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EMITTANCE GROWTH FROM RADIATION FLUCTUATIONS*

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1. INTRODUCTION

As an electron bunch travels through a transport system, fluctuations in the energy loss of individual electrons cause the size of the bunch to grow. I give here a calculation of the quantum-induced growth of the emittance of a beam in one transverse coordinate, making the following approximations: (1) that the transport system is linear; (2) that there is no coupling between the two transverse motions; and (3) that the radiation effects can be described by their values on the central design trajectory. This last assumption means that I am considering systems in which the quantum effects from bending magnets are much larger than from the focussing lenses.

It will be convenient to assume that the central design trajectory of the transport system lies in a plane - taken to be horizontal. This trajectory can be characterized by its curvature function $G(s) = 1/\rho(s)$, where s is the distance along the trajectory. I will take that the longitudinal coordinate s extends from zero at some specified "input" of the transport system to $s=L$ at a specified "output" point - which for SLC we may think of as the collision point. The transverse coordinate $x(s)$ will specify the horizontal displacement of any particular trajectory from the design one. I will be concerned only with motion in this coordinate, and in the x -emittance - namely the phase area in the x - x' plane.

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2. PHOTON INDUCED TRAJECTORIES

Suppose that an electron travels along the design trajectory until it reaches the position s_1 at which point it emits a photon of energy u . Its subsequent motion will depart from the design trajectory, and it will arrive at the end of the transport system - at $s=L$ - with some displacement Δx_L and slope $\Delta x'_L$. (By $\Delta x'_L$ I mean $d(\Delta x)/ds$ evaluated at $s=L$.) Both Δx_L and $\Delta x'_L$ are proportional to the photon energy u and depend on the focussing matrix from s_1 to L . So let me write

$$\Delta x_L(u, s_1) = (u/E_1) D(s_1, L), \quad (1a)$$

$$\Delta x'_L(u, s_1) = (u/E_1) D^{\dagger}(s_1, L), \quad (1b)$$

where E_1 is the electron energy at s_1 . The functions $D(s, L)$ and $D^{\dagger}(s, L)$ are, in effect, the dispersion function and its derivative (at L) of the partial transport system that extends from s to L . In the notation of K. Brown,¹¹ $D(s, L) = R_{16}$ and $D^{\dagger}(s, L) = R_{26}$ where the R 's refer to the transport matrix element for the partial transport system that extends from s to L .

Because of my assumption that the transport system is linear, an electron that had had some displacement from the design trajectory when it emitted the photon would have a change in x_L and x'_L at the output which would still be given by Eq. (1).

It follows that an electron which starts along the design trajectory at the beginning of the transport system and emits a number of photons along the way will have the total displacements in phase space x_L and x'_L , which are just the sums of the individual contributions. I must, however, subtract off the average value of such sums, because of my initial assumption that the design orbit takes into account the average rate of energy loss. I get then that total phase coordinates

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at L are

$$x_L = \sum_i \Delta x_L(u_i, s_i) - \langle \Sigma \Delta x \rangle, \quad (2a)$$

$$x'_L = \sum_i \Delta x'_L(u_i, s_i) - \langle \Sigma \Delta x' \rangle, \quad (2b)$$

where I use the angled brackets $\langle \rangle$ to indicate an average - or expectation - value.

The increments Δx_L and $\Delta x'_L$ are, of course, stochastic quantities, each depending on the chance event of the emission of a photon. So also the sums x_L and x'_L are stochastic quantities, and we can make, at most, some statistical conclusion about them.

Say, for instance, that we were to launch a very small pencil beam into the system on the design orbit. We already know that the averages $\langle x \rangle$ and $\langle x' \rangle$ will be zero by design. So we will be primarily interested the second moments: $\langle x_L^2 \rangle$ (the mean value of the square of the displacement); $\langle x'^2_L \rangle$ the mean square slope; and $\langle x x' \rangle$ the covariance. From these quantities we will be able to determine the nature of the quantum-induced emittance.

