Electromagnetic Kinetic Toroidal Eigenmodes for
General MHD Equilibria

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ABSTRACT

A comprehensive analysis of low-frequency, high-toroidal-mode-number linear eigenmodes for tokamaks is presented. The most significant new features of this stability study are that the calculation is interfaced with a general numerical magnetohydrodynamic (MHD) equilibrium, and that it is fully electromagnetic. The ballooning formalism is employed and all important kinetic effects, including those of trapped particles, are retained. In particular, the familiar trapped-electron-drift-wave frequency regime is considered and results are presented for three sequences of artificial equilibria: one of increasing β ($\equiv$ plasma pressure/magnetic pressure) values; one of varying equilibrium shape, from inverse-D to circular to normal-D; and one of increasing vertical ellipticity. The analysis is then applied to a realistic (self-consistent) high- β equilibrium generated with data obtained from the ISX-B tokamak experiment. Here it is found that, for the usual trapped-electron branch, kinetic microinstabilities appear to be present over a wide range of toroidal mode numbers.

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I. INTRODUCTION

In this paper we present an improved calculation of low-frequency, high-toroidal-mode-number linear eigenmodes. Compared with our earlier calculations in Refs. 1 and 2, the present analysis is generalized in a number of significant ways. First, this stability study is interfaced with a general numerical tokamak magnetohydrodynamic (MHD) equilibrium, and it is not restricted to the previous model equilibrium with circular, concentric magnetic surfaces and large aspect ratio. Second, the calculation is fully electromagnetic, retaining the perturbed magnetic vector potential components parallel and perpendicular to the equilibrium magnetic field, as well as the perturbed electrostatic potential. Hence, the present analysis is applicable to modes in the shear-Alfven branch as well as to those in the drift branches. Third, we employ a more general form of the perturbed distribution function, which is valid for mode frequencies of the order of or below the particle bounce or transit frequencies, as well as above them. It also includes finite gyroradius and finite banana width effects to all orders. Multiple ion species have been taken into account in these calculations, but results are presented here only for a single ion species. With regard to collisional effects, an improved Krook model collision operator,³ which can reproduce the result of a Lorentz operator for collision frequencies both large and small compared to the mode frequency, is employed. As in previous work,² the present study applies the ballooning formalism to the analysis of large toroidal mode number instabilities in an axisymmetric toroidal geometry. All important kinetic effects, including the effects of trapped electrons and trapped ions, are retained in this calculation.

In Sec. II, the derivation of the perturbed distribution function from

the gyrokinetic equation is presented, and the solution method employed to generate the resulting eigenmode equations is described. In Sec. III, microinstabilities in the familiar trapped-electron-drift branch are studied, and results are presented for three sequences of equilibria: a flux-conserving tokamak (FCT) sequence⁴ of equilibria with increasing β ($\equiv$ plasma pressure/magnetic pressure), a sequence with varying "D-ness," and a sequence with varying vertical ellipticity. These sequences of equilibria are all artificial in the sense that the profiles and other physical parameters here do not correspond to those from an actual experiment. We also present results for a realistic equilibrium representing the Impurity Study Experiment (ISX-B) device at high β . A kinetic mode, associated with the trapped-electron-drift branch, is found to be unstable over a wide range of toroidal mode numbers. Finally, conclusions are discussed in Sec. IV, and the model collision operators employed are described in detail in the appendix.

II. FORMULATION OF THE BASIC EQUATIONS

We use the ψ, θ, ζ coordinate system of the FEST-I equilibrium code⁵ in toroidal geometry, where ψ is related to the poloidal flux and acts as a radial coordinate ($\psi = 0$ on the magnetic axis, $\psi = 1$ at the plasma boundary), θ is an angle-like poloidal coordinate with period 2π , and ζ is the toroidal angle. The equilibrium magnetic field is given by

$$\underline{B} = B_0 g(\psi) \nabla \zeta \times \nabla \psi, \quad (1)$$

where $\xi \equiv \zeta - q(\psi)(\theta - \theta_0)$; $q(\psi)$, the usual safety factor, and $g(\psi)$ are computed as part of the equilibrium, θ_0 is a parameter to be determined at higher order in the ballooning hierarchy of equations² (with $\nabla \theta_0 \equiv 0$), and B_0 is the magnetic field on the magnetic axis. With these coordinates, $\nabla \psi$ and $\nabla \theta$ are orthogonal to $\nabla \zeta = \underline{e}_\zeta / R$, but $\nabla \psi$ is not orthogonal to $\nabla \theta$. The Jacobian is⁵

$$J \equiv (\nabla \psi \cdot \nabla \zeta \times \nabla \theta)^{-1} = v R^2 / 2\pi R_0, \quad (2)$$

where $v = (R_0 / 2\pi) \int d^3x / R^2$, R is the major radius, and R_0 is the major radius of the magnetic axis.

We will employ the ballooning representation for the perturbed electrostatic potential in the form²

$$\Phi(\psi, \theta, \zeta, t) = \exp[-i\omega t + i n \zeta - i n q(\psi) \theta]$$

$$\times \sum_{p'=-\infty}^{\infty} \hat{\phi}(\theta - 2\pi p', \psi) \exp[i n q(\psi) 2\pi p'] \quad (3)$$

where n is the toroidal mode number. In this form, Φ is automatically periodic in θ and ζ , while $\hat{\phi}(\theta)$ is defined in the infinite domain from $-\infty$ to ∞ in θ and is not periodic. Similar representations apply for $A_{\parallel}$ and $A_{\perp}$, the perturbed vector potential components parallel and perpendicular to $\tilde{B}$. Here we adopt the gauge condition $\tilde{A} \cdot (\tilde{n} \times \tilde{e}_{\psi}) = 0$, so that $\tilde{A} = A_{\parallel} \tilde{n} + A_{\perp} \tilde{e}_{\psi}$, where $\tilde{n} \equiv \tilde{B}/B$, $B \equiv |\tilde{B}|$, and $\tilde{e}_{\psi} \equiv \nabla\psi/|\nabla\psi|$.

The perturbed distribution function is given by

$$f = \frac{e}{m} \Phi \frac{\partial F^{(0)}}{\partial E} + h, \quad (4)$$

where e and m are the particle charge and mass, E is the particle kinetic energy per unit mass, and $F^{(0)}$ is the lowest order part (in ρ/r_n) of the equilibrium distribution function. Also, ρ is the particle gyroradius and r_n is the equilibrium density gradient scale length. Species subscripts have been suppressed, but will be added later.

