TITLE: LECTURES ON FERMI LIQUID THEORY

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Lectures on Fermi Liquid Theory

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Introduction to the lectures on Fermi Liquid Theory

The Fermi liquid theory was first introduced by Landau\(^1\) in 1956 to provide a theoretical basis for the properties of strongly correlated Fermi systems. This theory has proven to be crucial for our understanding of a broad range of materials. These include liquid \(^3\)He, \(^4\)He, \(^3\)He-\(^4\)He mixtures, simple metals, heavy fermions, and nuclear matter to name a few. In the high temperature superconductors questions have been raised regarding the applicability of Fermi liquid theory to the normal state behavior of these materials. I will not address this issue in these lectures.

My focus will be to summarize the foundations of this theory and to explore the consequences. These lectures are in part a summary of the excellent review article by Baym and Pethick\(^2\) and the books by Pines and Nozières\(^3\) and Baym and Pethick.\(^4\) They include as well a summary of some articles that I have authored and co-authored.\(^5,6\) In the main body of the lectures I will not make any additional references to the books or articles. It should be emphasized, however, that these lectures are enormously influenced by references 1-6 and I would strongly encourage the readers of my lectures to read these books and articles on this subject. In the absence of reading the original materials, my lectures should provide the essentials of a mini course in Fermi liquid theory.


I. Thermodynamics and Stability of a Fermi Liquid

A. Free Fermi Gas:

The free Fermi gas is characterized by the single particle spectrum, \( \varepsilon_{p\sigma} = p^2/2m \). For a given momentum \( p \) there are two degenerate levels for \( \sigma = +1(\uparrow) \) and \( -1(\downarrow) \). Given the spectrum we can determine completely the thermodynamics of \( N \) identical free Fermions of spin \( \sigma \).

We begin by first writing down the entropy \( S \),

\[
S = \frac{S}{V} = -k_B \sum_{p\sigma} \left[ n_{p\sigma}^0 \ln n_{p\sigma}^0 + (1 - n_{p\sigma}^0) \ln (1 - n_{p\sigma}^0) \right] 
\]  

(1)

where \( V \) is the volume, \( k_B \) is Boltzmann's constant and \( n_{p\sigma}^0 \) is the momentum distribution function.

How do we obtain this formula? This comes from counting:

i) Divide the states into \( \uparrow \) groups with \( G_j \) the number of states in the \( j^{\text{th}} \) group.

ii) Divide the particles into groups with \( N_j \) particles in the \( j^{\text{th}} \) group.

We then ask, How many ways can we put \( N_j \) identical particles into \( G_j \) states? The steps that give Eq. (1) can be found in standard textbooks and we will not discuss them here.

The important message to take from this is the counting argument. This depended on our ability to specify the number of states in a given group and the number of particles in this group.

To determine the form of \( n_{p\sigma}^0 \), we could maximize the entropy or minimize the thermodynamic potential \( \Omega = \Omega(T,V,\mu) \) for fixed temperature, \( T \), \( V \), and chemical potential \( \mu \). Here we will outline the minimization of \( \Omega \) to determine the essential steps to get \( n_{p\sigma}^0 \). The thermodynamic potential \( \Omega \) is given by,

\[
\frac{\Omega}{V} = TS - \mu n - P
\]

(2)

with \( P \) the pressure, \( n \) the density and \( \epsilon \) the energy density. The quantities \( n \) and \( \epsilon \) are given by,

\[
n = \frac{N}{V} \sum_{p\sigma} n_{p\sigma}^0 
\]

(3)

and

\[
\epsilon = \frac{F}{V} \sum_{p\sigma} n_{p\sigma}^0 \varepsilon_{p\sigma} 
\]

(4)

with \( F \) the total energy.
For fixed $T$, $V$, and $\mu$ the equilibrium condition requires,

$$d \left( \frac{\Omega}{V} \right) = dV - TdS - \mu d\nu = 0 . \quad (5a)$$

From Eqs. (3) and (4) we have,

$$d\nu - \mu d\nu = \sum_{p\sigma} (\varepsilon^\sigma_{p\sigma} - \mu) \delta n^\sigma_{p\sigma} \quad (5b)$$

and from Eq. (1) we obtain,

$$Tds = -Tk_B \sum_{p\sigma} \ln \left( \frac{n^\sigma_{p\sigma}}{1 - n^\sigma_{p\sigma}} \right) \delta n^\sigma_{p\sigma} . \quad (5c)$$