It turns out, as I will show later, that these same quantities will also allow us to find the emittance growth of a beam that enters the system with already a finite phase area.

I turn now to an evaluation of these useful statistical quantities.

3. STATISTICAL QUANTITIES

As an electron moves along its trajectory it is emitting photons of energy between u and $u + du$ at the rate $n(u, s)du$. The s -dependence enters because both the power radiated and the spectrum shape depend on $G(s)$, the local curvature of the design orbit.^[1] (It is here that I make the approximation that the radiation depends on $G(s)$ rather than the actual curvature of the trajectory.) The probability that a photon will be emitted in the path increment $ds = cdt$ with an energy in the interval du is then

$$n(u, s)du \, ds/c$$

The expected number $\langle N \rangle$ of photon of all energies to be emitted as an electron traverses the whole system is

$$\langle N \rangle = \frac{1}{c} \int_0^L ds \int_0^\infty du \, n(u, s). \quad (3)$$

The integral over u has been worked out in Ref. 2. Taking the results given there, we get that

$$\langle N \rangle = C_1 \int_0^L G(s)E(s)ds \quad (4)$$

with

$$C_1 = 20.61 (GeV)^{-1}.$$

For a Poisson sum, such as we have in Eq. (2), the standard deviation in x is given by

$$\sigma_x^2 = \langle N \rangle \langle \Delta x^2 \rangle, \quad (5)$$

where $\langle N \rangle$ is the mean number of events, and $\langle \Delta x^2 \rangle$ is the mean square of the increment sizes (an intuitive justification for this formula is given in Appendix

A.) In the present context $\langle x \rangle$ is zero, so σ_x^2 is also $\langle x^2 \rangle$. And I can express $\langle \bar{\Delta x}^2 \rangle$ as $1/\langle N \rangle$ times an integral over the frequency distribution of the Δx , so that

$$\langle x_L^2 \rangle = \int_0^L \int_0^\infty \bar{\Delta x}_L^2(u, s) n(u, s) du \frac{ds}{c}. \quad (6)$$

Taking Δx_L from Eq. (1), I can express this result as

$$\langle x_L^2 \rangle = \frac{1}{c} \int_0^L \frac{D^2(s, L)}{E^2(s)} \int_0^\infty u^2 n(u, s) du ds. \quad (7)$$

The integral over u can now be performed. Following Ref. 2 we get that

$$\int u^2 n(u, s) du = c C_2 G^3(s) E^7(s), \quad (8)$$

;

where

$$C_2 = \frac{55}{24\sqrt{3}} \frac{r_e \hbar c}{(mc^2)^6} = 4.13 \times 10^{-11} m^2 (GeV)^{-5}.$$

Then the evaluation of $\langle x_L^2 \rangle$ is reduced to a single integral over the trajectory.

$$\langle x_L^2 \rangle = C_2 \int_0^L G^3 E^5 D^2 ds, \quad (9)$$

where G , E , and D are all functions of s . In some cases where E varies little, it will usually be satisfactory to replace it by some average value E_0 and take it out of the integral. Then

$$\langle x_L^2 \rangle \approx C_2 E_0^5 \int G^3 D^2 ds. \quad (10)$$

Clearly, the variance of the slope x_L' can be obtained in exactly the same way

- replacing Δx_L by $\Delta x'_L$ in the integral of Eq. (5). Then

$$\langle x_L^2 \rangle = C_2 \int E^5 G^3 D^{12} ds. \quad (11)$$

And the covariance of x_L and x'_L is obtained by replacing Δx_L^2 in Eq. (5) by $\Delta x_L \Delta x'_L$. (This formula is also justified in Appendix A.) Then

$$\langle x_L x'_L \rangle = C_2 \int E^5 G^3 D D^\dagger ds. \quad (12)$$

Finally, I should remark that these results have been obtained ignoring any radiation anti-damping - which I assume to be small. Also, this result does not take into account any radiation in quadrupole (or other) lenses, since radiation is assumed to occur only according to the curvature of the design trajectory.