Employing the results of earlier studies,^{2,6-8} the gyrokinetic equation for h can be written as

$$\begin{aligned} & \frac{v_{\parallel}(\theta)}{B(\theta)} \frac{\partial \hat{h}_{\sigma}}{\partial \theta} - i(\omega + i\nu_F - \omega_d(\theta)) \hat{h}_{\sigma}(\theta) \\ &= -i(\omega - \omega_b^T) \frac{e}{T} F^{(0)} \left\{ (\hat{\phi}(\theta) - \frac{v_{\parallel}(\theta)}{c} \hat{A}_{\parallel}) J_0(\theta) \right. \\ & \quad \left. - i \frac{v_{\perp}(\theta)}{c} \hat{A}_{\perp}(\theta) J_1(\theta) \right\}. \end{aligned} \quad (5)$$

Here, h_{σ} has been decomposed like Φ in Eq. (3), with $\sigma = \pm 1$ labeling the sign of $v_{\parallel}$. We have employed a Krook collision operator³ $C(h) = -\nu_F(E, \mu) h$; the detailed functional form for ν_F is specified in the

appendix. Also, $v_{\parallel}$ is the velocity component along B , $v_{\perp}$ is the magnitude of the velocity perpendicular to B , $F^{(0)}$ is taken to be a Maxwellian with temperature T , $\mu = v_{\perp}^2/2B$ is the magnetic moment per unit mass,

$$J_{\lambda} = J_{\lambda}(\theta) \equiv J_{\lambda} \left(\frac{nv_{\perp}}{\Omega} |\nabla \xi| \right) ,$$

is the usual Bessel function for $\lambda = 0$ or 1 , $\Omega \equiv eB/mc$ is the gyrofrequency,

$$|\nabla \xi|^2 = \left(\frac{1}{R^2} + q^2 |\nabla \theta|^2 \right) + \left(\frac{dq}{d\psi} \right)^2 (\theta - \theta_0)^2 |\nabla \psi|^2 \\ + 2q \frac{dq}{d\psi} (\theta - \theta_0) \nabla \psi \cdot \nabla \theta ,$$

$|\nabla \theta|^2$, $|\nabla \psi|^2$, and $\nabla \psi \cdot \nabla \theta$ are computed as part of the equilibrium,⁵ and

$$\omega_k^T = - \frac{ncT}{e} \frac{1}{B_0 g} \frac{\partial}{\partial \psi} \ln F^{(0)} . \quad (6)$$

The magnetic (curvature and gradient) drift frequency ω_d is defined as $\omega_d \equiv n \nabla \xi \cdot \underline{v}_d$, where $\underline{v}_d \equiv \underline{n} \times (\mu \nabla B + v_{\perp}^2 \underline{n} \cdot \nabla \underline{n}) / \Omega$, neglecting the equilibrium electric field. After considerable algebra, it can be shown that

$$\omega_d(\theta) = - \frac{nmc\pi}{e} \left[(v_{\perp}^2 + 2v_{\parallel}^2) \kappa_w - \frac{v_{\perp}^2 B_0}{4\pi g B^2} \frac{d\beta}{d\psi} \right] , \quad (7)$$

where β is the ratio of plasma pressure to magnetic pressure on the chosen magnetic surface, and

$$\kappa_w \equiv - \frac{1}{4\pi B_0 g R^2} \left[g \frac{\partial}{\partial \psi} \left(\frac{R^2}{g} \right) + B_0^2 g^2 \frac{\partial}{\partial \theta} \left(\frac{\nabla \psi \cdot \nabla \theta}{B^2} \right) \right]$$

$$- \frac{4\pi^2 R_o^2 q}{U^2 B^2} (\theta - \theta_o) \frac{dq}{d\psi} - \frac{R_o^2 B_o^2}{2B^2} \frac{dB}{d\psi} . \quad (8)$$

A. Circulating particles

Note that σ is a constant of the motion for circulating particles. The boundary conditions on $\hat{h}_\sigma(\theta)$ are simply that $\hat{h}_\sigma(\theta) \rightarrow 0$ sufficiently rapidly as $\theta \rightarrow \pm\infty$ that the sum corresponding to Eq. (3) for h exists. We assume $\gamma \equiv \text{Im}(\omega) > 0$. Then Eq. (5) for circulating particles may be integrated to give

$$\begin{aligned} \hat{h}_+(\theta) = & -i \int_{-\infty}^{\theta} d\theta' \frac{e}{T} F^{(0)} \frac{B}{|v_\parallel|} (\omega - \omega_\star^T) \left[\left(\hat{\phi} - \frac{|v_\parallel|}{c} \hat{A}_\parallel \right) J_o \right. \\ & \left. - i \frac{v_\perp}{c} \hat{A}_\perp J_1 \right] \exp[-i \int_{\theta}^{\theta'} d\theta'' (\omega + i\nu_f - \omega_d) B/|v_\parallel|] , \end{aligned} \quad (9)$$

and

$$\begin{aligned} \hat{h}_-(\theta) = & -i \int_{\theta}^{\infty} d\theta' \frac{e}{T} F^{(0)} \frac{B}{|v_\parallel|} (\omega - \omega_\star^T) \left[\left(\hat{\phi} + \frac{|v_\parallel|}{c} \hat{A}_\parallel \right) J_o \right. \\ & \left. - i \frac{v_\perp}{c} \hat{A}_\perp J_1 \right] \exp[i \int_{\theta}^{\theta'} d\theta'' (\omega + i\nu_f - \omega_d) B/|v_\parallel|] . \end{aligned} \quad (10)$$

It is convenient to establish the following definitions for circulating particles

$$\tau_t \equiv \sigma \int_{-\pi}^{\pi} d\theta'' B/|v_\parallel| , \quad (11)$$

$$\omega_t \equiv 2\pi/\tau_t , \quad (12)$$

$$\hat{\theta}(\theta) \equiv |\omega_t| \int_0^{\theta} d\theta'' B/|v_\parallel| , \quad (13)$$

$$w_d(\theta) \equiv - \int_0^{\theta} d\theta'' \omega_d(\theta'') B g / |v_{\parallel}| \quad (14)$$

Here τ_t is the transit period and ω_t is the transit frequency. In terms of these quantities

$$\begin{aligned} \int_{\theta}^{\theta'} d\theta'' (\omega + i\nu_f - \omega_d) B g / |v_{\parallel}| &= \frac{\omega + i\nu_f}{|\omega_t|} [\hat{\theta}(\theta') - \hat{\theta}(\theta)] \\ &+ w_d(\theta') - w_d(\theta) \quad (15) \end{aligned}$$

The θ' integrations in Eqs. (9) and (10) may be carried out by means of a Fourier transformation in terms of the new variable $\hat{\theta}(\theta)$ defined in Eq. (13). Specifically, the transform pair is

$$\begin{aligned} \Psi_{\sigma}(\theta) &\equiv \{ [\hat{\phi}(\theta) - \sigma \frac{|v_{\parallel}(\theta)|}{c} \hat{A}_1(\theta)] J_0(\theta) - i \frac{v_{\perp}(\theta)}{c} \\ &\times \hat{A}_1(\theta) J_1(\theta) \} \exp[-i\sigma w_d(\theta)] \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\kappa \exp[i\kappa \hat{\theta}(\theta)] \Psi_{\sigma}(\kappa) \quad (16) \end{aligned}$$

and

$$\begin{aligned} \Psi_{\sigma}(\kappa) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\hat{\theta} \exp[-i\kappa \hat{\theta}] \Psi_{\sigma}(\hat{\theta}(\theta)) \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\theta \frac{|\omega_t| B g}{|v_{\parallel}|} \exp[-i\kappa \hat{\theta}(\theta)] \Psi_{\sigma}(\theta) \quad (17) \end{aligned}$$

