ELONGATION UPON TORSION
IN A THEORY FOR THE INELASTIC
BEHAVIOR OF METALS

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ABSTRACT

The torsion of a thin-walled cylinder is studied using the low temperature or visco-plastic limit of Hart's state variable theory for the inelastic behavior of metals. For values of the shear stress lower than the "hardness" of the material, reversible elongation is found to accompany the recoverable twist. For shear stress exceeding the hardness, there are, in addition, a permanent twist and elongation. These predictions agree qualitatively with existing experiments. Detailed quantitative results, for several loading histories, are obtained for Nickel.

I. INTRODUCTION

The test of any proposed set of constitutive relations is its ability to predict a wide range of experimentally observed phenomena. One such experimental observation in metals is the axial extension of hollow cylinders and solid bars of circular cross-section under applied axial torsion. In 1947, Swift¹ published experimental results on length changes of metals plastically deformed in torsion. His results on torsion of solid bars show that all of the metals tested, with one exception, experienced elongation under pure torsional loading. The exception was lead, which contracted. Swift also conducted an experiment on a cylindrical tube of 70-30 brass and found the same qualitative effects for a solid bar of the same material. However, the less restrained metal of the tube exhibited greater elongation than the solid bar. In a more recent study, Bailey, Hass, and Nawab² give results for hollow, circular cylinders of 1100 Aluminum. Their results give additional evidence of axial elongation in plastic torsion. Their study also presents evidence that circumferential contraction accompanies the axial elongation. In the present paper the axial elongation accompanying plastic torsion is predicted using a constitutive theory proposed by Hart³ for the nonelastic deformation of metals.

The axial elongation of circular cylinders under torsion is in some sense a second order effect of the type described by Reiner⁴ and Freudenthal and Ronay^{5,6}. The effect has been explained most commonly by the assumption of explicit non-linearity in the constitutive equations or by the assumption of deformation induced anisotropy⁷. In the present treatment it is shown to be a natural consequence of the constitutive equations proposed by Hart³. The effect arises from the rotation of an internal stress tensor due to the material rotations that accompany the torsion.

In the next section the constitutive equations will be reviewed. A limiting form of the equations that is valid at low temperatures (below about 15% of the melting temperature) will be described. This so-called low temperature limit of the theory describes the mechanical behavior of a large class of metals at room temperature. However such a theory would not apply to lead because it recrystallizes at room temperature. The low temperature limit of the theory will then be applied to the problem of torsional deformation of a thin walled cylinder. In particular, numerical simulations of this problem will be shown for Nickel tubes at room temperature.

II. CONSTITUTIVE RELATIONS

Hart³ has recently proposed a set of constitutive relations for the nonelastic deformation of metals. He developed the form of these relations from experimental evidence and a heuristic consideration of micro-mechanisms in the material. First a brief description of the constitutive relations as they apply to homogeneous, isotropic metals is presented. Following this is a discussion of the limiting process and the resulting set of constitutive relations valid for temperatures that are small compared with the melting point. Finally, the inclusion of elastic deformation is considered in order to relate the constitutive variables to observable deformation.

A. Constitutive Relations for Homogeneous, Isotropic Metals.

The constitutive relations proposed by Hart for the nonelastic deformation of metals are relations that express the real time nonelastic strain rate $\dot{\epsilon}$ as a function of the applied stress deviator $\underline{\sigma}$, the absolute temperature T , and the current values of two internal state variables σ^* and $\underline{a}$. The constitutive equations also include equations for the time rate of change of σ^* and $\underline{a}$. The state variable σ^* , called the "hardness", is a scalar and is a measure of the isotropic hardening. The variable $\underline{a}$ is a symmetric, deviatoric tensor and measures a stored anelastic strain. This time dependent, internal strain reflects the direction of prior straining and is the basis, in Hart's theory, for the Bauschinger effect. It is this same variable that is responsible for the axial elongation during torsional deformation because of the rotations associated with such deformation.

In the present paper we have made a notational change from that used in ref. 3. Here we employ $\dot{\epsilon}$ for the variable denoted $\dot{\epsilon}$ in Hart's paper.