4. ALTERNATE FORM FOR THE FLUCTUATIONS

Instead of characterizing the transport system by its "residual dispersion" functions $D(s)$ and $D^\dagger(s)$, I could have used the horizontal betatron, and off-energy functions $\beta(s)$ and $\eta(s)$, together with the betatron phase function $\phi(s)$ (from 0 to s),

$$\phi(s) = \int_0^s ds / \beta(s).$$

I will now express D and $D^\dagger$ in terms of these other variables.

I assume that suitable $\beta(s)$ and $\eta(s)$ have been defined. To be quite explicit - and to avoid unnecessary complications - let me assume that at the end of the system $\beta'(L) = 0$, and that $\eta(L) = \eta'(L) = 0$. Also let me write β_L for $\beta(L)$.

If an electron suffers an energy change u at the position s_1 its coordinates in betatron phase space suffer the increments Δx_1 and $\Delta x'_1$ given by

$$\Delta x_1 = \frac{u}{E}\eta(s_1); \quad \Delta x'_1 = \frac{u}{E}\eta'(s_1). \quad (13)$$

Thereafter, it executes betatron oscillations with respect to a displaced central trajectory located at $-(u/E)\eta(s)$. If I write the betatron displacement (only) as

$$\Delta x(s) = a\sqrt{\beta(s)}\cos(\phi - \theta),$$

the amplitude and phase constants a and θ are given in terms of the starting values of Δx and $\Delta x'$ at s_1 . Following Reference 2, Sections 2-6 and 5-5, we get that

$$a = \frac{u}{E} \left[H(s_1) \right]^{1/2} \quad (14)$$

and

$$\theta = \phi(s_1) - \alpha(s_1), \quad (15)$$

where $H(s)$ is the familiar "curly H" of Reference 2,

$$H(s) = \frac{1}{\beta} \left[\eta^2 + (\beta\eta' - \frac{1}{2}\beta'\eta)^2 \right], \quad (16)$$

and $\alpha(s)$ is a new function,

$$\alpha(s) = \arctan\left(\frac{\beta'}{2} - \frac{\beta\eta'}{\eta}\right). \quad (17)$$

At the end of the trajectory (at $s = L$) this oscillation (which may be superposed on others) will have displacement and slope given by

$$\Delta x_L = a\sqrt{\beta_L}\cos\{\phi(L) - \theta\}, \quad (18a)$$

$$\Delta x'_L = \frac{a}{\sqrt{\beta_L}}\sin\{\phi(L) - \theta\}. \quad (18b)$$

(I have made use of the fact that $\beta'(L) = 0$.) Since I have also postulated that $\eta(L)$ and $\eta'(L)$ are both zero the total displacement at $s = L$ is what has just

been given. Comparing this result with Eq. (1), we see that

$$D(s, L) = \sqrt{\beta_L} \sqrt{H(s)} \cos \Phi(s) \quad (19a)$$

and

$$D^{\dagger}(s, L) = \frac{1}{\sqrt{\beta_L}} \sqrt{H(s)} \sin \Phi(s) \quad (19b)$$

in which the angle Φ is given by

$$\Phi(s) = \phi(L) - \phi(s) + \alpha(s) = \Delta\phi(s, L) + \alpha(s), \quad (20)$$

so that $\Delta\phi(s, L)$ is the betatron phase advance from s to L .