If we now make use of Eqs. (5b and c) in Eq. (5) we find

$$(\varepsilon^\sigma_{p\sigma} - \mu) \cdot Tk_B \ln \left( \frac{n^\sigma_{p\sigma}}{1 - n^\sigma_{p\sigma}} \right) = 0$$

or

$$n^\sigma_{p\sigma} = \frac{1}{1 + e^{\frac{\varepsilon^\sigma_{p\sigma} - \mu}{k_B T}}}. \quad (5d)$$

The point to note here is that only the changes, $\delta n$, $\delta t$, and $\delta s$ are needed to derive Eq. (5d).

The momentum distribution function, $n^\sigma_{p\sigma}$, has the structure, for $T = 0$, shown in Fig. 1.

The Fermi momentum $p_F$ separates the occupied from the unoccupied states. This is also where $\varepsilon^\sigma_{p\sigma} - \mu$ changes sign. For the free Fermi gas $\mu = p^2_F/2m - F$ and near $p_F$ we have,

$$\varepsilon^\sigma_{p\sigma} = \mu + \frac{p^2}{2m} \quad \frac{p^2_F}{2m} \quad \varepsilon^\sigma_F(\mu, p_F)$$

where $\varepsilon^\sigma_F = p_F/m$. The density and $p_F$ are related,

$$n \sum_{p\sigma} n^\sigma_{p\sigma} = \int \frac{d\Omega}{(\pi \sigma)^2} \int_0^{p_F} \frac{p^2}{\pi^2} dp$$

$$\rho^2_F/(3\pi^2) \cdot (h = 1).$$
For the energy density we have,

\[ \epsilon = \sum_{p^a} \varepsilon_{p^a}^0 n_{p^a}^0 = 2 \int \frac{d\Omega}{4\pi^2} \int \frac{dp}{2\pi} \frac{p^1}{2m} \]

\[ = \frac{3}{5} n \varepsilon_F \]

For the specific heat \( c_v \) with \( V \) and \( N \) constant we have,

\[ c_v = T \left( \frac{\partial s}{\partial T} \right)_{V,N} \]  \( (6a) \)

where \( v = V/N \). The change in the entropy is given by,

\[ \delta s = - \sum_{p^a} \left[ \ln \left( \frac{n_{p^a}^0}{1 - n_{p^a}^0} \right) \right] \delta n_{p^a} \]

\[ = - \sum_{p^a} \left( \frac{\varepsilon_{p^a}^0 - \mu^1}{T} \right) \delta n_{p^a} \]

For the change in the momentum distribution function we have,

\[ \delta n_{p^a}^0 = n_{p^a}^0(T) - n_{p^a}^0(T = 0) \]

\[ = \left( \frac{n_{p^a}^0(T = 0)}{T} \right) \frac{\partial n_{p^a}^0}{\partial T} \delta T \]

\[ \frac{\delta s}{\delta T} = - \sum_{p^a} \left( \frac{\varepsilon_{p^a}^0 - \mu^1}{T} \right)^2 \frac{\partial n_{p^a}^0}{\partial \varepsilon_{p^a}^0} \]  \( (6b) \)

For small \( T \) the leading contribution to the integral in Eq. (7a) comes from \( |p| = p_F \). This allows us to write the integral in the following way,

\[ \frac{\delta s}{\delta T} = - \int \frac{d\Omega}{4\pi^2} \int \frac{dp}{2\pi} \frac{p^2}{T} \left( \frac{n_{p^a}^0}{T} \right)^2 \frac{\partial n_{p^a}^0}{\partial \varepsilon_{p^a}^0} \]

\[ = \frac{p_F^2}{\pi} \int dp \left( \frac{n_{p^a}^0}{T} \right)^2 \frac{\partial n_{p^a}^0}{\partial \varepsilon_{p^a}^0} \]

If we now let \( x = \frac{\varepsilon_{p^a}^0 - \mu^1}{T} \), then,
\[
\frac{\delta s}{\delta T} = -\frac{\mu_F^2}{\pi^2} \int_0^\infty \frac{T}{v_F} \, dx \, x^2 \, \frac{1}{T} \, \frac{\partial n}{\partial x}
\]

\[
\approx -\frac{\mu_F^2}{\pi^2 v_F^2} \int_{-\infty}^\infty dx \, x^2 \, \frac{\partial n}{\partial x} = +\frac{\mu_F^2}{\pi^2 v_F^2} \int_{-\infty}^\infty dx \, 2x \, n(x)
\]

\[
= \frac{\pi^2}{3} \, N^0(0)
\]