The reason for this is to emphasize that $\underline{\dot{N}}$ (or $\underline{\dot{\epsilon}}$) is a strain rate and is not a time derivative of anything. The same comment applies to the variable $\underline{P}$ introduced below which is the same as $\underline{\dot{q}}$ in ref. 3. In what follows, all tensors are deviatoric, symmetric, and denoted by boldface type. We define an associated scalar invariant, denoted by the corresponding light face character, according to the equation

$$a \equiv + \sqrt{\text{tr } \underline{a}^2} = + \sqrt{a_{ij} a_{ij}}, \quad (1)$$

where repeated indices are summed in the component form of the equation.

The total nonelastic strain rate $\underline{\dot{N}}$ is composed of two parts: the nonrecoverable (permanent or plastic) strain rate $\underline{P}$ and the corrotational or Jaumann time derivative of the stored, recoverable anelastic strain $\underline{a}$,

$$\underline{\dot{N}} = \underline{P} + \frac{D}{Dt} \underline{a}. \quad (2)$$

In addition, it is convenient to introduce two auxiliary stress variables $\underline{\sigma}_a$ and $\underline{\sigma}_f$ such that

$$\underline{\sigma} = \underline{\sigma}_a + \underline{\sigma}_f, \quad (3)$$

with their other properties given below. Then under the assumption of material isotropy, the following isotropic tensor relations are prescribed,

$$\underline{\sigma}_a = M \underline{a}, \quad (4)$$

$$\underline{P} = (P/\sigma_a) \underline{\sigma}_a, \quad (5)$$

$$\underline{N} = (N/\sigma_f) \underline{\sigma}_f. \quad (6)$$

The remaining equations are among the scalar invariants. They are

$$P = \dot{\epsilon}^* [\ln(\sigma^*/\sigma_a)]^{-1/\lambda}, \quad (7)$$

$$\dot{\epsilon}^* \equiv (\sigma^*/G)^m f \exp(-Q/RT), \quad (8)$$

$$N = \dot{a}^*(\sigma_f/M)^M, \quad (9)$$

and

$$\frac{D}{Dt} (\ln \sigma^*) = P\Gamma(\sigma^*, \sigma_a) - R(\sigma^*, T). \quad (10)$$

where D/Dt denotes the material time derivative.

In the foregoing equations the quantities Q , M , m , f , and λ are constants. Of those Q , M (the anelastic modulus), f , and m are material constants. The quantity λ seems to be the same for all materials. The coefficient $\dot{a}^*$ in Eq. (9) is a function of T and must be determined experimentally. The functions Γ and R are also to be determined experimentally. However, at moderate temperatures R is substantially zero, and there are fairly good analytical representations for Γ . Finally G is the modulus of rigidity at temperature T , and R is the gas constant. The measured values of the material parameters for Nickel at room temperature are shown in Table I.⁸

The material time derivative of a tensor ψ of any rank is given in terms of its spatial gradients $\nabla\psi$ and the velocity $\underline{v}$ of the material by

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + (\underline{v} \cdot \nabla)\psi.$$

The corotational derivative of a second rank tensor $\underline{a}$ is then,

$$\frac{D\underline{a}}{Dt} = \frac{D\underline{a}}{Dt} + \underline{a} \cdot \underline{W} - \underline{W} \cdot \underline{a}, \quad (11)$$

where $\underline{W}$ is the skew-symmetric part of the spatial gradients of the velocity. The use of the corotational and the material derivatives insure that the evolution laws for $\underline{a}$ and σ^* are independent of rigid motions of the material and, thus, have the material significance intended by Hart³.

B. Low Temperature Limit

The above set of constitutive relations can be modified for temperatures that are small compared to the melting point. Such a limit was previously suggested by Hart³ to model viscoplasticity and is often called the viscoplastic limit. Utilizing the fact that $\dot{\epsilon}^*$ decreases with temperature allows us to neglect P (in Eq. (7)), formally setting it equal to zero, as long as σ^* is greater than the scalar σ_a . Only when σ_a essentially equals σ^* does P become different from zero. In this limit, P is not determined by Eq. (7), but by the constraint $\sigma_a = \sigma^*$. Values of $\dot{\epsilon}^*$ for which this approximation is reasonable can be estimated using Eq. (7). For P less than the smallest practically observable strain rate of $10^{-10} \text{ sec}^{-1}$ and for σ_a within a tenth of a percent of σ^* , $\dot{\epsilon}^*$ must be less than $10^{-30} \text{ sec}^{-1}$. Values of $\dot{\epsilon}^*$ measured for nickel and 304 stainless steel at room temperature (25° C) are less than $10^{-35} \text{ sec}^{-1}$ for a typical hardness value of 20,000 psi.⁸

The constitutive relations in the low temperature or viscoplastic limit are given below.

