$ can be easily determined by computing the exterior derivative, then $\underline{p}$ found by inverting $\underline{w}$. If one defines $\xi^1 \equiv T$, $\xi^2 \equiv W$, it can be shown that

$$\underline{p}_i = 1/2 \epsilon^{-1} \gamma_{ab} \gamma^b \frac{\partial y^a}{\partial z^i} + 1/2 \gamma_{ab} \xi^b \frac{\partial \xi^a}{\partial z^i} + v^b \frac{\partial \tilde{x}}{\partial z^i}. \quad (32)$$

Next, a Darboux transformation taking $\{\underline{y}, \ell, E_0, T, W\}$ to $\{\underline{Y}, \psi, J, T, \tau\}$ is designed by requiring

$$\{\psi, J\} = 1, \quad (33a)$$

$$\{\underline{Y}, \psi\} = \{T, \psi\} = \{\tau, \psi\} = 0, \quad (33b)$$

$$\{\underline{Y}, J\} = \{T, J\} = \{\tau, J\} = 0, \quad (33c)$$

and $T = W$, $\underline{Y} = \underline{y}$, and $J = 0$ on the initial value surface $U = 0$, $E_0 = \kappa_0$, where κ_0 is defined in App. C. The details of solution of eqs. (33) are not entirely trivial, and cannot be recorded here. One finds that the Hamiltonian takes the form

$$\Lambda(\underline{Y}, \psi, J, \theta, \mu, T, \tau) = \kappa(\underline{Y}, J, \mu, T) - T + m\epsilon \{D(\underline{Y}, \psi, T, \kappa) - \omega_b(\underline{Y}, J, \mu, T)\} \\ \times [\gamma^{ab} F_a(\underline{Y}, \psi, T, \kappa) \frac{\partial}{\partial y^b} I(\underline{Y}, \mu, T, \kappa) + G(\underline{Y}, \psi, T, \kappa)] + m^{-1} \delta \Lambda(\underline{Y}, \psi, J, \theta, \mu, T) + O(\epsilon^2). \quad (34)$$

Here κ is the lowest order energy E_0 expressed in terms of the action I ,

$$\kappa(\underline{Y}, J, \mu, T) \equiv \int_0^J dJ' \Omega_b(\underline{Y}, J', \mu, T), \quad (35)$$

and $\delta \Lambda$ is the $O(\epsilon)$ part of (21) expressed in the longitudinal Darboux coordinates; to lowest order, $\underline{Y} = \underline{y}$ and $J = I$. The θ -independent functions D , F_a , and G and the remaining Poisson brackets of the Darboux variables are recorded in App. C.

We are now prepared to remove angle dependence by Lie transformation. Several ordering schemes are possible. We first assume that all angle derivatives maintain their nominal ordering. As we will see, such ordering misses the resonant interaction between multiple periodicities. It also necessitates a modification of formula (16). Because the fundamental Poisson bracket $\{\theta, \mu\}$ is of order ϵ^{-1} , the Lie operator $L_1 \equiv [\omega_1, \dots]$ contains a term $\epsilon^{-1} L_1^0$, where

$$L_n^\theta \equiv \frac{\partial w}{\partial \theta} \frac{\partial}{\partial \mu} - \frac{\partial w}{\partial \mu} \frac{\partial}{\partial \theta} ;$$

functions L_n^ψ and L_n^T are defined similarly. Unless the term in L_1^θ is removed, the transformation will not be near-identity. Littlejohn³⁴ removed it by setting $w_1 \equiv 0$. Here, however, a more general procedure is necessary, and it is sufficient to choose

$$L_1^\theta \Lambda_0 = 0, \text{ or } \partial w_1 / \partial \theta = 0. \quad (36)$$

The first two equations defining the Lie transformation are then

$$\Gamma_0 = \Lambda_0, \quad \Gamma_1 = \Lambda_1 + L_2^\theta \Lambda_0 + L_1^\theta \Lambda_1 + (L_1^\psi + L_1^T) \Lambda_0 ; \quad (37,38)$$

explicitly, Eq. (38) is

$$\Gamma_1 = \Lambda_1 + \bar{\Omega}_c \frac{\partial w_2}{\partial \theta} - \frac{\partial w_1}{\partial \mu} \frac{\partial \Lambda_1}{\partial \theta} + \Omega_b \frac{\partial w_1}{\partial \psi} - \frac{\partial w_1}{\partial T} - \frac{\partial w_1}{\partial T} \frac{\partial \Lambda_0}{\partial T}. \quad (39)$$