The quantity \(N^0(0)\) is the density of states at the Fermi energy and is given by

\[
N^0(0) = \frac{\mu_F^2}{\pi^2 v_F^2} = \frac{m}{2} \frac{\mu}{\varepsilon_F}
\]

(7)

for a spherical Fermi surface. The low temperature specific heat is then given by,

\[
c_v = T \left( \frac{\partial s}{\partial T} \right)_v = \frac{\pi^2}{3} \, N^0(0) \, T = \gamma T
\]

(8)

There are a couple of simple points to make here that will be useful in our discussion of Fermi liquids. The first is that the calculation of the energy density required a knowledge of the single particle spectrum for all \(p\). However, for the specific heat we only needed the spectrum close to \(p_F\). Moreover, it was not necessary for us to know how to calculate the chemical potential. All we needed to know is that the single particle spectrum evaluated at \(|p| = p_F\) is just \(\mu\). Of course this is all trivial in the free Fermi gas case, however, when the interactions are turned on these become important observations.

B. Landau Fermi Liquid Theory

We have spent some time discussing the free Fermi gas since it can be used to model some of the qualitative properties of several strongly interacting systems, including liquid \(^3\)He, electrons in metals, and nuclear matter. This observation led Landau to postulate the notion of a quasiparticle as an elementary excitation of the interacting Fermi liquid. The notion of a quasiparticle had been introduced earlier by Landau in the context of superfluid \(^4\)He. The important difference is that the Fermi liquid quasiparticles obey Fermi statistics, whereas, the quasiparticles of superfluid \(^4\)He obey Bose statistics.

The quasiparticle concept is both simple and profound. As we will see in the case of the Fermi liquid with a few basic assumptions the quasiparticle picture will allow us to determine a number of thermodynamic properties of a Fermi liquid. With this picture we will also discover new collective modes, at \(T = 0\) that we could not have anticipated from a study of the free Fermi gas. Thus, the quasiparticle picture proposed by Landau tells us when we should expect properties of the interacting system to be like the free Fermi gas as well as when and how they will differ.

The Fermi liquid theory of Landau is based on the assumption of one to one correspondence between the interacting and non-interacting system. There are two parts to this
assumption: The first is that the number of states in the interacting system is equal to the number in the non-interacting system. Consider the case of a spin 1/2 free Fermi gas where each state is uniquely specified by the momentum $p$ and the spin $\sigma$. The one-to-one correspondence implies that for each state in the non-interacting system of momentum $p$ and spin $\sigma$ there is exactly one state of momentum $p$ and spin $\sigma$ in the interacting system. The other part of the assumption is that for each particle in the non-interacting system there is a quasiparticle of the interacting system. From this we can immediately write down the expression for the entropy,

$$ s = -k_B \sum_{p\sigma} \left[ n^{(a)}_{p\sigma} \ln n^{(a)}_{p\sigma} + (1 - n^{(a)}_{p\sigma}) \ln (1 - n^{(a)}_{p\sigma}) \right] $$

with

$$ n = \sum_{p\sigma} n^{(a)}_{p\sigma} = \frac{p_F^3}{3\pi^2} $$

where $p_F^3/(3\pi^2)$ is the density of the non-interacting system. The form of Eq. (9) follows since entropy is just a counting problem and we have not changed the number of states or quasiparticles. The number of quasiparticles is equal to the number of particles by assumption and so we get Eq. (10). The equilibrium momentum distribution, $n^{(a)}_{p\sigma}$, function is obtained by minimizing the thermodynamic potential, $\Omega$,

$$ \delta \left( \frac{\Omega}{V} \right) = \delta \epsilon - T \delta s - \mu \delta n = 0 $$

To do this we need $\delta \epsilon$, note that $\epsilon$ is not needed. Landau assumed that the energy density, $\epsilon$, is given by

$$ \epsilon \approx \epsilon_n \sum_{p\sigma} \left\{ \frac{1}{p^3} \right\} n^{(a)}_{p\sigma} n^{(a)}_{p\sigma} $$

where

$$ \delta \left( \frac{\Omega}{V} \right) = \frac{\delta \epsilon}{\delta n_{p\sigma}}. $$

As a consequence of the interactions the quasiparticle Hamiltonian $H^{(a)}_{p\sigma}$ (where we drop the dependence on $n^{(a)}_{p\sigma}$) is a functional of $n^{(a)}_{p\sigma}$. This implies that the state of a given quasiparticle will depend on the state of all of the other quasiparticles.