1.) Below "yield", $\sigma_a < \sigma^*$

$$\begin{aligned} \underline{P} &= 0, \\ \underline{N} &= \dot{a}^*(\sigma_f/M)^{M-1}(\underline{\sigma}/M-\underline{a}), \end{aligned} \quad (12)$$

$$\frac{D\underline{a}}{Dt} = \underline{N}, \quad (13)$$

and,

$$\frac{D\sigma^*}{Dt} = 0. \quad (14)$$

2.) At "yield", $\sigma_a = \sigma^*$

$$\underline{P} = \underline{P} \underline{M} \underline{a} / \sigma^*, \quad (15)$$

$$\underline{N} = \dot{a}^* (\sigma_f / M)^{M-1} (\underline{\sigma} / M - \underline{a}), \quad (16)$$

$$\frac{D\underline{a}}{Dt} = \underline{N} - \underline{P}, \quad (17)$$

and,

$$\frac{D\sigma^*}{Dt} = P \sigma^* \hat{\Gamma}(\sigma^*), \quad (18)$$

where,

$$\hat{\Gamma}(\sigma^*) = \Gamma(\sigma^*, \sigma^*) = (kG/\sigma^*)^\delta. \quad (19)$$

The work hardening parameters, k and δ , as measured for Nickel at 25°C, are given in Table I.

The scalar P is determined using the constraint $\sigma_a = \sigma^*$ as follows. With Eqs. (15) and (16), Eq. (17) becomes,

$$\underline{P} \underline{M} \underline{a} / \sigma^* + \frac{D\underline{a}}{Dt} + \underline{a} \cdot \underline{W} - \underline{W} \cdot \underline{a} = \dot{a}^* (\sigma_f / M)^{M-1} (\underline{\sigma} / M - \underline{a}).$$

Taking the trace of the matrix product of this expression with $\underline{a}$ and using the fact that,

$$\text{tr } \underline{a}^2 = a^2 = (\sigma^* / M)^2,$$

the following expression results,

$$P + \frac{1}{M} \frac{D\sigma^*}{Dt} = \dot{a}^*(\sigma_f/M)^{M-1} (\text{tr } \underline{\underline{\sigma}} \cdot \underline{\underline{a}} / \sigma^* - \sigma^*/M) .$$

Using Eq. (18) then gives P in terms of the state variables, σ^* and $\underline{\underline{a}}$, and the applied deviatoric stress tensor, $\underline{\underline{\sigma}}$,

$$P = \dot{a}^*(\sigma_f/M)^{M-1} (\text{tr } \underline{\underline{\sigma}} \cdot \underline{\underline{a}} / \sigma^* - \sigma^*/M) / (1 + \sigma^* \hat{\Gamma}(\sigma^*)/M) . \quad (20)$$

C. The Elastic Strain and the Observable Deformations

The variable $\underline{\underline{N}}$ that characterizes the nonelastic strain rate in Hart's theory, is not the total deformation rate. To this the elastic deformation must be added, and $\underline{\underline{N}}$ must then be related to the observed velocity field and to the stress.

The elastic strain $\underline{\underline{e}}$ is measured with respect to the current configuration of the body and is assumed to be given in terms of the applied stress by Hooke's law.

The simplest relation between the spatial gradients of the velocity field, the rate of change of the elastic strains, and the nonelastic deformation rate which insures that the nonelastic deformation proceeds without dilatation, and which is consistent with the order of approximation already introduced is,

$$\underline{\underline{D}} = \underline{\underline{N}} + (\underline{\underline{e}} \cdot \underline{\underline{N}} + \underline{\underline{N}} \cdot \underline{\underline{e}}) + \frac{\delta \underline{\underline{e}}}{\delta t} , \quad (21)$$

where the last term is the convected derivative of the elastic strain,

$$\frac{\delta e}{\delta t} = \frac{D e}{D t} + e \cdot D + D \cdot e \quad (22)$$

The convected derivative, like the corotational derivative, is unaffected by rigid motions of the material. Convected derivatives of strain are rates of deformation.⁹