Here

$$\bar{\Omega}_c \equiv \frac{\partial \kappa}{\partial \mu | Y, J, T} = \frac{\int d\lambda' \Omega(\lambda') / [2(\kappa - \mu \Omega(\lambda') - e\phi_0)]^{1/2}}{\int d\lambda' / [2(\kappa - \mu \Omega(\lambda') - e\phi_0)]^{1/2}} \quad (40)$$

is the bounce-averaged gyrofrequency. Equation (39) is unusual in that the second order generating function appears in the first order transformation equation. Nevertheless, Eq. (39) can be solved for w_1 . It is consistent to take $\partial w_1 / \partial T = 0$. We choose Γ_1 to be independent of θ , ψ , and T . Averaging Eq. (39) successively over θ , ψ , and T gives

$$\Gamma_1 = \langle \Lambda_1 \rangle_\theta + \frac{\partial w_1}{\partial \psi} - \frac{\partial w_1}{\partial T}, \quad \Gamma_1 = \langle \Lambda_1 \rangle_{\theta, \psi} - \frac{\partial \langle w_1 \rangle}{\partial T}, \quad \Gamma_1 = \langle \Lambda_1 \rangle_{\theta, \psi, T}. \quad (41a-c)$$

Choosing a Fourier decomposition of the form $\exp[i(\ell\theta + m\psi + n\omega T)]$, we see that Eq. (41b) defines the $(0,0,n \neq 0)$ component of w_1 , and that Eq. (41a) (formally) defines the $(0,m \neq 0, n \neq 0)$ component of w_1 . Because of Eq. (36), w_1 is completely determined if we demand $\langle w_1 \rangle_{\theta, \psi, T} = 0$. Combining Eq. (41a) with Eq. (39) gives

$$\bar{\Omega}_c \frac{\partial w_1}{\partial \theta} = -\left(1 - \frac{\partial w_1}{\partial \mu} \frac{\partial}{\partial \theta}\right)(\Lambda_1 - \langle \Lambda_1 \rangle_\theta).$$

We see that w_2 is responsible for removing θ dependence from Γ_1 , just as in Littlejohn's theory.

The function w_1 removes ψ dependence. The formal solution of Eq. (41a) is

$$w_1 = - \sum_{\substack{m \neq 0 \\ n \neq 0}} \frac{\langle \Lambda_1 \rangle_{0, m, n}}{i(m\Omega_b - n\omega)} \exp[i(m\psi + n\omega T)], \quad (42)$$

where $\langle \Lambda_1 \rangle_{0, m, n}$ is the Fourier amplitude of Λ_1 . Away from resonance, w_1 is well-defined; the Lie transformation thus constructed generates the first correction to the longitudinal invariant.^{45,47} Resonance overlap in some part

of phase space signifies development of stochastic regions and destruction of μ and J in that region.

An important example of destruction of J has been given by Smith⁴⁸ in his study of the trapped ion mode. In our notation, Smith computed, in a simple tokamak geometry, the amplitudes $(\Lambda_1)_{0,m,\pm 1}$ for the Hamiltonian (21), taking $\phi_0 = 0$ and retaining only the ϕ_1 part of Λ_1 . He then used the Chirikov criterion for resonance overlap^{23,25} to estimate the extent of the stochastic region which exists around the separatrix between trapped and passing particles. He found, for mode frequencies of the order of the bounce frequency, that destruction of the adiabatic invariant occurred in quite a large region around the separatrix. This effect would greatly affect the dispersion relation for trapped ion modes and is a subject of continuing investigation.

By taking $\partial/\partial\psi$ to be $O(1)$, we can not correctly describe the effects of high harmonics of the bounce frequency: we thus miss the resonant interaction between gyration and bouncing. To describe this, we may take L_1^{ψ} and L_1^T to be $O(\epsilon^{-1})$. One must then choose: $w_1 = 0$, and more conventional Lie equations emerge:

$$\Gamma_0 = \Lambda_0, \quad \Gamma_1 = \Lambda_1 + (L_2^{\theta} + L_2^{\psi} + L_2^T) \Lambda_0. \quad (43a, b)$$

The solution

$$w_2 = - \sum_{(l,m,n) \neq 0} \frac{(\Lambda_1)_{l,m,n} \exp[i(l\theta + m\psi + n\omega T)]}{i(l\bar{\Omega}_c + m\bar{\Omega}_b - n\omega)} \quad (44)$$

displays the characteristic resonant denominators.