Then Eq. (9) becomes

$$\begin{aligned} \hat{h}_+(\theta) &= -i \frac{e}{T} F^{(0)} \frac{\omega - \omega_p^T}{|\omega_c|} \int_{-\infty}^{\hat{\theta}(\theta)} d\hat{\theta}' (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\kappa \exp(i\kappa\hat{\theta}') \\ &\quad \Psi_+(\kappa) \exp[-i(\hat{\theta}' - \hat{\theta})(\omega + i\nu_f)/|\omega_c| + i\omega_d(\theta')] \\ &= \frac{e}{T} F^{(0)} (\omega - \omega_p^T) (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\kappa \frac{\exp[i\kappa\hat{\theta}(\theta) + i\omega_d(\theta)] \Psi_+(\kappa)}{\omega + i\nu_f - \kappa\omega_c} . \end{aligned} \quad (18)$$

Here $\hat{\theta}$ means $\hat{\theta}(\theta)$ and $\hat{\theta}'$ means $\hat{\theta}(\theta')$. Likewise, from Eq. (10),

$$\hat{h}_-(\theta) = \frac{e}{T} F^{(0)} (\omega - \omega_p^T) (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\kappa \frac{\exp[i\kappa\hat{\theta}(\theta) - i\omega_d(\theta)] \Psi_-(\kappa)}{\omega + i\nu_f - \kappa\omega_c} . \quad (19)$$

B. Trapped particles

For trapped particles, the procedure for solving Eq. (5) is more complicated. Consider trapped particles with turning points (or orbit reflection points) at θ_1 and θ_2 , where $\theta_1 < \theta_2$. Then the boundary conditions are that

$$\hat{h}_+(\theta_1) = \hat{h}_-(\theta_1) \equiv \hat{h}_1 , \quad (20)$$

and

$$\hat{h}_+(\theta_2) = \hat{h}_-(\theta_2) \equiv \hat{h}_2 . \quad (21)$$

With the ballooning formalism, there are an infinite number of pairs of turning points for trapped particles. The relevant pair for some given value of θ is the one with $\theta_1 \leq \theta \leq \theta_2$. The general solution of Eq. (5) is

$$\begin{aligned} \hat{h}_\sigma(\theta) = & \exp(i\sigma\alpha_{\theta_3}^{\theta}) [\hat{h}_\sigma(\theta_3) - i\sigma \int_{\theta_3}^{\theta} d\theta' T_\sigma(\theta')] \\ & \cdot \exp(-i\sigma\alpha_{\theta_3}^{\theta'}) \end{aligned} \quad (22)$$

where

$$I_\theta^{\theta'} \equiv \int_{\theta}^{\theta'} d\theta'' [\omega + i\nu_F - \omega_d(\theta'')] B\mathcal{J}/|v_\parallel| \quad (23)$$

$$T_\sigma \equiv T_\phi + \sigma T_A \quad (24)$$

$$T_\phi \equiv \frac{e}{T} F^{(0)} \frac{B\mathcal{J}}{|v_\parallel|} \left[(\omega - \omega_\phi^T) [\hat{\phi}(\theta') J_0(\theta') - i \frac{v_\parallel(\theta')}{c} A_1(\theta') J_1(\theta')] \right] \quad (25)$$

and

$$T_A \equiv \frac{e}{T} F^{(0)} \frac{B\mathcal{J}}{|v_\parallel|} \left(\omega - \omega_\phi^T \right) \left[- \frac{v_\parallel(\theta')}{c} \hat{A}_1(\theta') J_0(\theta') \right] \quad (26)$$

In Eq. (22), θ_1 is arbitrary, except that $\theta_1 \leq \theta_3 \leq \theta_2$. Taking $\theta_3 = \theta_1$ and $\theta = \theta_2$ in Eq. (22) yields one equation, and taking $\theta_3 = \theta_2$ and $\theta = \theta_1$ yields a second equation. These two equations may be solved for the two unknowns $\hat{h}_1$ and $\hat{h}_2$. Using the result for $\hat{h}_1$ in Eq. (22) with $\theta_3 = \theta_1$ then gives

$$\begin{aligned} \hat{h}_\sigma(\theta) = & \exp(i\sigma\theta_{\theta_1}) \{ (2 \sin I_{\theta_1}^{\theta_2})^{-1} \int_{\theta_1}^{\theta_2} d\theta' [(T_\phi - T_A) \\ & \cdot \exp(-iI_{\theta'}^{\theta_2}) + (T_\phi + T_A) \exp(iI_{\theta'}^{\theta_2})] \\ & - i\sigma \int_{\theta_1}^{\theta} d\theta' (T_\phi + \sigma T_A) \exp(-i\sigma I_{\theta'}^{\theta'}) \} . \end{aligned} \quad (27)$$

For trapped particles, it is convenient to establish the following definitions

$$\tau_b \equiv \int_{\theta_1}^{\theta_2} d\theta'' B g / |v_\parallel| , \quad (28)$$

$$\omega_b \equiv 2\pi / \tau_b , \quad (29)$$

$$\hat{\theta}(\theta) \equiv \omega_b \int_{\theta_1}^{\theta} d\theta'' B g / |v_\parallel| - \pi , \quad (30)$$

$$\omega_d^{(0)} \equiv \frac{1}{\tau_b} \int_{\theta_1}^{\theta_2} d\theta'' \omega_d(\theta'') B g / |v_\parallel| , \quad (31)$$

$$\omega_d(\theta) \equiv - \int_{\theta_1}^{\theta} d\theta'' [\omega_d(\theta'') - \omega_d^{(0)}] B g / |v_\parallel| , \quad (32)$$

and

$$\Gamma \equiv [w + iv_F - \omega_d^{(0)}] / \omega_b . \quad (33)$$

Here, τ_b is the bounce period and ω_b is the bounce frequency. Note that with these definitions, $\hat{\theta}(\theta_1) = -\pi$, $\hat{\theta}(\theta_2) = \pi$, $\omega_d(\theta_1) = \omega_d(\theta_2) = 0$, and $d\hat{\theta} = d\theta \omega_b B g / |v_\parallel|$. Also,

$$I_{\theta}^{\theta'} = \Gamma[\hat{\theta}(\theta') - \hat{\theta}(\theta)] + w_d(\theta') - w_d(\theta) \quad (34)$$

The θ' integrations in Eq. (27) may be performed by means of a Fourier series transform in the new $\hat{\theta}$ variable defined by Eq. (30), rather than an integral transform. Specifically, the transform pair is

$$\begin{aligned} \Psi_{\sigma}(\theta) &= \left[\hat{\phi}(\theta) - \sigma \frac{v_{\parallel}(\theta)}{c} \hat{A}_{\parallel}(\theta) \right] J_0(\theta) - i \frac{v_{\perp}(\theta)}{c} \hat{A}_{\perp}(\theta) J_1(\theta) \\ &\quad \cdot \exp[-i\omega_d(\theta)] \\ &= \sum_{p=-\infty}^{\infty} \exp[ip\hat{\theta}(\theta)] \Psi_{\sigma}^p, \end{aligned} \quad (35)$$

with

$$\begin{aligned} \Psi_{\sigma}^p &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\hat{\theta} \exp(-ip\hat{\theta}) \Psi_{\sigma}[\hat{\theta}(\theta)] \\ &= \frac{1}{-\pi} \int_{\partial_1}^{\partial_2} d\theta \frac{\omega_p B \theta}{|v_{\parallel}|} \exp[-ip\hat{\theta}(\theta)] \Psi_{\sigma}(\theta) \quad (36) \end{aligned}$$