We minimize $\Omega/V$ as we did with the free particle case, thus

$$ n^{(a)}_{p\sigma} = \frac{1}{1 + \frac{\epsilon_n}{T_n}} $$

$$ n^{(a)}_{p\sigma} = \frac{1}{1 + \frac{\epsilon_n}{T_n}}. $$
where we drop for now the \( \alpha \). For the single quasiparticle spectrum we have

i) \( \varepsilon_{p\sigma} = \mu \)

ii) \( \varepsilon_{p\sigma} \approx \varepsilon_{pF} + \frac{\partial \varepsilon_{p\sigma}}{\partial p} \bigg|_{pF} (p - pF) \)

which we can write as,

\[ \varepsilon_{p\sigma} - \mu \approx v_F (p - pF) \]  \hspace{1cm} (13a)

with

\[ v_F = \frac{\partial \varepsilon_{p\sigma}}{\partial p} \bigg|_{pF} = \frac{pF}{m^*} \]  \hspace{1cm} (13b)

where \( m^* \) is the effective mass of the quasiparticle (qp).

In deriving Eq. (13) we had to assume that \( \partial \varepsilon_{p\sigma}/\partial p \) was finite as \( |p| \to pF \). For an interacting system we do not know the explicit form of \( \varepsilon_{p\sigma} \). In general we can say that the quasiparticle spectrum has the form given by Eq. (13a) near \( pF \) that \( \varepsilon_{p\sigma} = \mu \neq \varepsilon_F = pF^2/2m^* \). This is quite clear in liquid \(^3\)He at zero pressure where \( \mu \simeq -2.5 \) K and \( \varepsilon_F = pF^2/2m^* \simeq 1.5 \) K.

The effective mass characterizes the qp spectrum near \( pF \). Since we have \( \varepsilon_{p\sigma} - \mu \approx v_F (p - pF) \) then the low T specific heat is the “same” as the ideal gas:

\[ c_v = T \left( \frac{\partial s}{\partial T} \right)_{N, \nu} = \frac{\pi^2}{3} N(0) T \gamma T \]  \hspace{1cm} (14)

where

\[ N(0) = \frac{pFm^*}{\pi^2} = \frac{3}{2} \frac{n}{N} \cdot \frac{pF}{m^*} \]

\[ \gamma = \frac{\pi^2}{3} N(0) \cdot \frac{N}{N(0)} \]

We should note that \( \mu \) does not appear explicitly in Eq. (14). As we noted above \( \mu \) in general is not equal to \( -pF \). To see this we give here the results for the dilute Fermi gas values for \( \mu \) and \( pF \):

\[ \mu = \frac{pF^2}{2m} \left( 1 + \frac{4}{3\pi} \frac{pF}{m} + \mathcal{O}(pF/m)^2 \right) \]

\[ pF = \frac{pF^2}{2m} \left( 1 + \mathcal{O}(pF/m)^2 \right) \]

Here we see that at linear order in the scattering length, \( a \), the \( \epsilon \) is a difference.
To proceed further we must include the interaction effects. To do so Landau included terms to next order in the deviations from the equilibrium distribution function, \( \delta n_{p\sigma} = n_{p\sigma}^{(a)} - n_{p\sigma} \),

\[
\varepsilon_{p\sigma} \equiv \varepsilon_{p\sigma}(n_{p\sigma}) = \varepsilon_{p\sigma}^{(a)} + \sum_{p'\sigma'} f_{pp'}^{\sigma\sigma'} \delta n_{p'\sigma'}
\]

(15)

where the qp-interaction is given by

\[
f_{pp'}^{\sigma\sigma'} = \frac{\delta^2 \varepsilon}{\lambda n_{p\sigma} \delta n_{p'\sigma'}}.
\]

For the energy density \( \varepsilon \) we have,

\[
\varepsilon = \varepsilon_{p\sigma}^{(a)} \delta n_{p\sigma} + \frac{1}{2} \sum_{p\sigma, p'\sigma'} f_{pp'}^{\sigma\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'}
\]

(16)

As we will see the qp interaction, \( f_{pp'}^{\sigma\sigma'} \), has several non-trivial consequences, e.g., collective modes at \( T=0 \).