We may illustrate the use of (44) by estimating the degree of destruction of μ and J for a trapped particle in an axisymmetric tokamak. We consider the "theorist's equilibrium"⁴⁹ of static, nested flux surfaces concentric about a toroidal magnetic axis of radius R_0 . In terms of the radial, poloidal, and toroidal unit vectors $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$, we take $\mathbf{B} = [(r/R_0)\hat{\theta} + (R_0/R)\hat{\phi}]B_0$, where $R \equiv R_0(1 + \delta \cos\theta)$, $\delta \equiv r/R_0 < 1$, and $q(r) = O(1)$; we ignore electric fields. Defining $\Omega_0 \equiv eB_0/mc$, we find an approximate Hamiltonian from the time-independent version of (21):

$$H = m\mu\Omega_0(1 - \delta \cos\theta) + \frac{1}{2} m v^2 - m c \left[\frac{(2\mu\Omega_0)^{3/2}}{3\Omega_0 R_0} \right] [(3/2) \cos\theta \cos\phi + \sin\theta \sin\phi] + O(\epsilon^2). \quad (45)$$

In arriving at the form (45), a canonical transformation was used to remove an $O(\epsilon)$ term (related to the first correction to the magnetic moment) independent of θ . The zeroth order Hamiltonian describes a standard pendulum, characterized by the trapping parameter $M \equiv [E_0 - \mu\Omega_0(1 - \delta)] / (2\mu\Omega_0\delta)$, E_0 being the constant value of the energy. Trapped particles satisfy $0 < M < 1$. Expression (45) can be written straightforwardly in terms of the lowest order angle-action variables $\{\psi, J\}$, which are Jacobi elliptic functions. Fourier analysis of the perturbation reveals that a dense set of resonances "pile-up" and overlap near the separatrix $M = 1$. Although a resonance overlap criterion can be used to estimate the width of the stochastic region, it is more rigorous to follow Chirikov²⁵ and look directly at the evolution of the action near the separatrix. Since the bounce frequency and gyrofrequency are well-separated, it can be shown that the change in J occurs principally near $\psi = 0$ due

to kicks rapid compared to the bounce time, and that the change in μ is small compared to the change in J . The differential equations for J over each half bounce period can then be approximated by a standard mapping,²⁵

$$I_{n+1} = I_n + K \sin \theta_n, \quad \theta_{n+1} = \theta_n + I_{n+1}, \quad (46a,b)$$

where K is a known function. Using the well-known result^{25,50} that global stochasticity ensues for (46) when $K \gtrsim 1$ leads to the result⁵¹ that stochasticity exists in the region

$$|1 - M| \lesssim \left(\frac{\rho}{r}\right) \left(\frac{\bar{\omega}}{\omega_0}\right)^3 \exp\left(-\frac{\pi \bar{\omega}}{2\omega_0}\right), \quad (47)$$

where $\bar{\omega}_c \equiv \Omega_0(1+\delta)$ and $\omega_0 \equiv (\mu \Omega_0 \delta)^{1/2} / q R_0$. To within logarithmic corrections, this result has the familiar form $\exp(-\alpha/\epsilon)$. This is, in fact, a quite general result for analytic Hamiltonians, by virtue of an uncertainty principal of Fourier analysis.

V. DISCUSSION

Our principal contributions have been (1) to introduce a time-dependent Hamiltonian formulation of guiding center motion in the drift-kinetic ordering, and (2) to show from the Hamiltonian point of view how resonant interactions between multiple periodicities lead to adiabatic, as opposed to exact, invariants. As an example, we computed the extent of the stochastic region for a trapped particle in an axisymmetric tokamak. In itself, this calculation is academic, since the stochastic region is extremely small for typical device parameters. However, it serves as a very useful prototype for more relevant (and much more difficult) calculations of particle stochasticity in the magnetic fields of realistic devices.

Others have analyzed nonadiabatic effects by studying directly the equations of evolution for μ_0 or μ . (See, for example, Ref. 52.) The well-known feature of the Hamiltonian method is that it collects all of the physics into one scalar function, an advantage not substantially vitiated by the noncanonical formulation. We believe that the techniques and point of view discussed here will have wide applications to further studies of regular and stochastic particle motion.