Using Eq. (35), the θ' integrations in Eq. (27) may be carried out, yielding, after some algebra,

$$\begin{aligned} \hat{h}_{\sigma}(\theta) &= H_0^e F(0) \frac{\omega - \omega_b^T}{\omega_b} \sum_{p=-\infty}^{\infty} \left[\frac{\Psi_{\sigma}^p \exp[ip\hat{\theta}(\theta) + i\omega_d(\theta)]}{\Gamma - \omega p} \right. \\ &\quad \left. + \sigma(-1)^p \frac{\exp\{i\sigma[\Gamma\hat{\theta}(\theta) + w_d(\theta)]\}}{2 \cos \pi \Gamma} \left(\frac{\Psi_{\sigma}^p}{p - \Gamma} + \frac{\Psi_{\sigma}^p}{p + \Gamma} \right) \right] \quad (37) \end{aligned}$$

C. Mode equations and solution method

The equations in which the results for $\hat{h}_{\sigma}$ are to be used are just the

quasi-neutrality condition and the parallel and perpendicular components of Ampere's law. These are, respectively,

$$0 = \sum_j e_j \left\{ -\frac{e_j n_j}{T_j} \hat{\phi}(\theta) + \int d^3v J_0(\theta) [\hat{h}_{+j}(\theta) + \hat{h}_{-j}(\theta)] \right\}, \quad (38)$$

$$-\frac{n^2 |\nabla \xi|^2}{4\pi} \hat{A}_{\parallel}(\theta) = -\frac{1}{c} \sum_j e_j \int d^3v v_{\parallel}(\theta) J_0(\theta) [\hat{h}_{+j}(\theta) - \hat{h}_{-j}(\theta)] \quad (39)$$

and

$$-\frac{n^2 |\nabla \xi|^2}{4\pi} \hat{A}_{\perp}(\theta) = -\frac{1}{c} \sum_j e_j \int d^3v v_{\perp}(\theta) J_1(\theta) [\hat{h}_{+j}(\theta) + \hat{h}_{-j}(\theta)] \quad (40)$$

We have now restored the species subscript j , and n_j is the equilibrium number density. From Eq. (37)

$$\begin{aligned} [\hat{h}_{+j}(\theta) \pm \hat{h}_{-j}(\theta)] &= \frac{e}{T} F(0) \frac{\omega - \omega_{*}}{\omega_b} \sum_{p=-\infty}^{\infty} [\exp[ip\theta(\theta)] \\ &\quad \cdot \left(\frac{\Psi_{+}^p \exp[i\omega_d(\theta)]}{\Gamma - p} \pm \frac{\Psi_{-}^p \exp[-i\omega_d(\theta)]}{\Gamma + p} \right) \\ &\quad - \frac{(-1)^p}{\cos \pi \Gamma} \left\{ \frac{1}{\cos} \right\} [\Gamma \theta(\theta) + \omega_d(\theta)] \left(\frac{\Psi_{+}^p}{\Gamma - p} - \frac{\Psi_{-}^p}{\Gamma + p} \right)] \quad (41) \end{aligned}$$

for trapped particles. For circulating particles, $\hat{h}_{+j}$ and $\hat{h}_{-j}$ are given by Eqs. (18) and (19), respectively.

We will employ as our velocity space coordinates E , Λ , and σ , where $\Lambda \equiv \mu B_0 / E$. In terms of these variables

$$\int d^3v = \sqrt{2} \pi \int_0^\infty dE E^{1/2} \int_0^{h(\theta)} d\Lambda \{h(\theta) [1 - \Lambda/h(\theta)]^{1/2}\}^{-1}, \quad (42)$$

where $h(\theta) \equiv B_0/B(\theta)$. Circulating particles correspond to $0 \leq \Lambda < h_m$ and trapped particles to $h_m < \Lambda \leq h(\theta)$ at some given θ , where $h_m \equiv \min_{0 \leq \theta \leq 2\pi} h(\theta)$.

In order to solve the system of three linear, homogeneous, non-Hermitian integral equations given by Eqs. (38), (39), and (40) in the three unknown functions $\hat{\phi}(\theta)$, $\hat{A}_\parallel(\theta)$, and $\hat{A}_\perp(\theta)$, we employ a Ritz method. We decompose $\hat{\phi}(\theta)$ as

$$\hat{\phi}(\theta) = \sum_{\ell=0}^{L-1} \hat{\phi}_\ell h_\ell(\theta) [\hat{\rho}(\theta)]^{-1/2}, \quad (43)$$

where $h_\ell(\theta) \equiv H_\ell(\alpha^{1/2}\theta) \exp(-\alpha\theta^2/2) M_\ell^{-1/2}$, H_ℓ is the Hermite polynomial of order ℓ , $M_\ell \equiv (\pi/\alpha)^{1/2} 2^\ell \ell!$, $\hat{\rho}(\theta) \equiv \rho(\theta) B_0/q(\phi) R_0$ is dimensionless, and α is an adjustable parameter with $\text{Re}(\alpha) > 0$. The other potentials $\hat{A}_\parallel$ and $\hat{A}_\perp$ are decomposed in the same fashion. Then the operator

$$\frac{T_e}{|e| n_e} \int_{-\infty}^{\infty} d\theta [\hat{\rho}(\theta)]^{1/2} h_{\ell'}(\theta) \quad (0 \leq \ell' \leq L-1)$$

is applied to the three equations. The resulting set of algebraic equations, after some algebra, can be written in matrix form schematically as

$$M \begin{bmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{A}_{\parallel 1} \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{bmatrix} = \begin{bmatrix} M_{\ell' \ell}^{11} & M_{\ell' \ell}^{12} & M_{\ell' \ell}^{13} \\ M_{\ell' \ell}^{21} & M_{\ell' \ell}^{22} & M_{\ell' \ell}^{23} \\ M_{\ell' \ell}^{31} & M_{\ell' \ell}^{32} & M_{\ell' \ell}^{33} \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{A}_{\parallel 1} \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \quad (44)$$

Here $M(\omega)$ is a 3×3 block matrix, with each block being an $L \times L$ matrix.

Also

$$\begin{aligned}
 M_{l', l}^{m', m}(\omega) = & \sum_j \frac{n_j}{n_e} \frac{T_e}{T_j} Z_j^2 \delta_{m'1} \delta_{m'1} \delta_{ll'} + [\delta_{m'2} \delta_{m'2} \\
 & + \delta_{m'3} \delta_{m'3}] \frac{2T_e n^2}{\beta |e| B_0 n_e} \sum_j n_j T_j \int_{-\infty}^{\infty} d\theta |\nabla \xi|^2 h_l(\theta) h_{l'}(\theta) \\
 & - \sum_j \frac{n_j}{n_e} \frac{T_e}{T_j} Z_j^2 \frac{1}{4\pi^{3/2}} \int_0^h d\lambda \frac{|\omega_e| q R_0}{v} \int_{-\infty}^{\infty} d\kappa \\
 & \times \int_0^{\infty} d\hat{E} \hat{E}^{1/2} \exp(-\hat{E}) \{ \omega - \omega_{*j} [1 + \eta_j (\hat{E} - 3/2)] \} \\
 & \times \sum_{\sigma=\pm 1} [\omega + i\nu_f - \kappa \omega_e]^{-1} \left\{ \int_{-\infty}^{\infty} d\theta \frac{h_{l'}(\theta) [\hat{g}(\theta)]^{1/2}}{h(\theta) [1 - N/h(\theta)]^{1/2}} \right. \\
 & \times \exp[i\kappa \hat{\theta}(\theta) + i\sigma \omega_d(\theta)] [\delta_{m'1} J_L(\theta) + \delta_{m'2} \frac{v_{||}(\theta)}{c} J_0(\theta) \\
 & + \delta_{m'3} \frac{v_{\perp}(\theta)}{c} J_1(\theta)] \left. \left\{ \int_{-\infty}^{\infty} d\theta' \frac{h_l(\theta') [\hat{g}(\theta')]^{1/2}}{h(\theta') [1 - N/h(\theta')]^{1/2}} \right. \right. \\
 & \times \exp[-i\kappa \hat{\theta}(\theta') - i\sigma \omega_d(\theta')] [\delta_{m1} J_0(\theta') - \delta_{m2} \frac{v_{||}(\theta')}{c} J_0(\theta')
 \end{aligned}$$

