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ERROR ANALYSIS OF THE CHIRP-Z TRANSFORM WHEN IMPLEMENTED
USING WAVEFORM SYNTHESIZERS AND FFTS

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ABSTRACT

This report analyzes the effects of finite-precision arithmetic on discrete Fourier transforms (DFTs) calculated using the chirp-z transform algorithm. An introduction to the chirp-z transform is given together with a description of how the chirp-z transform is implemented in hardware. Equations for the effects of chirp rate errors, starting frequency errors, and starting phase errors on the frequency spectrum of the chirp-z transform are derived. Finally, the maximum possible errors in the chirp rate, the starting frequencies, and starting phases are calculated and used to compute the worst case effects on the amplitude and phase spectrums of the chirp-z transform.

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Introduction to the Chirp-z Transform

"The chirp-z transform is an efficient algorithm for evaluating the z transform of a finite-duration sequence along certain general contours in the z plane" [1]. It is derived by evaluating the z transform, $X(z)$, of a given N-point sequence, $x(n)$, where

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \quad (1)$$

along the general contour

$$z = AW^{-k}, \quad k = 0, 1, \dots, M-1, \quad (2)$$

where

$$A = A_0 e^{j2\pi\theta_0} \quad (3)$$

$$W = W_0 e^{-j2\pi\phi_0}. \quad (4)$$

In [1], the minus sign is missing from equation (4). The minus sign is necessary so that the contour defined in (2) will rotate in a counter-clockwise direction for positive values of ϕ_0 .

Figure 1 shows a sample contour in the z plane indicating the significance of A_0 , W_0 , θ_0 , and ϕ_0 . As shown, A_0 determines the initial radius of the contour, W_0 determines whether the contour spirals in, out, or has a constant radius, ϕ_0 determines the sample spacing and θ_0 determines the fractional frequency of the first output

sample where fractional frequency is frequency normalized by the sampling frequency.

Figure 1. Contour for CZT in the z Plane.

If we let X_k be the z transform at $z = z_k$, i.e.,

$$X_k = \sum_{n=0}^{N-1} x(n) z_k^{-n}, \quad (5)$$

then using (2), (3), and (4) we find

$$X_k = W^{k^2/2} \sum_{n=0}^{N-1} y(n) v(k-n), \quad (6)$$

where

$$y(n) = x(n) A^{-n} W^{n^2/2} \quad (7)$$

$$v(n) = W^{-n^2/2} \quad (8)$$

The details of obtaining equation (6) from equation (5) are given in [1].

By restricting the contour defined in (2) to the unit circle, for which $A_0 = W_0 = 1$, the chirp-z transform can be used to calculate the DFT of a sequence as is done in the signal processing of the Strip radar. Although the chirp-z transform is not as computationally efficient as the FFT, it can calculate DFTs anywhere on the unit circle with a different number of input and output points. It is for these reasons that a chirp-z transform is used in the fine-resolution azimuth processing of the Strip radar.

Implementation of the Chirp-z Transform

Equation (6) shows that X_k may be calculated as the convolution of the sequences $y(n)$ and $v(n)$ and thus may be evaluated efficiently by high speed convolution using FFTs as shown in Figure 2.

To implement the convolution in (6) with FFTs, $v(n)$ is generated as two separate chirps as shown in Figure 3 and equation (9).

Figure 2. Implementation of the Chirp-z Transform Algorithm.

Figure 3. $v(n)$ Waveform.

$$v(n) = \begin{cases} W^{-n^2/2} & 0 \leq n \leq M - 1 \\ W^{-(L-n)^2/2} & L - N + 1 \leq n \leq L - 1 \\ \text{arbitrary} & \text{other } n, \text{ if any} \end{cases} \quad (9)$$

In (9), N is the number of input samples, M is the number of output samples, and L is the smallest power of two greater than or equal to $(N+M-1)$. The values of $v(n)$ labeled as "arbitrary" in (9) are labeled as such because they do not contribute to the first $M-1$ values of X_k .

The four chirps needed in the chirp-z transform algorithm, $A^{-n} W^{n^2/2}$, $W^{-n^2/2}$, $W^{-(L-n)^2/2}$, and $W^{k^2/2}$ are all generated digitally with the hardware configuration in Figure 4. As shown, the chirp rate, the starting frequency, and the starting phase need to be specified to generate a chirp. The required chirp rate, starting frequency, and starting phase for the four chirps used in the chirp-z transform algorithm can be calculated by substituting equations (3) and (4) into the chirp equations.