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APPENDIX A: Poisson Brackets

In the coordinates $\{\underline{x}, \underline{v}, s, w\}$,

$$[F, G] = \frac{1}{m} \left(\frac{\partial F}{\partial \underline{x}} \cdot \frac{\partial G}{\partial \underline{v}} - \frac{\partial F}{\partial \underline{v}} \cdot \frac{\partial G}{\partial \underline{x}} \right) + \underline{R} \cdot \frac{\partial F}{\partial \underline{v}} \times \frac{\partial G}{\partial \underline{v}} - \frac{e}{mc} \frac{\partial A}{\partial s} \cdot \left(\frac{\partial F}{\partial \underline{v}} \frac{\partial G}{\partial w} - \frac{\partial F}{\partial w} \frac{\partial G}{\partial \underline{v}} \right) + \frac{\partial F}{\partial s} \frac{\partial G}{\partial w} - \frac{\partial F}{\partial w} \frac{\partial G}{\partial s}. \quad (A1)$$

In the coordinates $\{\underline{x}, v_{\parallel}, \theta, v_{\perp}, s, w\}$, all Poisson brackets except the following vanish:

$$[\underline{x}, v_{\parallel}] = \hat{b}/m, \quad [\underline{x}, \theta] = -\hat{a}/mv_{\perp}, \quad [\underline{x}, v_{\perp}] = \hat{c}/m, \quad (A2a, b, c)$$

$$[\theta, v_{\perp}] = m^{-1} [\Omega/v_{\perp} + F_3 + (v_{\parallel}/v_{\perp})Z_0], \quad [v_{\parallel}, v_{\perp}] = m^{-1} [v_{\perp} (1/2 Z_1 - S_1) + v_{\parallel} F_1], \quad (A2d, e)$$

$$[v_{\perp}, w] = -\hat{c} \cdot \left(\frac{e}{mc} \frac{\partial A}{\partial s} + v_{\parallel} \frac{\partial \hat{b}}{\partial s} \right), \quad [v_{\parallel}, \theta] = m^{-1} [1/2 Z_0 - S_0 - Z_2 - (v_{\parallel}/v_{\perp})F_0], \quad (A2f, g)$$

$$[\theta, w] = \hat{a} \cdot \left\{ \left(\frac{e}{mc v_{\perp}} \right) \frac{\partial A}{\partial s} + \left[\left(\frac{v_{\parallel}}{v_{\perp}} \right) \frac{\partial \hat{b}}{\partial s} + \frac{\partial \hat{c}}{\partial s} \right] \right\},$$

$$[v_{\parallel}, w] = -\left(\frac{e}{mc} \right) \hat{b} \cdot \frac{\partial A}{\partial s} + v_{\perp} \hat{c} \cdot \frac{\partial \hat{b}}{\partial s}, \quad [s, w] = 1, \quad (A2h, i)$$

where

$$\hat{a} \equiv \cos \theta \hat{r}_1 - \sin \theta \hat{r}_2, \quad \hat{c} \equiv -\sin \theta \hat{r}_1 - \cos \theta \hat{r}_2, \quad (A3a, b)$$

$$Z_0 \equiv \hat{b} \cdot \underline{v} \times \hat{b}, \quad Z_1 \equiv \underline{v} \cdot \hat{b}, \quad Z_2 \equiv \hat{b} \cdot \underline{R}, \quad \underline{R} \equiv (\underline{v} \hat{c}) \cdot \hat{a} = (\underline{v} \hat{r}_1) \cdot \hat{r}_2. \quad (A4a-c, A5)$$

$$F_0 \equiv \hat{b} \cdot (\underline{v} \hat{b}) \cdot \hat{a}, \quad F_1 \equiv \hat{b} \cdot (\underline{v} \hat{b}) \cdot \hat{c}, \quad F_2 \equiv \hat{a} \cdot \underline{R}, \quad F_3 \equiv \hat{c} \cdot \underline{R}, \quad (A6a-d)$$

$$S_0 \equiv 1/2 [\hat{a} \cdot (\underline{v} \hat{b}) \cdot \hat{c} + \hat{c} \cdot (\underline{v} \hat{b}) \cdot \hat{a}], \quad S_1 \equiv 1/2 [\hat{a} \cdot (\underline{v} \hat{b}) \cdot \hat{a} - \hat{c} \cdot (\underline{v} \hat{b}) \cdot \hat{c}], \quad (A7a, b)$$