$$\begin{aligned}
 & - i \delta_{m3} \frac{v_1(\theta)}{c} J_1(\theta) \} = \sum_j \frac{n_j}{n_e} \frac{T_e}{T_j} z_j^2 \frac{1}{2\pi^{1/2}} \\
 & \times \int_{-\infty}^{\infty} d\theta h_{\lambda,1}(\theta) \hat{\rho}(\theta)^{1/2} \int_{h_m}^{h(\theta)} d\lambda \{h(\theta) [1 - \lambda/h(\theta)]^{1/2}\}^{-1} \\
 & \times \int dE \hat{E}^{1/2} \exp(-E) \{ \omega - \omega_{ej} [1 + \eta_j (\hat{E} - 3/2)] \} \\
 & \times \{ \delta_{m,1} J_0(\theta) + \delta_{m,2} \frac{v_1(\theta)}{c} J_0(\theta) + \delta_{m,3} \frac{v_1(\theta)}{c} J_1(\theta) \} \\
 & \times \sum_{p=-\infty}^{\infty} \left[\exp[ip\hat{\theta}(\theta)] \left(\frac{\Psi_{+1}^{pm} \exp[i\omega_d(\theta)]}{\omega + i\nu_f - \omega_d^{(0)} - p\omega_b} \right. \right. \\
 & \left. \left. - (-1)^m \frac{\Psi_{-1}^{pm} \exp[-i\omega_d(\theta)]}{\omega + i\nu_f - \omega_d^{(0)} + p\omega_b} \right) - \frac{(-1)^p}{\cos \pi \Gamma} \right. \\
 & \times \{ (\delta_{m,1} + \delta_{m,3}) i \sin[\Gamma\hat{\theta}(\theta) + \omega_d(\theta)] + \delta_{m,2} \\
 & \times \cos[\Gamma\hat{\theta}(\theta) + \omega_d(\theta)] \} \left(\frac{\Psi_{+1}^{pm}}{\omega + i\nu_f - \omega_d^{(0)} - p\omega_b} \right. \\
 & \left. \left. - \frac{\Psi_{-1}^{pm}}{\omega + i\nu_f - \omega_d^{(0)} + p\omega_b} \right) \right] , \tag{45}
 \end{aligned}$$

where

$$v_{\sigma l}^{pm} = \frac{\omega_p q R_0}{2\pi v} \int_{\theta_1}^{\theta_2} d\theta'' \frac{h_l(\theta'') [\hat{\rho}(\theta'')]^{1/2}}{h(\theta'') [1 - N/h(\theta'')]^{1/2}} \\ \times \exp[-ip\theta(\theta'') - i\omega_d(\theta'')] \{\delta_{m1} J_0(\theta'') \\ - \delta_{m2} \frac{v_{||}(\theta'')}{c} J_0(\theta'') - i\delta_{m3} \frac{v_{\perp}(\theta'')}{c} J_1(\theta'')\} \quad (46)$$

Also, $\delta_{ll'} = 1$ for $l = l'$, $\delta_{ll'} = 0$ for $l \neq l'$, $\hat{E} \equiv n_j E/T_j$,
 $\omega_{*j} \equiv -[ncT_j/e_j B_0 g(\psi)] \{\partial \ln n_j / \partial \psi\}$, $\eta_j \equiv [\partial \ln T_j / \partial \psi] / [\partial \ln n_j / \partial \psi]$,
 $e_j \equiv Z_j |e|$, and $\beta \equiv (8\pi/B_0^2) \sum_j n_j T_j$.

The matrix M in Eq. (44) has $3L$ eigenvalues λ_i ($i = 1, \dots, 3L$), and $3L$ corresponding eigenvectors (and eigenfunctions). Equation (44) can be satisfied only if $\text{Det}(M) = 0$, and this will happen only if at least one of the λ_i 's is zero, since $\text{Det}(M) = \prod_{i=1}^{3L} \lambda_i$. Now each of the λ_i 's is a function of ω . Therefore the eigenvalue condition on ω is

$$\lambda_i(\omega) = 0 \quad , \quad (47)$$

for some single λ_i corresponding to a chosen eigenvector. The eigenvalue of ω can be found by standard root-finding methods, such as the method of false position. The number of basis functions, L , must be chosen sufficiently large that the eigenvalue of ω is independent of L to some acceptable accuracy.

In addition to a code which implements this procedure for solving Eq. (44), we have a code containing essentially identical physical assumptions, but using a model MHD equilibrium with circular, concentric magnetic surfaces; the magnetic field is given by

$$\vec{B} = B_0 \left(\vec{e}_\zeta + \frac{r}{R_0} \vec{e}_\theta \right) / \left(1 + \frac{r}{R_0} \cos \theta \right) , \quad (48)$$

where r is the toroidal minor radius and θ is now the actual geometrical poloidal angle.

III. RESULTS

We consider first the effects of changing MHD equilibria on the familiar trapped-electron modes belonging to the drift branch. In particular, we consider three sequences of equilibria which have a common starting point. The first sequence is a flux-conserving-tokamak (FCT) sequence⁴ with varying β . The second sequence is one of varying "D-ness," at fixed, low β . The third sequence is one of varying ellipticity, at fixed, low β . These sequences are all artificial in that they do not represent physical profiles and parameters from any real experiment.

The equilibrium which serves as the starting point for all three sequences is shown in Fig. 1. We concentrate on a particular surface, surface 25 of 49, where $\psi = 0.5$, which is shown darkened in on Fig. 1. The general parameters of this equilibrium are as follows [the subscript zero labels quantities at the magnetic axis ($\psi = 0$), the subscript a labels quantities at the plasma boundary ($\psi = 1$)]: $B_0 = 36.55$ kgauss, $R_0 = 132.6$ cm, $a = 18.67$ cm, $m_i/m_e = 36/2$, $Z_i = 1$, $q_0 = 0.701$, $q_a = 3.47$, and $\beta_0 = 0.015\%$. The local parameters on surface 25 are: $q = 2.50$, $\beta = 0.0086\%$, $n_e = 0.187 \times 10^{13} \text{ cm}^{-3}$, $T_e = 1.01$ keV, $T_i = 0.507$ keV, $\eta_i = \eta_e = 0$, and $v_e^* = 0.023$. With this low β value, and a circular plasma boundary, the flux surfaces are circular and concentric to a good approximation.