Figure 4. Digital Chirp Synthesizer.

For the $A^{-n} W^{n^2/2}$ chirp,

$$A^{-n} W^{n^2/2} = e^{-j2\pi n\theta_0} e^{-j\pi\phi_0 n^2}, \quad (10)$$

for which

$$\text{phase} = -\pi\phi_0 n^2 - 2\pi n\theta_0, \quad (11)$$

$$\text{frequency} = \frac{d\text{phase}}{dn} = -2\pi n\phi_0 - 2\pi\theta_0, \quad (12)$$

$$\text{chirp rate} = \frac{d\text{frequency}}{dn} = -2\pi\phi_0. \quad (13)$$

The starting phase and the starting frequency for the $A^{-n} W^{n^2/2}$ chirp, determined by setting $n = 0$ in (11) and (12), are 0 and $-2\pi\theta_0$, respectively.

For the $W^{-n^2/2}$ chirp,

$$W^{-n^2/2} = e^{j\pi\phi_0 n^2}, \quad (14)$$

for which

$$\text{phase} = \pi\phi_0 n^2 \quad (15)$$

$$\text{frequency} = \frac{d\text{phase}}{dn} = 2\pi\phi_o n \quad (16)$$

$$\text{chirp rate} = \frac{d\text{frequency}}{dn} = 2\pi\phi_o \quad (17)$$

The starting phase and the starting frequency for the $W^{-n^2/2}$ chirp, evaluated by setting $n = 0$ in (15) and (16), are both zero.

For the $W^{-(L-n)^2/2}$ chirp,

$$W^{-(L-n)^2/2} = e^{j\pi\phi_o(L-n)^2}, \quad (18)$$

for which

$$\text{phase} = \pi\phi_o(L-n)^2 \quad (19)$$

$$\text{frequency} = \frac{d\text{phase}}{dn} = -2\pi\phi_o(L-n) \quad (20)$$

$$\text{chirp rate} = \frac{d\text{frequency}}{dn} = 2\pi\phi_o \quad (21)$$

The starting phase for the $W^{-(L-n)^2/2}$ chirp, evaluated from (19) for $n = (L-N+1)$, is $\pi\phi_o(N-1)^2$. The starting frequency, evaluated from (20) for $n = (L-N+1)$, is $-2\pi\phi_o(N-1)$.

For the $W^{k^2/2}$ chirp,

$$W^{k^2/2} = e^{-j\pi\phi_o k^2}, \quad (22)$$

for which

$$\text{phase} = -\pi\phi_o k^2 \quad (23)$$

$$\text{frequency} = \frac{d\text{phase}}{dk} = -2\pi\phi_o k \quad (24)$$

$$\text{chirp rate} = \frac{df_{\text{frequency}}}{dk} = -2\pi\phi_o \quad (25)$$

The starting phase and the starting frequency for the $W^{k^2/2}$ chirp, evaluated by setting $k = 0$ in (23) and (24), are both zero.

Two additional changes are made to the chirp rate, starting frequency, and starting phase equations listed above. First, 2π is factored out and is incorporated in the sine and cosine ROMs shown in Figure 4. Second, the starting frequency equations are modified to account for the frequency offset inherent in the digital chirp synthesizer as shown below.

The required starting frequency offset in the digital chirp synthesizer can be calculated as follows. First, let $F_{out}(n)$ be the output of the frequency accumulator and $P_{out}(n)$ be the output of the phase accumulator as shown in Figure 4. Then, for the two chirps in which the exponent of W has a negative sign, $F_{out}(n)$ and $P_{out}(n)$ can be calculated from equations (26) and (27), respectively.

$$F_{out}(n) = n\phi_o + \theta_o \quad (26)$$

$$P_{out}(n) = \psi_o + \sum_{i=0}^{n-1} [F_{out}(i)] \quad (27)$$

where ψ_o , ϕ_o , and θ_o are defined in Figure 4.

