THE ASYMPTOTIC DEGREES
OF FREEDOM OF FLUID FLOWS

Final Report
for the Period July 1, 1986 - December 31, 1989

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September 1990

Prepared for

THE U.S. DEPARTMENT OF ENERGY
AGREEMENT NO. DE-FG02-86ER25020

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Abstract

We have obtained rigorous estimates for the attractors of some basic dissipative differential equations which are within the physical or numerical ranges (e.g. 2D Navier-Stokes equations). We have shown that the ring laser cavity equations have a finite dimensional attractor. We have constructed inertial manifolds for a large class of dissipative differential equations (e.g. Kuramoto-Sivashinsky and Ginzberg-Landau equations). For a large class of equations including the 2D Navier-Stokes equations we have introduced several approximate inertial manifolds which yield new approximative ordinary differential equations with better error estimates than those of the usual Galerkin approximations. We have evidence that the new approximating schemes lead to computational improvements upon the Galerkin schemes. We have given a normal form for the Navier-Stokes which allows the explicit asymptotic integration of the equations. We have also proposed a new theoretical approach to decaying homogeneous turbulence. We also made some contribution to robust control theory which may be relevant to fluid dynamics.

Our research work supported by the Grants DE-FG02-86ER25020 was oriented in 6 directions of research, 5 connected to the dissipative (partial or ordinary) differential equations (DDE) occurring in fluid mechanics and quite interconnected. We give a brief presentation of the results we obtained in each of those directions.

(a) Estimates of the dimension of the global attractors of DDE.

We have rigorously shown that the attractor of the 2D Navier-Stokes equation has a fractal dimension $\leq c_0(l_0/l_d)^2(\log(l_0/l_d)+1)^{1/3}$ where l_0 is the macroscale length while l_d is Kraichnan's dissipation microscale length [a.1]. This is the first rigorous connection between Kraichnan's theory of the inertial range and the mathematical theory of the 2D Navier-Stokes equations. We have also proven that the Kolmogorov-Spiegel-Sivashinsky equation [a.2], the nonlocal Burgers equations [a.3] and the ring laser cavity equations have finite dimensional attractors [a.4] (for other equations see [a.9], [a.10]) and we have found a new argument in favor of the classical $(Re)^{9/4}$ estimate of the number of the asymptotic degrees of freedom for homogeneous turbulence [a.5]. Also we have proven that the Lyapunov dimension of the global attractor of the Lorenz equations is in the range $[2.401..., 2.408...]$ [a.6].

Concerning the 2D Navier-Stokes equations we have again rigorously proven an empirical fact, namely that the energy spectrum falls exponentially in the dissipation range [a.7]. This also allowed us to give a rigorous basis to the Lumley-Sirovich approximation of DDE's [a.8].

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(b) Inertial manifolds for DDE and estimates of their dimension.

We have developed three methods of constructing inertial manifolds for a DDE. An inertial manifold is an invariant finite dimensional Lipschitz manifold M which attracts exponentially all the solutions of the DDE. On M , the DDE reduces to an ordinary differential system with $\dim M$ unknowns. The method used in [b.1] for the Kuramoto-Sivashinsky equations is based on a functional Perron type approach. In [b.2] we use a functional Cauchy type approach for constructing inertial manifolds, while in [b.3] we use a Hadamard type approach. We showed that our methods apply to the Kuramoto-Sivashinsky equation, Kolmogorov-Spiegel-Sivashinsky equation, Ginzburg-Landau equation, Cahn-Hilliard equation, etc. [b.2], [b.3]. These results show in a rigorous way that for any practical purposes

, these partial differential equations are actually ordinary differential equations albeit with a great number of unknowns. Besides, under mild additional conditions, every solution is exponentially tracked by an appropriate solution on M [b.2], [b.3] and [c.1].

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(c) Construction of approximate inertial manifolds.

Most of the 1 dimensional examples of DDE's have inertial manifolds, but the theoretical estimates of their dimension is still high; also most of the 2 dimensional DDE's (e.g. the periodic 2D Navier-Stokes equations are not known to have inertial manifolds. Therefore for practical reasons the attractors or inertial manifolds (which are elusive mathematical entities) have to be replaced by well chosen approximations, the so called approximate inertial manifolds. By compressing a DDE to such manifolds one obtains nonlinear Galerkin approximation of the DDE's (NLGA). The first NLGA was considered in [c.1]. Its computational utility was proven in [c.2]. A theoretically better NLGA was introduced in [c.3]. It was shown later by Jolly-Kevrekidis and Titi that Titi's NLGA in [c.3] is, for the Kuramoto-Sivashinsky equation, computationally more accurate than any other previously used approximation scheme. We also developed two new methods of constructing NLGA. One based on the introduction of an

effective viscosity in the Navier-Stokes equations [c.4], the other on several canonical ways of approximating the global attractor of a DDE by algebraic sets [c.5], [c.6]. All these approximations seem to have a very real practical value as indicated by the recent computational evidence obtained by Jolly-Kevrekidis-Titi and Temam-Rosier-Jouberteau. Although all these results use as basic variables a finite number of modes corresponding to low wave numbers (i.e. they yield spectral schemes), the methods can be easily adapted to finite difference schemes or finite elements schemes, as we have shown in [c.7].

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(d) Normal forms for quadratic DDE's.

We have introduced a new method for reducing the 2D or 3D to a normal form which can be integrated by trivial quadratures. We have given explicit formulas for the normal form of the equations as well as for the formal inverse of the normalizing map (see [d.1]). The method can be applied to any quadratic (or even algebraic) DDE. For instance when applied to the classical Riccati equation $\dot{u} + au + u^2 = 0$ it yields the "multiplicative integral"

$$a. \exp[a \int (au + u^2)^{-1} du] .$$

In the general case our method gives the asymptotic quadrature of a similar but functional multiplicative integral associated with the DDE. Hopefully, this approach may lead to an explicit asymptotic integration of the Hopf equation in the theory of fully developed turbulence.

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(e) Functional approach to the theory of decaying homogeneous turbulence.

We tried to set the theory of decaying homogeneous turbulence on a more rigorous mathematical basis [e.1]. Namely we studied some remarkable families (the existence of which is posited) of homogeneous statistical distributions (on the space of divergence-free vector fields with finite space average energy and enstrophy) which are invariant under the change of scales and the time evolution of the Navier-Stokes equations. Then we derived rigorously certain basic empirical or heuristical results in the theory of decaying homogeneous turbulence (see [e.1]; also see [a.]); in particular we obtained in a

consistent way the von Karman-Howarth-Dryden equation and the Kolmogorov energy spectrum.

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(f) Robust control theory for distributed systems.

Because of its possible relevance to fluid dynamics, we have considered several problems in nonlinear and robust control. Specifically, we have developed nonlinear generalizations of the well-known control H^∞ -theory which allows one to synthesize feedback controllers for a broad class of nonlinear plants. This work has caught the interest of some engineers in the aerospace industry, especially at Honeywell where these ideas were used in the control of the NAS (National Aerospace Plane). See [f.1], [f.2].

Moreover in relation to robust control (i.e. the control of systems with parameter and modelling uncertainty) we developed an effective algorithm for distributed parameter systems [f.3], [f.4], and we have also worked out a novel interpolation scheme called "spectral interpolation". This allows one to design controllers for systems in the presence of the most commonly used uncertainty models [f.5], [f.6].

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