For a toroidal mode number $n = 25$, we obtain the eigenfunction shown in Fig. 2. This is an ordinary trapped-electron mode, with $\hat{\phi}$ and $\hat{A}_\perp$ even around $\theta = 0$ and $\hat{A}_\parallel$ odd. In this calculation we keep 21 bounce harmonics for trapped particles ($p = -10$ through 10), employ the "new" collision operator discussed in the appendix, choose $\alpha = 1.4 + 0i$, and set the ballooning parameter $\theta_0 = 0 + 0i$. The corresponding eigenfrequency is

$\omega_{\text{sh}} = 0.167 + 0.226 i$ or $\omega = (1.82 + 2.48 i) \times 10^5 \text{ sec}^{-1}$. It is this eigenfunction and eigenvalue that will be followed in the three sequences of equilibria.

A. β Variation

The last equilibrium in the FCT sequence, the one with the highest β value, is shown in Fig. 3. It can be seen that the magnetic axis has moved out substantially in the usual Shafranov shift,⁹ and surface 25, which is darkened in, has deformed slightly. This equilibrium has $\beta_0 = 7.28\%$ and $R_0 = 138 \text{ cm}$. Since this is an FCT sequence, the $q(\psi)$ profile is the same as for the initial equilibrium. In the FCT sequence, we adopt a scenario where n_j and T_j at any ψ vary as the square root of β , so that β_0 can remain fixed in a self-consistent fashion (recall that $\beta \equiv 8\pi \int n_j T_j / B_0^2$). This scenario was chosen as a reasonable compromise; the results for $\gamma(\beta)$ are quite scenario-dependent. Also the shapes of the $n_j(\psi)$, $T_j(\psi)$, and $\beta(\psi)$ profiles are kept fixed in this FCT sequence. In particular, we take $T_j(\psi) = \text{const.}$, so that $\beta(\psi) \propto n_e(\psi) = n_i(\psi)$. In this final equilibrium in the FCT sequence, the parameters on surface 25 are: $q = 2.50$, $\beta = 3.56\%$, $n_e = 3.78 \times 10^{13} \text{ cm}^{-3}$, $T_e = 20.8 \text{ keV}$, $T_i = 10.4 \text{ keV}$, $\eta_i = \eta_e = 0$, and $v_e^* = 0.0020$.

The variation of the eigenvalue as a function of the local β on surface 25 is shown in Fig. 4. The effect of increasing β is seen to be destabilizing for local β values up to 0.6%, and stabilizing for local β values above that value. The destabilization is due mainly to the increase in T_e and $T_i \propto \beta^{1/2}$, while the stabilization is due mainly to changes in the θ -variation of the magnetic drift velocity components [defined before Eq. (7)] which is due to the Shafranov shift and the surface deformation.

This is the same effect that is responsible for the so-called second stability region¹⁰ for ideal-MHD ballooning modes. This effect is illustrated in Figs. 5 and 6, which show the θ and ϕ components of the magnetic drift velocity versus the (periodic) poloidal angle θ , for $\beta = 0.0086\%$ and $\beta = 3.56\%$ respectively. In Fig. 5, $v_{d\theta}$ and $v_{d\phi}$ show the $-\cos\theta$ and $-\sin\theta$ variation characteristic of circular, concentric magnetic surfaces, with a negative or "bad curvature" (destabilizing) region for $v_{d\theta}$ around $\theta = 0$. In Fig. 6, at higher β , it can be seen that the "bad curvature" region has become narrower and shallower, and the "good curvature" (stabilizing or positive) region has become higher and wider. This figure also illustrates the order unity changes that can occur in the equilibrium at high β , and the consequent inaccuracy of a circular, concentric model equilibrium at high β .

Further comparing the low- β and high- β equilibria, it can be seen that $(2\epsilon_0)^{1/2}$ (see appendix), which is roughly the fraction of trapped particles, decreases from 0.43 to 0.36, and the average shear, which corresponds to $r(dq/dr)/q$ for the model equilibrium, increases from 0.92 to 0.93 (although the local shear at $\theta = 0$ increases much more, from 1.1 to 3.6). These are also stabilizing effects, but appear to be less important than the magnetic drift velocity changes. The connection length, obtained by direct integration along the field line, stays fixed at 329 cm.

The real frequency, ω_r , is seen in Fig. 4 to make a transition from the electron to the ion diamagnetic direction as β increases, and then back again. This is associated with the change in the ion gyroradius $\rho_i = T_i^{1/2} \propto \beta^{1/4}$, and the consequent change in $k_{\perp}\rho_i$, as discussed in Ref. 11.

B. D-ness variation

The second sequence of equilibria has varying "D-ness" parameter Δ , which is mainly related to the triangularity of the plasma boundary. The equilibrium in Fig. 1 has $\Delta = 0$. Figure 7 shows an inverse-D equilibrium with $\Delta = -0.5$, and Fig. 8 shows a normal-D equilibrium with $\Delta = +0.5$. Except for Δ , these three equilibria are identical, with very low β values. In both cases, surface 25 is darkened in.

The variation of the eigenvalue $\omega = \omega_r + i\gamma$ for the eigenfunction shown in Fig. 2 as a function of Δ is shown in Fig. 9. Now the "old" collision operator, discussed in the appendix, is used and $\alpha = 5.6$. It can be seen that, relative to the circular case, increasing normal D-ness of the equilibrium is slightly destabilizing and then saturates. Increasing inverse D-ness is moderately stabilizing. The effect on ω_r is slight in both cases. Thus varying D-ness may be expected to have only a small effect on trapped-electron modes.

C. Ellipticity variation

The third sequence of equilibria has varying ellipticity, b/a , of the plasma boundary, where a is the horizontal minor radius of the plasma boundary and b is the vertical minor radius. The original equilibrium has $b/a = 1$. The cross-sectional area, πab , is constrained to remain fixed in this sequence, so that a decreases as b increases. Except for a and b , the equilibria in this sequence are the same, with fixed low β . The final equilibrium in the sequence, with $b/a = 2.5$, is shown in Fig. 10, with surface 25 darkened in. The variation of the eigenvalue $\omega = \omega_r + i\gamma$ for the eigenfunction shown in Fig. 2 as a function of b/a is shown in Fig. 11. Again the "old" collision operator, discussed in the appendix, is used,

and $\alpha = 5.6$. The effect of increasing ellipticity is seen to be strongly stabilizing for $b/a < 2$, but the growth rate curve starts to level off for $b/a > 2$. The real frequency is much less affected by ellipticity. Thus vertical ellipticity has a much stronger stabilizing effect than D-ness on the trapped-electron mode. The stabilizing effect of ellipticity on this mode has been noted before in Ref. 12. Note that even for the largest ellipticity considered here, $b/a = 2.5$, the magnetic field strength $B(\theta)$ has not yet acquired multiple minima as a function of poloidal angle (cf. Ref. 13). However, the computer code employed here is capable of treating such a situation.