Substituting (26) into (27) and evaluating the summations yields

$$P_{out}(n) = \psi_o + \sum_{i=0}^{n-1} [F_{out}(i)] = \psi_o + \sum_{i=0}^{n-1} (\theta_o + i(\phi_o))$$

$$\begin{aligned}
&= \psi_0 + \sum_{i=0}^{n-1} (\theta_0) + \sum_{i=0}^{n-1} i(\phi_0) = \psi_0 + (n)\theta_0 + \frac{n(n-1)}{2} (\phi_0) \\
&= \psi_0 + n \left[\theta_0 - \frac{\phi_0}{2} \right] + n^2 \left[\frac{\phi_0}{2} \right] .
\end{aligned} \tag{28}$$

As shown in (28), the starting phase of the phase accumulator output is ψ_0 , the starting frequency is $\theta_0 - \phi_0/2$ and the chirp rate is ϕ_0 . Notice the additional factor of $-\phi_0/2$ in the starting frequency. To compensate for this factor, $\phi_0/2$ must be added to the starting frequency equations for the two chirps in which the exponent of W has a negative sign. Conversely, for the chirps in which the exponent of W has a positive sign, $\phi_0/2$ must be subtracted from the starting frequency equation.

Table 1 summarizes the required chirp rates, starting frequencies, and starting phases for the four chirps used in the chirp-z transform algorithm. In Table 1, the chirp rate is in units of fractional frequency per output sample, the starting frequency is in units of fractional frequency, and the starting phase is in units of radians normalized by 2π .

Table 1. Chirp Rate, Starting Frequency and Starting Phase for the Chirp-z Transform Chirps.

<u>Chirp</u>	<u>Chirp Rate</u>	<u>Starting Frequency</u>	<u>Starting Phase</u>
$A^{-n} W^{n^2/2}$	$-\phi_0$	$-\phi_0/2 - \theta_0$	0
$W^{-n^2/2}$	ϕ_0	$\phi_0/2$	0
$W^{-(L-n)^2/2}$	ϕ_0	$\phi_0/2 - (N-1)\phi_0$	$0.5(N-1)^2\phi_0$
$W^{k^2/2}$	$-\phi_0$	$-\phi_0/2$	0

Derivation of Error Equations

At this point, equations for the effects of chirp rate errors, starting frequency errors, and starting phase errors on the amplitude and phase spectrum of the chirp-z transform will be derived. Errors in these parameters arise due to the finite number of bits used to represent them as shown in Figure 4. Chirp rate errors and starting frequency errors in the $A^{-n} W^{n^2/2}$, the $W^{-n^2/2}$, and the $W^{-(L-n)^2/2}$ chirps will be considered first since they affect both the amplitude and phase spectrums. Next, starting frequency errors in the $W^{k^2/2}$ chirp, which affect only the phase spectrum of the chirp-z transform, will be analyzed. Finally, the effects of starting phase errors will be considered.

Chirp Rate Errors

The first type of error to be considered will be a chirp rate error. As shown in Table 1, the magnitude of the chirp rate for all four chirps is equal to ϕ_0 . Notice that ϕ_0 is also the output sample spacing as shown in Figure 1. Thus, we reach the important conclusion that a chirp rate error is equivalent to an error in the output sample spacing, ϕ_0 . For the above conclusion to be valid, the same magnitude of ϕ_0 , including any truncation or round off errors, must be used everywhere it appears in Table 1. This requires that $-\phi_0$ be computed by first truncating or rounding ϕ_0 and then negating the result.

If the above condition is satisfied, the frequency translation in the output spectrum of the chirp-z transform, due to a chirp rate error, can be calculated as follows. In Figure 1, the output sample fractional frequency corresponding to frequency bin k can be calculated from (29).

$$\begin{array}{l} \text{Output Sample} \\ \text{Fractional Frequency} \end{array} = k(\phi_o) + \theta_o, \quad (29)$$

where ϕ_o and θ_o are defined in Figure 1.

If there is an error in ϕ_o , the output sample fractional frequency which corresponded to frequency bin k in (29), will now correspond to frequency bin k_e as shown in (30).

$$\begin{array}{l} \text{Output Sample} \\ \text{Fractional Frequency} \end{array} = k_e(\phi_o + \phi_{\text{error}}) + \theta_o, \quad (30)$$

where ϕ_{error} is the error in ϕ_o due to truncation or rounding.

Setting the output sample fractional frequency in (29) equal to that in (30) and solving for the difference between k_e and k results in equation (31), an expression for determining the frequency translation in bins that occurs because of an error in ϕ_o .

$$\begin{array}{l} \text{frequency translation} \\ \text{due to a chirp rate error} \end{array} = k_e - k = k \left(\frac{\phi_o}{\phi_o + \phi_{\text{error}}} - 1 \right) \quad (31)$$

The bandwidth distortion due to a chirp rate error will be calculated next. The bandwidth, in fractional frequency, can be calculated from equation (32) independent of the amplitude level used to specify the bandwidth.

$$\text{bandwidth} = (k_2 - k_1) \phi_o \quad (32)$$

where k_2 and k_1 are the bin numbers of the upper and lower edges of the bandwidth.