III. TORSION OF A THIN WALLED CYLINDER

In this section, the proposed constitutive relations are applied to what is perhaps the simplest multi-axial example, the deformation due to an imposed homogeneous shear stress. The resulting deformation corresponds to the quasi-static, isothermal torsion of a thin-walled, circular cylinder. The procedure used in solving this example will be to use the constitutive theory to find the explicit form of the spatial velocity gradients $\underline{V}_y$. Specifically, the velocity gradients are determined in terms of the deviatoric stress $\underline{\sigma}$, the state variables σ^* and $\underline{a}$, the material parameters M , $\dot{M}$, $\dot{a}^*$, G , and the material function $\hat{f}$ by Eq. (21). The state variables can then be determined using the appropriate rate laws, given by Eqs. (13, 14) and Eqs. (17, 18). The motion is described by the deformation gradients $\underline{F}$,

$$\underline{F} = \text{grad } \underline{x} ,$$

where $\underline{x}$ is the position of a material point initially located at $\underline{X}$, and the vector operator, grad, denotes the gradient with respect to $\underline{X}$. The deformation gradients can be determined by integrating the differential equations,

$$\frac{D\underline{F}}{Dt} = (\underline{V}_y) \cdot \underline{F} . \quad (23)$$

For this example, all quantities can be referred to a rectangular, cartesian coordinate system. The first, second and third coordinate axes correspond to the circumferential, axial and radial directions, respectively.

For this problem the applied stress deviator is a given function of time,

$$\underline{\underline{\sigma}} = \begin{Bmatrix} 0 & \sigma_{12}(t) & 0 \\ \sigma_{12}(t) & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}.$$

The corresponding deformation is uniform. Consequently, the velocity gradients have the form

$$\underline{\underline{\nabla v}} = \begin{Bmatrix} A(t) & B(t) & 0 \\ 0 & C(t) & 0 \\ 0 & 0 & D(t) \end{Bmatrix}, \quad (24)$$

where the components are functions of time to be determined by

Eq. (21). The elastic strain $\underline{\underline{e}}$ is related to the stress by Hooke's Law,

$$\underline{\underline{e}} = \frac{1}{2G} \begin{Bmatrix} 0 & \sigma_{12}(t) & 0 \\ \sigma_{12}(t) & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}. \quad (25)$$

The convected derivative of the elastic strain is given by Eq. (22).

$$\frac{\delta \underline{\underline{e}}}{\delta t} = \frac{\sigma'_{12}}{2G} \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} + \frac{\sigma_{12}}{2G} \begin{Bmatrix} 0 & A+C & 0 \\ A+C & 2B & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \quad (26)$$

where the partial derivative with respect to time is replaced by the corresponding ordinary derivative, denoted by a superposed prime. The

anelastic strain $\underline{\underline{a}}$ is a function of time and has the following components,

$$\underline{\underline{a}} = \begin{Bmatrix} a_{11}(t) & a_{12}(t) & 0 \\ a_{12}(t) & a_{22}(t) & 0 \\ 0 & 0 & a_{33}(t) \end{Bmatrix},$$

where $a_{11}(t) + a_{22}(t) + a_{33}(t) = 0$. The corotational derivatives of $\underline{\underline{a}}$ is given by Eq. (11),

$$\frac{D\underline{\underline{a}}}{Dt} = \underline{\underline{a}}' + \begin{Bmatrix} -a_{12}^B & B(a_{11}-a_{22})/2 & 0 \\ B(a_{11}-a_{22})/2 & a_{12}^B & 0 \\ 0 & 0 & 0 \end{Bmatrix}. \quad (27)$$

For convenience, the quantities $\underline{\underline{\sigma}}$, σ_f , σ^* are now replaced by the following non-dimensional variables,

$$\underline{\underline{s}} \equiv \underline{\underline{\sigma}}/M, \quad s_f \equiv \sigma_f/M, \quad s^* \equiv \sigma^*/M.$$

In terms of these quantities, the velocity gradients are given by Eq. (21), with Eq. (26) and Eq. (12) or Eq. (16).