D. ISX-B

Now we consider an equilibrium with profiles and physical parameters corresponding to an actual tokamak experiment, namely ISX-B, for a discharge achieving a high β value on the magnetic axis, $\beta_0 = 10.7\%$. The global parameters for this equilibrium are: $B_0 = 12.8$ kgauss, $m_i/m_e = 3672$, $Z_i = 1$, $R_0 = 93$ cm, $a = 27$ cm, $q_0 = 0.91$, and $q_a = 3.45$. The local parameters on surface 33 (of 73) are: $\beta = 1.4\%$, $q = 1.58$, $T_e = 0.74$ keV, $T_i = 0.68$ keV, $n_i = n_e = 1.07$, $v_e^* = 0.21$, $\psi = 0.44$, and $n_e = 3.99 \times 10^{13} \text{ cm}^{-3}$. This equilibrium is shown in Fig. 12 with surface 33 darkened in. For a toroidal mode number $n = 19$, $\theta_0 = 0$, using the "new" collision operator discussed in the appendix, an unstable kinetic mode is found with eigenvalue $\omega/\omega_{*e} = -0.029 + 0.273 i$ or $\omega = (-0.37 + 3.46 i) \times 10^5 \text{ sec}^{-1}$. By artificially varying the β parameter for the fixed equilibrium in Fig. 12 from very low values to the actual value, this mode can be identified as the usual trapped-electron mode, which is unstable even for $\beta = 0$. The corresponding eigenfunction is shown in Fig. 13. Here, $\hat{\phi}$ and $\hat{A}_1$ are even in

θ around $\theta = 0$ and $\hat{\lambda}_1$ is odd. The variation in $\omega = \omega_r + i\gamma$ with n is shown for this eigenfunction in Fig. 14, for $5 \leq n \leq 35$. The average ion bounce and transit frequencies, $\bar{\omega}_{b1}$ and $\bar{\omega}_{t1}$, respectively, are also indicated. As n increases, the growth rate is seen to rise rapidly, to reach a peak at $n = 19$, and then to fall off more slowly. The real frequency is in the electron diamagnetic direction for $n \leq 16$ but changes to the ion diamagnetic direction for $n \geq 17$.

All of the results presented up to this point have been for the lowest order (local) eigenvalue $\omega = \omega^{(0)}(n, \psi, \theta_0)$ in the ballooning formalism hierarchy of equations. As shown in Ref. 2, it is also possible to compute a global eigenvalue, ω , for the particular case of closely spaced turning points. This leads to the so-called "1/n correction term" in the form²

$$\begin{aligned} \omega &= \omega^{(0)}(n, \psi^0, \theta_0^0) + \frac{1}{2n(dq/d\psi)} \left[\frac{d^2 \omega^{(0)}}{d\theta_0^2} \frac{d\omega^{(0)}}{d\psi^2} \right]^{1/2} \bigg|_{\theta_0=\theta_0^0, \psi=\psi^0} \\ &= \omega^{(0)} + \delta\omega, \end{aligned} \quad (49)$$

where θ_0^0 and ψ^0 are determined by the conditions, for $\omega^{(0)} \equiv \omega_r + i\gamma$,

$$\left. \frac{\partial \gamma}{\partial \theta_0} \right|_{\theta_0=\theta_0^0} = 0, \quad (50)$$

and

$$\left. \frac{\partial \gamma}{\partial \psi} \right|_{\psi=\psi^0} = 0. \quad (51)$$

For the ISX-B equilibrium, for $n = 19$, $\theta_0^0 = 0$, and $\psi^0 = 0.56$ corresponds to surface 41. On surface 41, for $\theta_0 = 0$, $dq/d\psi = 2.59$,

$\partial^2 \omega^{(0)} / \partial \theta_0^2 = (-5.68 - 5.08 i) \times 10^6 \text{ sec}^{-1}$, and $\partial^2 \omega^{(0)} / \partial \phi^2$
 $= (1.50 - 2.18 i) \times 10^7 \text{ sec}^{-1}$. Thus $\omega^{(0)} = (0.80 + 3.75 i)$
 $\times 10^5 \text{ sec}^{-1}$, $\delta\omega = (-0.17 - 1.43 i) \times 10^5 \text{ sec}^{-1}$, and $\omega = (0.63 + 2.32 i)$
 $\times 10^5 \text{ sec}^{-1}$, according to Eq. (49). This procedure could be carried out for
 any n , although it is only valid when n is sufficiently large that
 $|\delta\omega| < |\omega^{(0)}|$, i.e., when the turning points are in fact closely spaced.
 For smaller values of n , a more general WKB procedure would need to be
 carried out. Such a calculation for the ideal MHD case is discussed in Ref.
 14.

IV. CONCLUSIONS

In this paper we have presented a comprehensive calculation of low-frequency modes in tokamaks. The analysis has been applied to eigenmodes from the familiar trapped-electron drift branch for three sequences of equilibria. The effect of increasing β , in an FCT sequence of equilibria, is found to be initially destabilizing, but then becomes stabilizing when the Shafranov shift and surface deformation become significant. The effect of varying the equilibrium from an inverse-D shape to circular to a normal-D shape appears to have a relatively minor effect on the trapped-electron mode growth rate, at fixed low β values. On the other hand, the effect of increasing vertical ellipticity on the trapped-electron mode is found to be quite strongly stabilizing. When a realistic equilibrium representing the ISX-B device at high β is analyzed, a kinetic mode, associated with the trapped-electron drift branch, is found to be unstable over a wide range of toroidal mode numbers. Also, for the most unstable mode number, the so-called " $1/n$ correction term" appears to be insufficient to stabilize the mode.

The comprehensive kinetic code described in the present paper can also be used to study kinetic corrections to the usual ideal MHD ballooning mode.^{7,15,16} This general subject will be considered in a future paper.

Finally, we note that in future work we will eliminate the assumption that the equilibrium distribution functions for all species can be taken to be Maxwellian. This is in fact a very good approximation for the electrons and for all of the ordinary ion species. However, it cannot properly represent a neutral beam ion species which is in the process of slowing down.¹⁷ For such a beam species, an equilibrium distribution function, which is an arbitrary function of energy and pitch angle, is required. When

this is implemented, the kinetic analysis described here can be properly applied to experiments utilizing perpendicular (as well as parallel) neutral beam injection heating.

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Appendix: COLLISION OPERATORS

The collision operators employed are Krook operators, $C(\hat{h}) = -\nu_f(E, \Lambda)\hat{h}$, with an effective collision frequency $\nu_{fj}(E, \Lambda)$, where j labels the species. For electrons the form used is

$$\nu_{fe}(E, \Lambda) = \left(\nu_f^l + \nu_f^h \frac{\nu_e^*}{X^3 Z_f} [Z_f + H_{ee}(X)] \right) \times \left(1 + \frac{\nu_e^*}{X^3 Z_f} [Z_f + H_{ee}(X)] \right)^{-1} \quad (A1)$$

Two different forms of ν_f^l are used. For the "old" operator

$$\nu_f^l \equiv \frac{\nu_e}{X^3} \frac{Z_f + H_{ee}(X)}{|h_m - \Lambda|} \quad (A2)$$

while for the "new" operator

$$\nu_f^l \equiv \frac{\nu_e \epsilon_0}{X^3} \frac{Z_f + H_{ee}(X)}{|h_m - \Lambda|^2} \frac{0.111(|\omega|/\nu_e Z_f) + 1.31}{11.79(|\omega|/\nu_e Z_f) + 1} \quad (A3)$$