Let me put all of these results in a final form which will probably be the most convenient for understanding the dominant contributions to the fluctuations. Since it is now clear that I am concerned only with the fluctuations at the output of the transport system (for SLC, the interaction point) let me now drop the subscript "L" and use instead "q" to identify the quantum source of the effect. Using Eqs. (19) together with Eqs. (9), (11), and (12) my final result becomes:

$$\langle x_q^2 \rangle = C_2 \beta_L \int E^5(s) G^3(s) H(s) \cos^2 \Phi(s) ds \quad (21a)$$

$$\langle x_q'^2 \rangle = \frac{C_2}{\beta_L} \int E^5(s) G^3(s) H(s) \sin^2 \Phi(s) ds \quad (21b)$$

and

$$\langle x_q x_q' \rangle = C_2 \int E^5(s) G^3(s) H(s) \cos \Phi(s) \sin \Phi(s) ds \quad (21c)$$

As mentioned before, it may often be convenient to replace $E(s)$ by its typical value E_0 and take it out of the integral.

A complete description of the distribution in x and x' would require calculating higher moments of the distributions – a more difficult program, which I will not tackle. The second moments are generally sufficient, however, so long as there is sufficient bending that the average number of photons emitted is not too small. If there are, say, ten or more photons emitted we may expect the distribution in x and x' to be nearly gaussian, and completely characterized by the second moments.

Incidentally, the method outlined here can be used to investigate the quantum-induced energy spread, time spread, and the correlations of these with x . I will, however, defer these matters to a later note.

5. BEAM EMITTANCE

The three variances calculated in Eqs. (21) can be used to obtain the magnitude of the emittance of the beam for x -motion. For beams with a gaussian distribution in phase space the magnitude of the emittance e of the beam as usually defined is related to the variances by

$$e^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \quad (22)$$

Even if the distributions are not gaussian, I can choose to define the x -emittance by the same relation, and I choose to do so.

If a pencil beam of negligible emittance is introduced into a beam transport system the magnitude of its emittance at the output is obtained from Eq. (22) by inserting the values of $\langle x_q^2 \rangle$, $\langle x'_q{}^2 \rangle$, and $\langle x_q x'_q \rangle$ from Eqs. (21).

If the input beam has already a significant emittance at the input, we can obtain its emittance at the output by folding it with the quantum-induced emittance according to a procedure I will outline.

Let me first introduce a vocabulary I find convenient. I will mean by the words "the emittance" and the symbol e the set of the three elements $\langle x^2 \rangle$, $\langle x'^2 \rangle$,

and $\langle xx' \rangle$. And with them I also include the concept of the phase ellipse in the x - x' plane. I may then speak of an "upright" or "tilted" emittance according to whether $\langle xx' \rangle$ is zero or not. I will use the expression "the magnitude of the emittance", and the symbol $\underline{e}$ (note underline) to refer to the single quantity defined in Eq. (22).

My parameters can be put in correspondence with those of the phase ellipse described in **TRANSPORT**^[1] as follows

$$\begin{aligned}\langle x^2 \rangle &\longleftrightarrow \sigma_{11} \\ \langle x'^2 \rangle &\longleftrightarrow \sigma_{22} \\ \langle xx' \rangle &\longleftrightarrow \sigma_{12} = \sigma_{21}\end{aligned}$$

I choose not to use the sigma-matrix notation, however, because of its possible confusion with traditional notation for standard deviations σ_x^2 and $\sigma_{x'}^2$.

Suppose now that a beam with a given emittance enters a linear transport system. Its emittance at the output, in the absence of radiation effects, is obtained by the usual optics methods. Let the elements of this emittance e_a at the output be $\langle x_a^2 \rangle$, $\langle x_a'^2 \rangle$, and $\langle x_a x_a' \rangle$. Next suppose that the quantum effects are turned on. The final emittance will be increased by the additional "smear" of each ray arising from quantum effects. That is, the quantum-induced emittance e_q calculated above must be folded with the primitive emittance e_a to get the resultant emittance e_r - all evaluated at $s = L$. (Remember that this folding is only correct under the assumption that the transport system is sensibly linear.)