Taking the ratio of (32) with an error in ϕ_o to (32) without an error in ϕ_o and substituting in equation (31), one obtains

$$\begin{aligned} \text{bandwidth ratio} &= \frac{(k_{e2} - k_{e1})\phi_o}{(k_2 - k_1)\phi_o} = \frac{(k_2 - k_1) \frac{\phi_o^2}{\phi_o + \phi_{\text{error}}}}{(k_2 - k_1)\phi_o} \\ &= \frac{\phi_o}{\phi_o + \phi_{\text{error}}} \end{aligned} \quad (33)$$

where the bandwidth ratio given by (33) is valid for any given amplitude used to define the measurement of bandwidth. Note that the entire frequency spectrum of the chirp-z transform, including nulls and sidelobes, will be broadened/compressed by the ratio given in (33).

Starting Frequency Errors

The effect of starting frequency errors in the $A^{-n} W^{n^2/2}$, the $W^{-n^2/2}$, and the $W^{-(L-n)^2/2}$ chirps will be considered next. Notice that ϕ_o appears in the formulas given in Table 1 for calculating the starting frequencies. Since an error in ϕ_o has already been accounted for in the formulas presented above, the following analysis is for starting frequency errors that are not due to errors in ϕ_o .

A starting frequency error can be modeled as an extra factor of $e^{j2\pi(n)(f_{\text{error}})}$. Therefore, a starting frequency error in the $A^{-n} W^{n^2/2}$ chirp can be simulated by multiplying the right side of (7) by $e^{j2\pi(n)(f_{\text{error}})}$. Substituting this new value of (7) into (6) gives

$$X_k = W^{k^2/2} \sum_{n=0}^{N-1} z(n) A^{-n} W^{n^2/2} v(k-n) , \quad (34)$$

where

$$z(n) = x(n) e^{j2\pi(n)(f_{\text{error}})} \quad (35)$$

As shown in (35), $z(n)$ is just the original $x(n)$ translated in frequency by a factor of f_{error} . Therefore, the chirp-z transform of $z(n)$ is just the chirp-z transform of $x(n)$ translated in frequency by f_{error} . This corresponds to a frequency translation of

$$\text{frequency translation due to a starting frequency error} = k_e - k = \frac{f_{\text{error}}}{\phi_0} \text{ bins.} \quad (36)$$

A starting frequency error in $v(n)$ can be analyzed in the same way if the starting frequency error is the same in both parts of $v(n)$. This is necessary to avoid the introduction of a frequency discontinuity in $v(n)$. As will be shown later, this condition is satisfied.

Therefore, a starting frequency error in $v(n)$ can be simulated by multiplying the right side of (8) by $e^{j2\pi(n)(f_{\text{error}})}$. Substituting this new value of (8) into (6) gives

$$X_k = W^{k^2/2} \sum_{n=0}^{N-1} y(n) v(k-n) e^{j2\pi(k-n)f_{\text{error}}} \quad (37)$$

Since k is independent of n , the factor $e^{j2\pi(k)(f_{\text{error}})}$ can be taken outside of the summation as shown in (38).

$$X_k = W^{k^2/2} e^{j2\pi(k)(f_{\text{error}})} \sum_{n=0}^{N-1} y(n) e^{-j2\pi(n)(f_{\text{error}})} v(k-n) \quad (38)$$

As before, the effect of a starting frequency error in $v(n)$ is to shift the amplitude and phase spectrum of the chirp-z transform of $x(n)$ by a factor of $-f_{\text{error}}$. The frequency translation in bins is given by (36) with f_{error} replaced by $-f_{\text{error}}$. Also, the factor of $e^{j2\pi(k)(f_{\text{error}})}$ that appears in front of the summation will cause an additional error in the phase spectrum that increases linearly with the number of the output sample.