For both operators,

$$\nu_f^h \equiv 1.08 \nu_e \frac{Z_f + H_{ee}(1)}{1 + H_{ee}(1)} \quad (A4)$$

Here h_m is defined after Eq. (42), and $\epsilon_0 \equiv 1 - h_m$, $Z_f \equiv \sum_i Z_i^2 n_i/n_e$, $\sum_i$ implies a sum over all ionic species, $e_i \equiv Z_i |e|$, Z_i is the ion charge number, $\nu_e \equiv 1221.8 n_e \lambda_e^3 / T_e^{3/2}$, $\lambda_e = 15.94 - (1/2) \ln(n_e/T_e^2)$, n_e is in units of 10^{13} cm^{-3} , T_e is in units of keV, ν_e is in units of sec^{-1} , $\nu_e^* \equiv \nu_e Z_f / (\epsilon_0^{3/2} \bar{\omega}_{te})$, $\bar{\omega}_{te} \equiv (2n_e/m_e)^{1/2} / qR_0$, $\bar{\omega}_{be} \equiv \epsilon_0^{1/2} \bar{\omega}_{te}$, $H_{ee}(X) \equiv (\pi^{1/2} X)^{-1} \exp(-X^2) + [1 - 1/(2X^2)] \text{erf}(X)$, erf is the usual error

function, $\chi \equiv (m_e T_e)^{1/2}$, and $H_{ee}(1) = 0.6289$. The "old" operator for v_f^i , which is the same as that used in Ref. 1, reproduces the results of calculations using a Lorentz operator in the limit $|\omega| < v_e z_f < \bar{\omega}_{be}$. As shown in detail in Ref. 3, the "new" operator for v_f^i reproduces the result from Lorentz operator calculations in the limit $v_e z_f < |\omega| < \bar{\omega}_{be}$ as well as the limit $|\omega| < v_e z_f < \bar{\omega}_{be}$, with interpolation in between. In the limit $\bar{\omega}_{be} < v_e z_f$, the collision frequency v_f^h reproduces the effect of using a full Fokker-Planck operator through the Braginskii transport coefficients.¹⁸ A number-conserving term in the electron collision operator would have no effect for the parameter range considered here.¹

For the ions, the collision frequency for species i is

$$v_{fi} = \frac{v_i(X)}{\chi^3} |h_m - \Lambda|^{-1}, \quad (A5)$$

for the "old" operator and

$$v_{fi} = \frac{v_i(X) \epsilon_0}{\chi^3} |h_m - \Lambda|^{-2} \frac{0.111(|\omega|/v_i) + 1.31}{11.79 (|\omega|/v_i) + 1}, \quad (A6)$$

for the "new" operator, where

$$v_i(X) \equiv 1221.8 z_i^2 \frac{(m_e/m_i)^{1/2}}{T_i^{3/2}} \sum_j n_j z_j^2 \lambda_{ij} H_{ij}(X), \quad (A7)$$

$$\lambda_{ij} \equiv 18.39 - \ln \left[\frac{z_i z_j (m_i + m_j)}{m_i T_j + m_j T_i} \left(\frac{n_i z_i^2}{T_i} + \frac{n_j z_j^2}{T_j} \right)^{1/2} \right], \quad (A8)$$

$$H_{ij}(X) \equiv H_{ee} \left[\left(\frac{m_j T_i}{m_i T_j} \right)^{1/2} X \right], \quad (A9)$$

n_i and n_j are in units of 10^{13} cm^{-3} , T_i and T_j are in units of keV, v_i is in

units of sec^{-1} and $X \equiv (m_i E / T_i)^{1/2}$. The "old" ion operator reproduces the effects of using a Lorentz operator in the limit $|\omega| < v_i$ only, while the "new" operator reproduces these effects in the limit $v_i < |\omega|$ as well as the limit $|\omega| < v_i$, with interpolation in between, as shown in Ref. 3. Neither operator includes finite Larmor radius effects. However, for $k_{\perp} \rho_i \lesssim 1$, these effects will be small for the modes considered.

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FIGURE CAPTIONS

Fig. 1. The poloidal flux surfaces for the initial equilibrium in the R-Z plane, where R is the toroidal major radius and Z is the vertical coordinate. Surface 25 (of 49) is darkened in. Here $\beta = 0.0086\%$ on surface 25, $\Delta = 0$ and $b/a = 1$.

Fig. 2. Eigenfunction on surface 25 of the initial equilibrium for a trapped-electron mode for toroidal mode number $n = 29$. Here $\hat{\phi}$, the perturbed electrostatic potential decomposed according to the ballooning representation, is shown versus θ , the nonperiodic ballooning coordinate along the field line. Also, $\hat{A}_{\parallel}$ and $\hat{A}_{\perp}$ are the corresponding parallel and perpendicular components of the perturbed vector potential.

Fig. 3. Final equilibrium in the FCT sequence. Here $\beta = 3.56\%$ on surface 25, which is darkened in, $\Delta = 0$ and $b/a = 1$.

Fig. 4. Variation of the eigenvalue $\omega = \omega_r + i\gamma$ on surface 25 versus β for the FCT sequence for a trapped-electron mode. Here $n_j, T_j \propto \beta^{1/2}$ as β varies and the shapes of the $n_j(\psi)$ and $T_j(\psi)$ profiles are held fixed.

Fig. 5. Variation of the magnetic drift velocity components with θ , the (periodic) poloidal angle, for $\beta = 0.0086\%$ on surface 25.

Fig. 6. Same quantities as in Fig. 5, but for $\beta = 3.56\%$ on surface 25.

Fig. 7. Equilibrium with greatest inverse D-ness. Here $\beta = 0.0086\%$ on surface 25, which is darkened in, $\Delta = -0.5$, and $b/a = 1$.

Fig. 8. Equilibrium with greatest normal D-ness. Here $\beta = 0.0086\%$ on surface 25, which is darkened in, $\Delta = +0.5$, and $b/a = 1$.

Fig. 9. Variation of the eigenvalue $\omega = \omega_r + i\gamma$ on surface 25 versus the D-ness parameter Δ for a trapped-electron mode.

Fig. 10. Final equilibrium in a sequence with increasing (vertical) ellipticity. Here $\beta = 0.0086\%$ on surface 25, which is darkened in, $\Delta = 0$, and $b/a = 2.5$.

Fig. 11. Variation of the eigenvalue $\omega = \omega_r + i\gamma$ on surface 25 versus the ellipticity parameter b/a for a trapped-electron mode.

Fig. 12. Equilibrium for ISX-B. Here $\beta_0 = 10.7\%$ and β on surface 33, which is darkened in, is 1.4% .

Fig. 13. Eigenfunction for an unstable kinetic mode on surface 33 of the ISX-B equilibrium, for a toroidal mode number $n = 19$. Quantities are as in Fig. 2.

Fig. 14. Variation of the eigenvalue $\omega = \omega_r + i\gamma$, for the mode in Fig. 13, with the toroidal mode number n .

#81 T 0150

Fig. 1

Fig. 2

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Fig. 3

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Fig. 5

FIG. 6

Fig. 7

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Fig. 8

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Fig. 9

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Fig. 10

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Fig. 11

Fig. 12

#81T0200

Fig. 13

Fig. 14