APPENDIX B: Darboux Transform for Guiding Center Problem

The solutions of Eqs. (20) are

$$\underline{x} = \underline{x} - \epsilon \rho \hat{a} - \epsilon^2 \rho^2 \hat{a} \{ 1/2 \hat{a} \cdot \underline{v} \ln \Omega - [1/2 F_3 + (v_{\parallel}/v_{\perp})Z_0] \} + O(\epsilon^3), \quad (B1)$$

$$u = v_{\parallel} - \epsilon \rho [1/2 v_{\perp} (-1/2 Z_0 + S_0 + Z_2) + v_{\parallel} F_0] + O(\epsilon^2), \quad (B2)$$

$$w = w - \frac{e}{c} \rho \hat{a} \cdot \frac{\partial A}{\partial s} + \epsilon^2 \rho^2 \left\{ m \Omega \frac{v_{\parallel}}{v_{\perp}} \hat{a} \cdot \frac{\partial \hat{b}}{\partial s} + 1/2 m \Omega \hat{a} \cdot \frac{\partial \hat{c}}{\partial s} + \frac{e}{c} \hat{a} \cdot \frac{\partial A}{\partial s} [1/2 F_3 + (v_{\parallel}/v_{\perp})Z_0 - 1/2 \hat{a} \cdot \underline{v} \ln \Omega] + 1/2 \frac{e}{c} \hat{a} \cdot \underline{v} \left(\hat{a} \cdot \frac{\partial A}{\partial s} \right) \right\} + O(\epsilon^3). \quad (B3)$$

$$\mu = \frac{v^2}{2\Omega} + \epsilon p^2 \{1/3 v_{\perp} (F_3 + 1/2 \hat{a} \cdot \nabla \ln \Omega) - 1/2 v_{\parallel} Z_0\} + O(\epsilon^2), \quad T = s, \quad (B4,5)$$

where

$$\rho \equiv v_{\perp} / \Omega(x, s). \quad (B6)$$

The Poisson brackets of the Darboux variables not specified by Eqs. (20) are exactly

$$[\underline{X}, \underline{X}] = \epsilon \hat{b} \times \hat{I} / m \Omega^*, \quad [\underline{X}, U] = m \{ \hat{b} + \epsilon U [\hat{b} \times (\hat{b} \cdot \nabla) \hat{b}] / \Omega^* \}, \quad (B7a, b)$$

$$[X, W] = \epsilon b \times \left(\frac{e}{mc} \frac{\partial \hat{A}}{\partial T} + \epsilon U \frac{\partial \hat{b}}{\partial T} \right) / \Omega^*, \quad (B7c)$$

$$[U, W] = - \frac{e}{mc} \hat{b} \cdot \frac{\partial \hat{A}}{\partial T} - \epsilon U \{ \hat{b} \times (\hat{b} \cdot \nabla) \hat{b} \} \cdot \left(\frac{e}{mc} \frac{\partial \hat{A}}{\partial T} + \epsilon U \frac{\partial \hat{b}}{\partial T} \right) / \Omega^*, \quad [T, W] = 1, \quad (B7d, e)$$

where

$$\Omega^* \equiv \Omega + \epsilon U \hat{b} \cdot (\nabla \times \hat{b}). \quad (B8)$$

APPENDIX C: Quantities related to the longitudinal invariant

The following quantities appear in Eq. (34):

$$D(\underline{Y}, \psi, T, \kappa) \equiv \int_{\kappa_0}^{\kappa} dE_0 \left[\frac{\partial}{\partial E_0} \left(U \hat{b} \cdot \frac{\partial \underline{X}}{\partial T} \right) - \frac{\partial}{\partial T} \left(U \frac{\partial s}{\partial E_0} \right) \right],$$

$$F_a(\underline{Y}, \psi, T, \kappa) \equiv \int_{\kappa_0}^{\kappa} dE_0 \left[\frac{\partial}{\partial Y_a} \left(U \frac{\partial s}{\partial E_0} \right) - \frac{\partial}{\partial E_0} \left(U \hat{b} \cdot \frac{\partial \underline{X}}{\partial Y_a} \right) \right],$$

$$G(\underline{Y}, \psi, T, \kappa) \equiv \gamma_{ab} \int_{\kappa_0}^{\kappa} dE_0 \frac{\partial F_a}{\partial E_0} \frac{\partial F_b}{\partial \psi},$$

where $\kappa_0 \equiv \mu \Omega(\underline{Y}, \ell_{\min}) + e \phi_0(\underline{Y}, \ell_{\min})$ and $\ell_{\min}$ labels the position of minimum field strength along the line labelled by $\underline{Y}$.

The Poisson brackets unspecified by Eqs. (33) are

$$[Y_a, Y_b] = \epsilon \gamma_{ab}, \quad [Y_a, T] = 0, \quad [Y_a, T] = 0, \quad [T, T] = 1.$$

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