Since a chirp rate error and a starting frequency error in the $A^{-n} W^{n^2/2}$, the $W^{-n^2/2}$, and the $W^{-(L-n)^2/2}$ chirps both cause a frequency translation of the amplitude and phase spectrums, their effects can be combined into one equation as has been done in (39).

$$\begin{aligned} \text{total frequency} &= k_e - k = k \left(\frac{\phi_o}{\phi_o + \phi_{\text{error}}} - 1 \right) \\ \text{translation} &+ \frac{f_{s1\text{error}} - f_{s2\text{error}}}{\phi_o + \phi_{\text{error}}} \text{ bins,} \end{aligned} \quad (39)$$

where $f_{s1\text{error}}$ is the starting fractional frequency error in the $A^{-n} W^{n^2/2}$ chirp and $f_{s2\text{error}}$ is the starting fractional frequency error in the $W^{-n^2/2}$ and the $W^{-(L-n)^2/2}$ chirps.

The errors analyzed above affect both the amplitude and phase spectrums of the chirp-z transform. Next, a starting frequency error in the $W^{k^2/2}$ chirp, which affects only the phase spectrum, will be considered. A starting frequency error in the $W^{k^2/2}$ chirp can be analyzed by multiplying (6) by a factor of $e^{j2\pi(k)(f_{\text{error}})}$ as shown in (40).

$$X_k = W^{k^2/2} e^{j2\pi(k)(f_{\text{error}})} \sum_{n=0}^{N-1} y(n) v(k-n) \quad (40)$$

Since the magnitude of $e^{j2\pi(k)(f_{\text{error}})}$ is one, a starting frequency error in the $W^{k^2/2}$ chirp will not affect the amplitude spectrum but will cause an error in the phase spectrum that increases linearly with the output sample number. Notice that a starting frequency error in $v(n)$ causes the same type of error in the phase spectrum as shown in (38).

Starting Phase Errors

Finally, the effect of a starting phase error will be considered. As shown in Table 1, the starting phase of the $A^{-n} W^{n^2/2}$ chirp, the $W^{-n^2/2}$ chirp, and the $W^{k^2/2}$ chirp is zero which can be represented with no error. The starting phase of the $W^{-(L-n)^2/2}$ chirp is non-zero, however, and could have an error due to truncation or rounding. Also, the starting phase calculation for the $W^{-(L-n)^2/2}$ chirp does not take into account a possible starting frequency error in the $W^{-n^2/2}$ and the $W^{-(L-n)^2/2}$ chirps. Therefore, a starting frequency error in these chirps will cause a starting phase error in the $W^{-(L-n)^2/2}$ chirp. The magnitude of the error can be calculated by adding a starting frequency error of f_{error} to (20).

$$\text{frequency of } W^{-(L-n)^2/2} = \frac{d\text{phase}}{dn} = -2\pi\phi_o(L-n) + 2\pi(f_{\text{error}}) \quad (40)$$

Integrating (40) with respect to n gives

$$\text{phase of } W^{-(L-n)^2/2} = \pi\phi_o(L-n)^2 - 2\pi(L-n)f_{\text{error}} \quad (41)$$

The new starting phase of the $W^{-(L-n)^2/2}$ chirp, evaluated from (41) for $n = (L-N+1)$, is $2\pi(0.5(N-1)^2\phi_o - (N-1)f_{\text{error}})$. Therefore, not including the f_{error} term causes a starting phase error of $(N-1)f_{\text{error}}$ if the 2π is factored out. Since the $W^{-(L-n)^2/2}$ chirp and the $W^{-n^2/2}$ chirp are two parts of the same chirp, $v(n)$, a starting phase error in only the $W^{-(L-n)^2/2}$ chirp will cause a discontinuity in $v(n)$. Deriving an equation to predict the effect of the discontinuity is difficult. However, for the range of values of N and f_{error} that are encountered in the Strip radar, the discontinuity in $v(n)$ is so small that its effect on the amplitude and phase spectrums of the chirp-z transform can be ignored. This will be shown in the next section.

Derivation of Maximum Possible Errors

Now that formulas for the effects of chirp rate, starting frequency, and starting phase errors have been derived, an analysis will be done on the maximum possible errors.

As shown in Figure 4, only a finite number of bits are available to represent the chirp rate, the starting frequency, and the starting phase. Therefore, these quantities must be either truncated or rounded to the appropriate number of bits. For a system in which negative numbers are represented in two's complement notation, i.e. our case, binary truncation will always result in a number which is less than or equal to the original number. However, if a negative number is computed by negating and truncating a positive number, the sign of the error will depend on whether negation is done before or after truncation. For example, if a quantity like $-\theta_0$ is computed by first negating θ_0 and then truncating, the error in the resulting number will be less than or equal to zero. If, however, θ_0 is truncated first and then negated, the error in the resulting number will be greater than or equal to zero. Although this same phenomenon also occurs with negation and rounding, the change of sign of the error is usually not noticeable since rounding a number can generally produce either a positive or a negative error.