It is useful to stop at this point and comment on Eq. (16). The fact that Landau was brilliant hardly needs to be stated, however, the special nature of this brilliance is worth noting. To capture the low temperature properties of a Fermi liquid Landau realized that only terms to second order in \( \delta n_{p\sigma} \) were needed. If we wanted to go beyond this we know that non analytic terms appear in the thermodynamics of the quasiparticles. Clearly, this can not be accounted for by including terms in Eq. (16) that are third order in \( \delta n_{p\sigma} \). What we see here is that Landau knew when to stop to capture the essential physics. He was truly a special kind of genius.

In the absence of spin orbit coupling \([ (\sigma \cdot p)(\sigma' \cdot p') ] \) the interaction is a scalar under simultaneous rotations of the spin and momenta, thus

\[
f_{pp'}^{\sigma\sigma'} = f_{pp'}^{a} + f_{pp'}^{a} \sigma \cdot \sigma
\]

(17a)

where the spin symmetric interaction is given by,

\[
f_{pp'}^{a} = \frac{1}{2} f_{pp'}^{11} + f_{pp'}^{11}
\]

(17b)

and the antisymmetric one is given by,

\[
f_{pp'}^{a} = \frac{1}{2} f_{pp'}^{11} - f_{pp'}^{11}
\]

(17c)
In Eq. (16) only \(|p|, |p'| \approx p_1'\) contribute, thus
\[ f_{pp'}^{s,a} = \sum_{\ell} f_{\ell}^{s,a} P_{\ell}(\cos \theta) \]
\[ \cos \theta = \hat{p} \cdot \hat{p}' \]
The Landau parameters, \(f_{\ell}^{s,a}\), can be made dimensionless as follows.
\[ F_{\ell}^{s,a} = N(o) f_{\ell}^{s,a} \]
For \(^3\)He
\[ N(o) \sim \frac{n}{\varepsilon_F} \sim o K^{-1} \Lambda^{-3} \]
\[ f_{\ell} \sim o K \Lambda \]
The expansion in Eq. (16) is an expansion in the small parameter \(\hbar n_{p\sigma}\), thus it is exact as long as
\[ \sum_{p\sigma} \varepsilon_{(0)}^{(p\sigma)} \hbar n_{p\sigma} \ll |\varepsilon_o| \]
and
\[ \sum_{p\sigma, p'\sigma'} f_{pp'}^{p\sigma} \hbar n_{p\sigma} \hbar n_{p'\sigma'} \ll |\varepsilon_o| \]
These constraints are such that the theory is still valid for rather large values of the dimensionless Fermi liquid parameters and effective mass. For example in \(^3\)He with \(P = 34\) bar, \(F_{p}^3 \sim 10^2\) and \(m^*/m \sim 6\), these are not small parameters!

Let us now consider some of the equilibrium properties. The compressibility, \(K\) at \(T = 0\) is given by
\[ K^{-1} = n \frac{\partial P}{\partial n} + n^2 \frac{\partial P}{\partial \mu} \]
where we make use of the Gibbs-Duhem relation
\[ dP = s dT + n d\mu \]
with \(s = 0\) at \(T = 0\) to get the final expression.

To express this in terms of Fermi liquid parameters we need the change \(n, \mu_{p\sigma}\) due to a change in \(\mu\). This is given as follows.
\[
\delta n_{p_\sigma} = \frac{\partial n_{p_\sigma}}{\partial \varepsilon_{p_\sigma}} (\delta \varepsilon_{p_\sigma} - \delta \mu)
\]

where from Eq. (15) we have,

\[
\delta \varepsilon_{p_\sigma} = \sum_{p', \sigma' \mu'} f^{\sigma \sigma'}_{p \mu' \mu} \delta
\]

\[
-\frac{p^2}{\pi^2} \int \frac{dp}{2\pi} \left\{ -\delta(\varepsilon_{p_\sigma} - \mu) \right\} (f^{\sigma}_{\mu} \delta n - \delta \mu)
\]

\[
= -\frac{p^2}{\pi^2 v_F^2} \int dp \{ \delta(p - p_F) \} (f^{\sigma}_{\mu} \delta n - \delta \mu)
\]