The folding of two emittances is done simply by adding the corresponding elements of the two emittances:

$$\langle x_r^2 \rangle = \langle x_a^2 \rangle + \langle x_q^2 \rangle \quad (23a)$$

$$\langle x_r'^2 \rangle = \langle x_a'^2 \rangle + \langle x_q'^2 \rangle \quad (23b)$$

$$\langle x_r x_r' \rangle = \langle x_a x_a' \rangle + \langle x_q x_q' \rangle \quad (23c)$$

(For those not familiar with the theorem just stated, I give a demonstration in Appendix B).

In general, there is no simple relationship between the magnitude of the resultant emittance e_r and the magnitudes of its two contributions e_a and e_q . If, however, the two covariances $\langle x_a x'_a \rangle$ and $\langle x_q x'_q \rangle$ are each zero – that is, if both emittances happen to be upright ellipses – there is a rather simple connection. Let me indicate by $1/\bar{\beta}$ the aspect ration of an upright emittance; that is, $(\sigma'_x = \sigma_x/\bar{\beta})$. (Please note that $\bar{\beta}$ is a property of the beam, not of the lattice.) In general, the aspect ratios of the emittances e_a and e_q may be different. I can now write for the magnitudes of e_a and e_q

$$e_a = \langle x_a^2 \rangle / \bar{\beta}_a \quad (24a)$$

$$e_q = \langle x_q^2 \rangle / \bar{\beta}_q \quad (24b)$$

From these we can get a formula for the resultant magnitude. Using Eqs. (22) and (23),

$$e_r^2 = e_a^2 + e_q^2 + e_a e_q \left(\frac{\bar{\beta}_q}{\bar{\beta}_a} + \frac{\bar{\beta}_a}{\bar{\beta}_q} \right) \quad (25)$$

Notice that the quantity in parenthesis is always at least as large as 2, and only takes on its minimum value of 2 when $\bar{\beta}_q = \bar{\beta}_a$ – that is, when the emittances are geometrically similar. In that special circumstance Eq. (25) becomes simply

$$e_r = e_a + e_q. \quad (26)$$

For given magnitudes, e_a and e_q , the resultant magnitude is a minimum if the two emittances are similar; and that minimum value is just the sum of the two component magnitudes. Although I have demonstrated this last result only for upright emittances, it is not difficult to see that it must also be true even if the emittances are tilted, so long as they are similar.

6. ACKNOWLEDGEMENTS

The ideas in this report grew out of discussions with P. Bambade, K. Brown, H. DeStaebler, R. Servranckx, and W. Weng. In particular, much of the presentation of the material in Appendix A evolved from discussions with H. DeStaebler.

7. Appendix A – Statistics of Poisson sums

I try to give here a plausible justification for Eqs. (5) and (6) for $\langle x^2 \rangle$, and the related formula for $\langle x x' \rangle$. We have a random time sequence of events each of which I can characterize by the projected displacement and slope Δx and $\Delta x'$ the event will produce at the output point $s = L$. And we are interested in the total random walk that will result at a given time T (the time for an electron to traverse the system). Namely,

$$x = \sum_{i=1}^{N(T)} \Delta x_i - x_0 \quad (A1)$$

where the Δx_i are samples of a random variable δx , where N , the number of steps, is also a random variable with Poisson statistics, and where x_0 is the average value of the sum, which we subtract to make $\langle x \rangle$ equal to zero. It seems fairly clear that

$$x_0 = \langle N \rangle \langle \Delta x \rangle \quad (A2)$$

so I won't belabor the point.

The variance of x is a little more tricky. What I want to show is that

$$\sigma_x^2 = \langle N \rangle \langle \overline{\Delta x^2} \rangle \quad (A3)$$

with the average suitably defined.

First consider a Poisson process with a constant average rate ν and a fixed distribution of step sizes.