Bandwidth Distortion

Having said the above, the maximum bandwidth distortion caused by chirp rate errors can now be calculated. In Figure 4, the chirp rate, ϕ_0 , will either be rounded or truncated to 24 bits. Since ϕ_0 will always be greater than zero in our application, there will be no ambiguity on the sign of the error produced by truncating ϕ_0 . Therefore, for 24-bit truncation, $-2^{-24} < \phi_{\text{error}} \leq 0$. For 24-bit rounding, $-2^{-25} < \phi_{\text{error}} \leq 2^{-25}$. For both rounding and truncation, the maximum bandwidth

distortion, calculated from equation (33), will occur for ϕ_0 as small as possible, where ϕ_0 is calculated from (42).

$$\phi_0 = \frac{0.375}{M_{\max} - 1} \quad (42)$$

As shown in (42), the smallest ϕ_0 will occur for the largest $M_{\max}$. Assuming the largest $M_{\max}$ is 82 gives a ϕ_0 of 4.6296×10^{-3} . By substituting this value of ϕ_0 and the range of values of ϕ_{error} computed above into (33), the maximum bandwidth distortion for truncation and rounding can be computed. For 24-bit truncation of the chirp rate, $1 \leq \text{bandwidth ratio} \leq 1.0000128$ while for 24-bit rounding, $0.99999356 \leq \text{bandwidth ratio} \leq 1.0000064$. Note that truncation always produces a bandwidth broadening while rounding can produce a bandwidth broadening or a bandwidth compression.

If the specifications for the Strip radar require that the bandwidth ratio calculated from (33) be between 0.999 and 1.001, the corresponding upper and lower limits on ϕ_{error} , calculated from (33), are 4.6343×10^{-6} and -4.625×10^{-6} . Since the magnitude of the lower limit is smaller, the magnitude of the error produced by truncation or rounding must be less than 4.625×10^{-6} . For truncation, this requires a minimum of 18 bits of resolution since $2^{-18} < 4.625 \times 10^{-6} < 2^{-17}$. For rounding, only 17 bits of resolution are required.

Frequency Translation

To calculate the maximum frequency translation from (39), the maximum starting frequency errors, $f_{s1\text{error}}$ and $f_{s2\text{error}}$, are needed. The maximum value of $f_{s1\text{error}}$, which is the starting frequency error in the $A^{-n} W^{n^2/2}$ chirp, will be calculated first. Note that a factor of $-\phi_0/2$ appears in the starting frequency equation for the $A^{-n} W^{n^2/2}$ chirp.

Since ϕ_0 has already been either truncated or rounded to 24 bits, the additional error introduced by rounding or truncating $-\phi_0/2$ will be $\pm 2^{-25}$ with the sign of the error determined by how $-\phi_0/2$ is computed. To keep the magnitude of ϕ_0 constant everywhere, however, $-\phi_0/2$ must be computed by first negating the truncated or rounded ϕ_0 , then dividing by two and finally rounding or truncating the result. With this procedure, the additional error introduced by $-\phi_0/2$ is -2^{-25} for truncation and 2^{-25} for rounding. The starting frequency equation for the $A^{-n} W^{n^2/2}$ chirp also contains a factor of $-\theta_0$. For 24-bit truncation of $-\theta_0$, the maximum error is $\pm 2^{-24}$ with the sign being determined by how $-\theta_0$ is calculated. For 24-bit rounding, however, the maximum error in $-\theta_0$ will be $\pm 2^{-25}$ independent of how $-\theta_0$ is calculated. Therefore, for 24-bit truncation of the starting frequency of the $A^{-n} W^{n^2/2}$ chirp, the two selectable maximum values of $f_{s1error}$ are $-2^{-25} + 2^{-24}$, and $-2^{-25} - 2^{-24}$. For 24-bit rounding, the maximum value of $f_{s1error}$, which is not selectable, is $2^{-25} \pm 2^{-25}$.