Note that the delta function picks out the momenta near \(p_F\). We finally get that,

\[
\delta n = -N(\sigma)(f^{\sigma}_{\mu} \delta n - \delta \mu)
\]

or

\[
\frac{\delta \mu}{\delta n} = \frac{1 + F^a_\sigma}{N(\sigma)}.
\]

For the compressibility, \(K^1\) (or the incompressibility, \(K^{-1}\)) we find,

\[
K^{-1} = \frac{\hbar^2}{m v_F^2} \left( \frac{1 + F^a_\sigma}{N(\sigma)} \right) = \frac{2}{3} n v_F (1 + F^a_\sigma)
\]

\[
\text{nmv}_1^2
\]

where \(v_F^2 = (v_F^2/3)(m^*/m)(1 + F^a_\sigma)\) is the first sound velocity.

The magnetic susceptibility at constant \(N\) and \(V\), \(\chi_{N,V}(m)\), is given by

\[
\chi^1(m) = \left( \frac{\partial H}{\partial m} \right)_{N,V}
\]

\[
(20)
\]
For small magnetic fields, $\chi(m) = \chi + O(m^2)$, where $\chi = m/B$, where $B = |B|$. To determine $\chi$ we first compute $m_\z = m$, where we chose the $\hat{z}$ axis such that $\hat{z} \parallel B$, then,

$$m = \sum_{p\sigma} \sigma z n_{p\sigma} = \sum_{p\sigma} \sigma z (\tilde{n}_p^{(\sigma)} + \delta n_{p\sigma})$$

$$= \sum_{p\sigma} \sigma z \delta n_{p\sigma}$$

and

$$\delta n_{p\sigma} = \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} (\varepsilon_{p\sigma} - \delta \mu).$$

The chemical potential is just the Gibbs Free energy per particle, where for a paramagnetic material we have that

$$\mu = \frac{G}{N} = \frac{G(m = 0)}{N} + O(m^2)$$

Thus to linear order we can ignore $\delta \mu$.

$$\varepsilon_{p\sigma} - \varepsilon_{p\sigma}^{(\sigma)} = -\sigma \cdot B + \sum_{p'\sigma'} f_{p\sigma}^{\sigma'} \delta n_{p'\sigma'}$$

$$= -a \sigma z B$$

The last line follows since we apply a uniform magnetic field. If we now plug this last expression into the above equations for $\delta n_{p\sigma}$ and $m$ we have,

$$a \sigma \cdot B$$

$$a \sum_{p'\sigma'} (f_{p}^{\sigma} + f_{p'}^{\sigma} \sigma \cdot \sigma') \frac{\partial n_{p'}}{\partial \sigma} (\sigma' \cdot B)$$

$$a \sigma \cdot B + a F_{o}^{\alpha} \sum_{\sigma'} (\sigma \cdot \sigma') \sigma' \cdot B$$

$$a(1 + F_{o}^{\alpha}) \sigma \cdot B$$

For the coefficient $a$ we get,

$$a = \frac{1}{1 + F_{o}^{\alpha}}$$
The parameter $a$ can be viewed as a renormalization of the Zeeman splitting due to qp-interaction. The magnetization is then given by,

$$m = \sum_{p, \sigma} \sigma_z \delta n_{pa}$$

$$= \int \frac{d\Omega}{4\pi^2} \int p_f^2 \frac{d\epsilon}{2\pi} \sum_{\sigma} \sigma(\sigma \cdot B) \frac{1}{1 + F_{f}^a} \delta(p - p_f)$$

$$= \frac{N(a)}{1 + F_{f}^a} B,$$

where we have set the magnetic moment $g\mu_B = 1$. From this we find,

$$\chi = \frac{N(a)}{1 + F_{f}^a}. \quad (21)$$

**Stability Conditions for a Normal Fermi Liquid (Pomeranchuck)**

In equilibrium we used the extremal condition,

$$\delta \left( \frac{\Omega}{V} \right) = 0,$$

to determine $n_{pa}^{(e)}$. However, to insure that we have a stable minimum it is necessary to show that $\delta^2 \left( \frac{\Omega}{V} \right) \rightarrow 0$. This stability criterion will impose constraints on the Fermi liquid parameters (or Landau parameters). To see this we first expand $\Omega/V$ to second order in $\delta n_{pa}$.