$$A = -\dot{a}^*(s_f)^{M-1} [(M/G)s_{12}(s_{12}-a_{12}) + a_{11}], \quad (28)$$

$$B = 2\dot{a}^*(s_f)^{M-1}(s_{12}-a_{12}) + (M/G)s_{12}'[1-(Ms_{12}/G)^2]^{-1}, \quad (29)$$

$$C = \dot{a}^*(s_f)^{M-1} [(M/G)s_{12}(s_{12}-a_{12}) - a_{22}] + \left(\frac{M}{G}\right)^2 s_{12}s_{12}'[1-(Ms_{12}/G)^2]^{-1}, \quad (30)$$

$$D = -\dot{a}^*(s_f)^{M-1} a_{33}.$$

The rate laws for the state variables can now be written as a closed system of equations. In the low-temperature theory, there are the two regimes, $s^* > a$ ($\sigma^* > \sigma_a$) and $s^* = a$ ($\sigma^* = \sigma_a$).

1) $s^* > a$

Using Eqs. (12) and (27) in the evolution equations (13, 14), the following system of equations is obtained for the determination of s^* ,

$$a_{11}, a_{12}, a_{22}, a_{33}.$$

$$s^{*'} = 0,$$

$$a'_{11} = \dot{a}^*(s_f)^{M-1} [2a_{12}(s_{12}-a_{12})-a_{11}] + (M/G)a_{12}s'_{12}[1-(Ms_{12}/G)^2]^{-1},$$

$$a'_{22} = -\dot{a}^*(s_f)^{M-1} [2a_{12}(s_{12}-a_{12})+a_{22}] - (M/G)a_{12}s'_{12}[1-(Ms_{12}/G)^2]^{-1},$$

$$a'_{33} = -\dot{a}^*(s_f)^{M-1} a_{33},$$

and

$$a'_{12} = \dot{a}^*(s_f)^{M-1} (s_{12}-a_{12})(1-a_{11}+a_{22}) - (M/2G)(a_{11}-a_{22})s'_{12}[1-(Ms_{12}/G)^2]^{-1},$$

where

$$(s_f)^2 = 2(s_{12}-a_{12})^2 + a_{11}^2 + a_{22}^2 + a_{33}^2.$$

Initially, the components of $\underline{a}$ are zero and s^* has some positive value. Since a_{33} is initially zero, it will remain zero for all time. Since $\underline{a}$ is deviatoric, then $a_{11} = -a_{22}$. Using this result allows us to eliminate one equation and simplify the others. The original system of equation then reduces to the following two equations,

$$a'_{11} = \dot{a}^*(s_f)^{M-1} [2a_{12}(s_{12}-a_{12})-a_{11}] + (M/G)a_{12}s'_{12}[1-(Ms_{12}/G)^2]^{-1}, \quad (31)$$

and

$$a'_{12} = \dot{a}^*(s_f)^{M-1} (s_{12}-a_{12})(1-2a_{11}) - (M/G)a_{11}s'_{12}[1-(Ms_{12}/G)^2]^{-1}, \quad (32)$$

where

$$(s_f)^2 = 2(s_{12} - a_{12})^2 + 2a_{11}^2. \quad (33)$$

For $s_{12} = \text{constant}$, the system has fixed points at $a_{11} = 0$, $a_{12} = s_{12}$, stable in the sense that perturbations to the solution at this point decrease exponentially with time.

2) $s^* = a$

The system of equations valid in this case are obtained in a similar manner.

$$s^{*'} = Ps^*\hat{f}(s^*), \quad (34)$$

$$a'_{11} = \dot{a}^*(s_f)^{M-1} [2a_{12}(s_{12} - a_{12}) - a_{11}] + (M/G)a_{12}s'_{12} [1 - (Ms_{12}/G)^2]^{-1} - Pa_{11}/s^*, \quad (35)$$

$$a'_{12} = \dot{a}^*(s_f)^{M-1} (s_{12} - a_{12})(1 - 2a_{11}) - (M/G)a_{11}s'_{12} [1 - (Ms_{12}/G)^2]^{-1} - Pa_{12}s^*, \quad (36)$$

where s_f is given by Eq. (33). The scalar, P , is given by Eq. (20),

$$P = \dot{a}^*(s_f)^{M-1} [2s_{12}a_{12}/s^* - s^*] [1 + s^*\hat{f}(s^*)]^{-1}. \quad (37)$$