The maximum value of $f_{s2error}$, which is the starting frequency error of both the $W^{-n^2/2}$ chirp and the $W^{-(L-n)^2/2}$ chirp, will be calculated next. The starting frequency equations for the $W^{-n^2/2}$ and the $W^{-(L-n)^2/2}$ chirps both contain a factor of $\phi_0/2$ which has a maximum error of -2^{-25} for truncation or 2^{-25} for rounding as explained above. Since $\phi_0/2$ is always greater than zero in our application, there is no ambiguity in the sign of the truncation or the rounding error. The $W^{-(L-n)^2/2}$ chirp also contains an additional factor of $-(N-1)\phi_0$. However, since $(N-1)$ is an integer and ϕ_0 has already been truncated to 24 bits, the factor $-(N-1)\phi_0$ will not introduce any more error. Therefore, the maximum value of $f_{s2error}$ is -2^{-25} for truncation and 2^{-25} for rounding.

Since the sign and magnitude of $f_{s1error}$ can be altered by the radar designer if truncation is used, it is worthwhile to calculate the value

of $f_{s1error}$ which will minimize the maximum frequency translation calculated from (39). For 24-bit truncation of the chirp rate and the starting frequencies, the optimum value of $f_{s1error}$ can be calculated as follows. The first term on the right hand side of (39) is the frequency translation due to a chirp rate error. The magnitude of this term depends on k and its sign depends on the ratio $\phi_0/(\phi_0 + \phi_{error})$, which is just the bandwidth ratio calculated in (33). Since $k=0, 1, \dots, M-1$, where M is the number of output samples, the largest value of k will be $M_{max}-1$ and the minimum value will be zero. As mentioned above, M_{max} was assumed to be 82 which gives a maximum k of 81. Substituting in the maximum value of k of 81 and the maximum bandwidth ratio of 1.0000128 into the first term on the right hand side of (39) yields

$$\begin{aligned} \text{frequency translation} \\ \text{due to a chirp rate error} &= k[(\text{bandwidth ratio}) - 1] \\ &= (81)[1.0000128 - 1] = 0.001037 \text{ bins.} \quad (43) \end{aligned}$$

The second term on the right hand side of equation (39) is the frequency translation due to the starting frequency errors $f_{s1error}$ and $f_{s2error}$. The maximum value of $f_{s2error}$ is -2^{-25} . Since $f_{s2error}$ is subtracted in (39), it will cause a positive frequency translation like the chirp rate error caused. Therefore, to minimize the total frequency translation calculated from (39), the maximum negative value of $f_{s1error}$, $-2^{-24} - 2^{-25}$, should be chosen. This value of $f_{s1error}$ will be obtained if $-\theta_0$ is calculated by first negating θ_0 and then truncating. With $f_{s1error} = -2^{-24} - 2^{-25}$, $f_{s2error} = -2^{-25}$, $\phi_0 = 4.6296 \times 10^{-3}$, and $\phi_{error} = -2^{-24}$, the second term on the right hand side of (39) is equal to -0.00001287 bins. This gives a total frequency translation of $0.001037 - 0.00001287 = 0.001024$ bins. Notice that the frequency translation caused by the chirp rate error is 81 times larger than the frequency translation caused by the starting frequency errors.

For 24-bit rounding of the chirp rate and the starting frequencies, the maximum frequency translation can be calculated as follows. If the starting frequency of the $A^{-n} W^{n^2/2}$ chirp is rounded, the maximum possible value of $f_{s1error}$ is $2^{-25} + 2^{-25}$ which will produce a positive frequency translation error. Therefore, the maximum frequency translation, calculated from (39), will occur when ϕ_{error} and $f_{s2error}$ both cause positive frequency translations. Substituting a k of 81 and a maximum bandwidth ratio for rounding of 1.0000064 into (43) yields a frequency translation of 0.0005184 bins due to the chirp rate error. Substituting $f_{s1error} = 2^{-25} + 2^{-25}$, $f_{s2error} = 0$, $\phi_o = 4.6296 \times 10^{-3}$, and $\phi_{error} = -2^{-25}$ into the second term on the right hand side of (39) yields a frequency translation due to $f_{s1error}$ and $f_{s2error}$ of 0.00001287 bins. Therefore, the total worst case frequency translation is $0.0005184 + 0.00001287 = 0.00053127$ bins. Again, the chirp rate error is the dominant term in the frequency translation error. The maximum negative frequency translation, which will occur for $f_{s1error} = 0$, $f_{s2error} = 2^{-25}$, and $\phi_{error} = 2^{-25}$, is -0.00052808 bins.