In such a case it is well known (see, for example, Parzen⁽⁴⁾) that Eq. (A3) is correct, with $\langle N \rangle = \nu T$. You should notice that this answer is different from what you might, at first, guess. You might think that since the sum has N terms, and since the variance of a sum is the sum of the variances, we should take

$$\sigma_x^2 = \langle N \rangle \sigma_{\Delta x}^2 \text{ (wrong)}$$

This equation is wrong, however, because it does not take into account the fluctuations in N , which makes an additional contribution to σ_x^2 in the amount $\langle x \rangle^2 \sigma_N^2$. Since the N comes from a Poisson distribution $\sigma_N^2 = \langle N \rangle$, so that the right answer is

$$\begin{aligned} \sigma_N^2 &= \langle N \rangle \sigma_{\Delta x}^2 + \langle \Delta x \rangle^2 \sigma_N^2, \\ &= \langle N \rangle (\sigma_{\Delta x}^2 + \langle \Delta x \rangle^2). \end{aligned} \quad (A4)$$

Recalling the definition of

$$\sigma_{\Delta x}^2 = \langle (\Delta x - \langle \Delta x \rangle)^2 \rangle = \langle \overline{\Delta x^2} \rangle - \langle \overline{\Delta x} \rangle^2, \quad (A5)$$

Eq. (A4) gives just Eq. (A3).

Now consider that the average frequency of events varies with time so that ν becomes $\nu(t)$, and take now the contribution to σ_x^2 that comes from a small time interval Δt . Physically, we can use Eq. (A3) to write that

$$\Delta(\sigma_x^2) = \nu(t) \Delta t \langle \overline{\Delta x^2} \rangle_t, \quad (A6)$$

because $\nu \Delta t$ is just $\langle \Delta N \rangle$, the expected number of events in Δt . Now we can argue that the total variance in the sum x is the sum of the variances from the

definite number of intervals Δt . Which is to say that

$$\sigma_x^2 = \sum_{\Delta t} \Delta(\sigma_x^2) = \sum_{\Delta t} \nu(t) \langle \overline{\Delta x^2} \rangle_t.$$

Or, letting Δt be arbitrarily small,

$$\sigma_x^2 = \int_0^T \nu(t) \langle \overline{\Delta x^2} \rangle_t dt \quad (A7)$$

This is the theorem needed to justify Eqs. (5) or (6) in the text.

If we chose to describe the distribution of the Δx in terms of $n(u,t)$ the rate at time t of the emission of photons with energies between u and $u + du$, we would have that

$$\langle \overline{\Delta x^2} \rangle_t = \frac{1}{\nu(t)} \int_0^\infty \overline{\Delta x^2}(u,t) n(u,t) du. \quad (A8)$$

then Eq. (A7) becomes

$$\sigma_x^2 = \int_0^T \int_0^\infty \overline{\Delta x^2}(u,t) n(u,t) du dt. \quad (A9)$$

But that integral is precisely what I would normally take, and then divide by $\langle N \rangle$, to get a global average of $\overline{\Delta x^2}$. So Eq. (A3) follows, as was to be shown.

Note also that Eq. (A9) becomes Eq. (6) of the text, if we change the variable t to s ; and recognize that because $\langle x \rangle = 0$, $\sigma_x^2 = \langle x^2 \rangle$.

Clearly the same arguments can be applied to the variables x' and $\Delta x'$.

The covariance $\langle xx' \rangle$ is perhaps less familiar, but it turns out that a nearly identical set of arguments is valid. Here, however, let me point out that if you

buy Eq. (A3) – as well as the corresponding one for x' – you should also accept that

$$\langle xx' \rangle - \langle x \rangle \langle x' \rangle = \langle N \rangle \langle \Delta x \Delta x' \rangle \quad (\text{A10})$$

by the following argument.