For $s_{12} = \text{constant}$, this system of equations has a stable fixed point at $s^* = \sqrt{2}s_{12}$, $a_{12} = s_{12}$, $a_{11} = 0$. However, the situation changes if the work hardening parameter k , in Eq. (19), is zero. For a material that does not work harden, the hardness is constant and the reduced system of equations has two distinct sets of fixed points. To leading order in an s^* expansion for these points, $a_{11} = (s^*)^2(1 - a_{12}/s_{12})$ and $a_{12} = \pm s^*/\sqrt{2}$. For $s_{12} > 0$, the point associated with $a_{12} > 0$ is stable while the point associated

with $a_{12} < 0$ is unstable. For $s_{12} < 0$, the situation is reversed. The presence of stable, non-zero values of a_{11} manifests itself in a non-vanishing deformation rate at constant stress.

The two systems of differential equations, Eqs. (31, 32) and Eqs. (34, 35, 36) can be integrated numerically for different stress histories. The deformation is calculated by simultaneous integration of Eq. (23) which reduces to the following equations for this example,

$$\begin{aligned} F'_{11} &= A(t)F_{11} , \\ F'_{12} &= A(t)F_{12} + B(t)F_{22} , \\ F'_{22} &= C(t)F_{22} , \end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$ are obtained from Eqs. (28, 29, 30). The deformation gradients are related to the shear strain $\tan \gamma$, angle of twist ϕ , logarithmic axial strain ϵ_2 , and logarithmic circumferential strain ϵ_1 by the following equations,

$$\begin{aligned} \tan \gamma &= F_{12}/F_{22} , \\ \phi &= (L_0/R_0)F_{12}/F_{11} , \\ \epsilon_1 &= \ln(R/R_0) = \ln(F_{11}) , \\ \epsilon_2 &= \ln(L/L_0) = \ln(F_{22}) , \end{aligned} \tag{38}$$

where R , L are the radius and length of the tube and R_0 , L_0 are the initial values of these dimensions.

A system of differential equations which predicts the deformation gradients, stress, and the state variables for different angle of twist

histories is needed for comparison with existing experimental work. Using Eq. (38) and its time derivative, the following expressions are obtained

$$F_{12} = R_o \phi F_{11} / L_o ,$$

$$v_{1,2} = R_o \phi' F_{11} / (L_o F_{22}) ,$$

where ϕ is a specified functions of time. Using these results, a system of differential equations is obtained for the determination of F_{11} , F_{22} , s_{12} , a_{11} , a_{12} , and s^* in the two regimes, $s^* > a$ and $s^* = a$.

1) $s^* > a$

$$a'_{11} = R_o \phi' F_{11} a_{12} / (L_o F_{22}) - \dot{a}^*(s_f)^{M-1} a_{11} ,$$

$$a'_{12} = \dot{a}^*(s_f)^{M-1} (s_{12} - a_{12}) - R_o \phi' F_{11} a_{11} / (L_o F_{22}) ,$$

$$s^{*'} = 0 ,$$

$$F'_{11} = -\dot{a}^*(s_f)^{M-1} [(M/G) s_{12} (s_{12} - a_{12}) + a_{11}] F_{11} ,$$

$$F'_{22} = \dot{a}^*(s_f)^{M-1} [(M/G) s_{12} (s_{12} - a_{12}) + a_{11}] F_{22} + (M/G)^2 s_{12} s'_{12} F_{22} [1 - (M s_{12} / G)^2]^{-1}$$

$$s'_{12} = (G/M) [1 - (M s_{12} / G)^2] [R_o \phi' F_{11} / (L_o F_{22}) - 2 \dot{a}^*(s_f)^{M-1} (s_{12} - a_{12})] ,$$

where s_f is given by Eq. (33). For $\phi = \text{constant}$, this system has a stable fixed point at $a_{11} = 0$, $a_{12} = s_{12}$.

2) $s^* = a$

$$a'_{11} = R_o \phi' F_{11} a_{12} / (L_o F_{22}) - \dot{a}^*(s_f)^{M-1} a_{11} - P a_{11} / s^*$$

$$a'_{12} = \dot{a}^*(s_f)^{M-1} (s_{12} - a_{12}) - R_o \phi' F_{11} a_{11} / (L_o F_{22}) - P a_{12} / s^*$$

$$s^{*'} = P s^* \hat{\Gamma}(s^*) ,$$

where the remaining equations are unchanged and P is given by Eq. (37). For $\phi = \text{constant}$, this system has a stable fixed point at $s^* = \sqrt{2}s_{12}$, $a_{12} = s_{12}$, $a_{11} = 0$. However, the system has no fixed points if the work hardening parameter k is zero.