Similarly, since the starting frequency of the $W^{k^2/2}$ chirp is $-\phi_o/2$, the maximum starting frequency error for the $W^{k^2/2}$ chirp is -2^{-25} for truncation and 2^{-25} for rounding. This combined with a starting frequency error in the $v(n)$ chirp gives an error in the phase spectrum of the chirp-z transform of $2\pi(k)(-2^{-25} - 2^{-25})$ for truncation and $2\pi(k)(2^{-25} + 2^{-25})$ for rounding. Assuming a maximum value of k of 81 gives a maximum phase error of $\pm 2\pi(0.00000241)$ radians. This is equivalent to a phase truncation to 18 bits since $2^{-18} \leq 0.00000241 \leq 2^{-17}$. Since the phase is truncated to 11 bits at the output of the digital chirp synthesizer, the effect of a starting frequency error in the $W^{k^2/2}$ chirp and the effect of a starting frequency error in the $v(n)$ chirp on the phase spectrum of the chirp-z transform can be ignored.

Next, the maximum value of the starting phase error for the $W^{-(L-n)^2/2}$

chirp will be calculated. As stated in the previous section, the starting phase error will be $(N-1)(f_{s2error})$ which will be maximum for the maximum value of N . For this analysis, the maximum value of N will be assumed to be 173. As shown above, the maximum value of $f_{s2error}$ is $\pm 2^{-25}$ which gives a maximum starting phase error of $\pm 5.126 \times 10^{-6}$. This error is equivalent to that caused by a 17 bit truncation of the starting phase since $2^{-18} \leq 5.126 \times 10^{-6} \leq 2^{-17}$. Add to this the effect of truncating the starting phase to 16 bits in the digital chirp synthesizer and the total worst case starting phase error is approximately $2^{-16} + 2^{-17}$. Since this worst case starting phase error is less than the error caused by the truncation of the phase at the output of the phase accumulator to 11 bits, it can be ignored.

Noise Floor and Phase Error Effects

Finally, the effect of truncating the output of the phase accumulator in the digital chirp synthesizer to 11 bits will be considered. The 11 bit phase data truncation can be modeled as uniformly distributed random phase noise in the output of the digital chirp synthesizer over the range of $-2^{-11} < \text{phase noise} \leq 0$. The phase noise was applied to each of the four chirps used in the chirp-z transform. Figure 5 and Figure 6 show the effect of this phase noise on the amplitude and phase spectrums of the chirp-z transform of a complex sinusoid multiplied by a Taylor window function with -100 dB sidelobes and an $nbar$ of 20. Both the mean and the mean plus three standard deviations for 300 different runs are given in Figures 5 and 6. As shown in Figure 5, the phase noise raised the noise floor of the amplitude spectrum to -80 dB for the "mean" plot and to -72 dB for the "mean + 3 σ " plot while not effecting the bandwidth, amplitude, or fractional frequency of the main lobe. As shown in Figure 6, the phase noise produced mean phase errors of 0.0046 radians and mean plus three standard deviation phase errors of 0.0089 radians over the frequency region where the amplitude spectrum was greater than -30 dBc.

Figure 5. Amplitude Spectrum of the Chirp-z Transform With and Without Phase Noise.

Figure 6. Phase Spectrum of the Chirp-z Transform With and Without Phase Noise.

Conclusions

In conclusion, this report has presented the results of an analysis of the effects of finite-precision arithmetic on the chirp-z transform. Formulas for the frequency translation and bandwidth distortion caused by chirp rate, starting frequency, and starting phase errors were derived. For 24-bit truncation of the chirp rate, it was found that $1 \leq \text{bandwidth ratio} \leq 1.0000128$. For 24-bit rounding of the chirp rate, $0.99999356 \leq \text{bandwidth ratio} \leq 1.0000064$. Keeping the bandwidth ratio between 0.999 and 1.001 required a minimum of 18 bits for truncation and 17 bits for rounding. Additionally, it was found that the maximum frequency translation of the phase and amplitude spectrums for 24-bit truncation of the chirp rate and all the starting frequencies was 0.001024 bins. For 24-bit rounding, the maximum positive frequency translation was 0.00053127 bins while the maximum negative frequency translation was -0.00052808 bins. Finally, it was shown that phase truncation at the output of the digital chirp synthesizers raised the noise floor of the amplitude spectrum to -80 dB with no effect on the main lobe. The phase truncation also introduced an average phase error of 0.0046 radians and an average plus 3 standard deviations error of 0.0089 radians.

References

¹Lawrence R. Rabiner and Bernard Gold, Theory and Application of Digital Signal Processing, Prentice Hall, Inc., 1975, pp. 393-399.

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