Consider a new variable y made up of the sum of x and x' :

$$y = x + x' = \sum_{i=1}^N (\Delta x_i + \Delta x'_i).$$

You now believe that, by using the theorem of Eq. (A3), that

$$\begin{aligned} \sigma_y^2 &= \langle N \rangle \langle (\Delta x + \Delta x')^2 \rangle \\ &= \langle N \rangle \langle (\overline{\Delta x^2}) + \langle \overline{\Delta x'^2} \rangle + 2\langle \Delta x \Delta x' \rangle \rangle. \end{aligned} \quad (\text{A11})$$

On the other hand, we can also write, σ_y^2 in terms of the variance of x and x' . Using the general relation (A5) for y .

$$\begin{aligned} \sigma_y^2 &= \langle y^2 \rangle - \langle y \rangle^2 = \langle (x + x')^2 \rangle - \langle x + x' \rangle^2 \\ &= \langle x \rangle^2 + \langle x' \rangle^2 + 2\langle xx' \rangle - \langle x \rangle^2 - \langle x' \rangle^2 - 2\langle x \rangle \langle x' \rangle \\ &= 2\langle xx' \rangle - 2\langle x \rangle \langle x' \rangle \end{aligned} \quad (\text{A12})$$

Equating the right-hand-sides of Eqs. (A11) and (A12), we obtain Eq. (10). In the text, $\langle x \rangle$ and $\langle x' \rangle$ are zero, so there,

$$\langle xx' \rangle = \langle N \rangle \langle \Delta x \Delta x' \rangle, \quad (\text{A13})$$

the relation that was used to obtain Eq. (12) of the text.

8. Appendix B – Emittance and Convolution

For an x-emittance which is bi-gaussian in x and x' the magnitude of the emittance is usually defined as $1/\pi$ times the area of the contour along which the beam density has dropped to $1/e$ of its central value. It can be shown that for such an ellipse which is centered at $x = x' = 0$, the magnitude of the emittance is given by

$$\epsilon^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \quad (B1)$$

The correlation coefficient r is defined by

$$r = \frac{\langle xx' \rangle}{\sigma_x \sigma_{x'}}, \quad (B2)$$

so the emittance can also be expressed in terms of r by

$$\epsilon^2 = \sigma_x \sigma_{x'} \sqrt{1 - r^2}. \quad (B3)$$

This last form may be more familiar, but is less convenient for present purposes.

I show here that when two diffuse emittances are convoluted, the variances simply add linearly.

Suppose a beam of emittance e_a is smeared out in such a way that each pencil of rays is given an emittance e_b , and we want to know the resultant emittance e of the beam. The smearing has the property that the coordinates of each particle become

$$\begin{aligned} x &= x_a + x_b \\ x' &= x'_a + x'_b, \end{aligned} \quad (B4)$$

where x_a and x'_a are the coordinates in emittance a , and x_b and x'_b in emittance b . The variables x_a and x'_a may, in general, be correlated random numbers as well

as x_b and x'_b . But I want here to assume that coordinates in e_a are uncorrelated to those in e_b , which means that x_a and x'_a are independent of (uncorrelated with) x_b and x'_b . With these assumptions, when I take the second moment

$$\langle x^2 \rangle = \langle x_a^2 + x_b^2 + 2x_a x_b \rangle,$$

the mean of $\langle x_a x_b \rangle$ is equal to $\langle x_a \rangle \langle x_b \rangle$, which by our postulates is zero, so that

$$\langle x^2 \rangle = \langle x_a^2 \rangle + \langle x_b^2 \rangle. \quad (B5)$$

A similar result applies to $\langle x'^2 \rangle$. For the covariance,

$$\langle x x' \rangle = \langle x_a x'_a + x_b x'_b + x_a x'_b + x'_a x_b \rangle.$$

Here x'_b is uncorrelated to x_a , and x'_a is uncorrelated to x_b , so the means $\langle x_a x'_b \rangle = \langle x_a \rangle \langle x'_b \rangle$ and $\langle x'_a x_b \rangle = \langle x'_a \rangle \langle x_b \rangle$ are zero. It follows that

$$\langle x x' \rangle = \langle x_a x'_a \rangle + \langle x_b x'_b \rangle, \quad (B6)$$

Equations (B5) and (B6) express the theorem used in Section 5 of the text.

The results of this Appendix are presented in a more general form in a report by M. Della Negra⁽⁴⁾.

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