The above set of equations were integrated numerically for several angle of twist histories using the material parameters for Nickel as given in Table I. A second order predictor-corrector scheme was used for this purpose. The results for a constant twist rate are shown in Fig. 1. These curves are qualitatively similar to results described by Bailey, Hass and Nawab² for constant twist rate tests. The simulation of a cyclic twist test is shown in Fig. 2. These curves exhibit the same characteristics as the cyclic tests described by Swift¹.

"Elastic" tests can be simulated by using an angle of twist history which leaves $s^* > a$. Results from a half cycle twist test of this type are shown in Fig. 3. Note in Fig. 3b that the axial and circumferential strain did not recover completely by the end of the test. This is due in part to the fact that the anelastic strain components had not returned to their initial value by the end of the simulation. Of particular interest is the comparison in Fig. 3c of the axial strain versus shear strain for the loading part of the test with the parabola, $\epsilon_2 = 0.265(\tan \gamma)^2$, indicated by the dotted line. Foux¹⁰ and Poynting¹¹ obtained similar results from their measurements of elongation in torsion of solid wires.

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TABLE I

Measured values of material parameters for pure Nickel at 25° C

$$\dot{a}^* = 1.37 \times 10^{27} \text{ sec}^{-1}$$

$$\lambda = 0.15$$

$$M = 7.6$$

$$M = 1.03 \times 10^7 \text{ psi}$$

$$G = 1.1 \times 10^7 \text{ psi}$$

$$k = 9.5 \times 10^{-3}$$

$$\delta = 1.64$$

Captions

- Figure 1. Numerical simulation of a constant twist rate test ($\phi' = 3 \times 10^{-4}$ rad/sec) on a thin-walled, circular cylinder of Nickel at 25°C with initial dimensions $R_o = 0.175$ in. and $L_o = 0.125$ in. and initial hardness $\sigma^* = 10,000$ psi. (a) Angle of twist (ϕ) and shear stress (σ_{12}) versus time. (b) Axial (ϵ_2) and circumferential (ϵ_1) strain versus angle of twist.
- Figure 2. Numerical simulation of a cyclic twist test ($\phi(t) = 0.5 \sin[0.021 t]$) on a thin-walled, circular cylinder of Nickel at 25°C with initial dimensions $R_o = 0.375$ in. and $L_o = 2.0$ in. and initial hardness $\sigma^* = 10,000$ psi. (a) Angle of twist (ϕ) and shear stress (σ_{12}) versus time. (b) Axial (ϵ_2) and circumferential (ϵ_1) strain versus angle of twist.
- Figure 3. Numerical simulation of an "elastic", half cycle, twist test ($\phi(t) = 0.017[1 - \exp(-0.05t)]$; $0 \leq t \leq 300$ sec) on a thin-walled, circular cylinder of Nickel at 25°C with initial dimensions $R_o = 1.5$ in. and $L_o = 12.5$ in. and initial hardness $\sigma^* = 10,000$ psi. (a) Angle of twist (ϕ) versus time. (b) Axial (ϵ_2) and circumferential (ϵ_1) strain versus shear strain ($\tan\gamma$). (c) Axial strain (for $0 \leq t \leq 300$ sec) versus shear strain compared to the parabola, $\epsilon_2 = 0.265 (\tan\gamma)^2$.

Figure 1a. Angle of twist (ϕ) and shear stress (σ_{12}) versus time.

Figure 2b. Axial (ϵ_1) and circumferential (ϵ_2) strain versus angle of twist (ϕ).

Figure 2a. Angle of twist (ϕ) and shear stress (σ_{12}) versus time.

Figure 1b. Axial (ϵ_1) and circumferential (ϵ_2) strain versus angle of twist (ϕ).

Figure 3a. Angle of twist(ϕ) versus time.

Figure 3b. Axial (ϵ_2) and circumferential (ϵ_1) strain versus shear strain (γ_{xy}).

Figure 3c. Axial strain (for $0 < t < 300$ sec) versus shear strain compared to the parabola, $\epsilon_2 = 0.265 (\tan \gamma)^2$.