$$\delta \left( \frac{\Omega}{V} \right) = \sum_{p, \sigma} \left( \frac{n_{pa}}{\mu} \right) \delta n_{pa}$$

$$+ \frac{1}{2} \sum_{p, \sigma} \delta n_{pa}^{(e)} \delta n_{pa}^{(e)} \epsilon_{p} \delta \epsilon_{pa}, \quad T \delta \epsilon_{pa}$$

For the change in entropy we have,

$$\delta s = k_B \sum_{p, \sigma} \left\{ \delta n_{pa} \left[ \ln \frac{1}{n_{pa}} n_{pa}^{(e)} \right] + \frac{1}{2} \frac{\delta n_{pa}^{2}}{n_{pa}} \right\}$$

Collecting terms that are second order in $\delta n_{pa}$ we have

$$\sum_{p, \sigma} \delta n_{pa}^{(e)} \delta n_{pa}^{(e)} + T \sum_{p, \sigma} \frac{\delta n_{pa}^{2}}{n_{pa}^{(e)}} \rightarrow 0. \quad (22)$$
where the terms linear in $\delta n_{p\sigma}$ vanish in equilibrium.

If we consider now arbitrary distortions of the F.S. we have

$$\delta n_{p\sigma} = \frac{-\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \sum_i \nu_i^\sigma P_i(\cos \theta_p)$$

where $\cos \theta_p = \hat{p} \cdot \hat{q}$ and $\hat{q}$ is the $z$ direction defined by the external disturbance. Here again we note that the distortions are confined to be near $\rho_F$. If we plug this $\delta n_{p\sigma}$ into Eq. (22) and carry out the integrations we obtain the Pomeranchuk stability conditions,

$$\left(1 + \frac{F_{\rho}^{\alpha \beta}}{2\ell + 1}\right) > 0.$$  \hfill (23)

To see how these conditions can be used we consider a uniform magnetic field

$$\delta m = N(\rho)\nu_{\sigma}^\alpha = N(\rho)(\nu_{\sigma}^1 - \nu_{\sigma}^0)$$

then

$$\frac{\Omega(m) - \Omega(\rho)}{v} \approx \left(1 + \frac{F_{\rho}^{\alpha \beta}}{2N(\rho)}\right) m^2$$

(Small fields)

Suppose $1 + F_{\rho}^{\alpha \beta} < 0$, then increasing $m$ lowers the Grand potential which would imply that the ferromagnetic phase is the stable phase. From the Fermi liquid side we see that as $F_{\rho}^{\alpha \beta} \to 1$ from above the Fermi liquid is unstable since large changes in $m$ make small changes in $\Omega(m)$. Liquid $^3$He is close to a ferromagnetic instability and $F_{\rho}^{\alpha \beta} \approx 0.76$ at $P = 34$ bar, however, it solidifies before becoming a ferromagnet!

Galilean Invariance and the Effective Mass:

For a Galilean invariant (GI) system the effective mass is related to the Fermi liquid parameter $F_{\rho}^{\alpha \beta}$. To see this consider observing the Fermi liquid from a frame (the $'l'$ frame in Fig. 2) moving with a velocity $u = (u_x, u_y, u_z)$. In the rest frame we add a quasiparticle of momentum $p$ and energy $\varepsilon_{\rho \sigma}$. This is the same as adding a bare particle of momentum $p$ and $\varepsilon_{\rho \sigma} = p^2/2m$ since there is a one to one correspondence between quasiparticles and particles. We can picture this as follows:

From GI we know that the form of the physical laws are unchanged going from the rest to the moving frame. In the free particle case this gives,

$$\rho' = \rho \quad (p' = p + mu) \quad (\varepsilon' = \varepsilon + \frac{p^2}{2m} - \frac{u^2}{2m} - p \cdot u),$$

where we have dropped for now the spin label and we are considering $u \cdot v_F$. If we now make use of the one to one correspondence between quasiparticles and bare particles and the GI we have that,
\[ \varepsilon'_{p-\mu} - \varepsilon_p = \varepsilon'_{p-\mu} - \varepsilon'_p = -p \cdot u. \]

We now expand \( \varepsilon'_{p-\mu} \) to linear order in \( u \),

\[ \varepsilon'_{p-\mu} \approx \varepsilon'_p - mu \cdot \frac{\partial \varepsilon'_p}{\partial p} \bigg|_{u=0} \]

\[ = \varepsilon'_p - \frac{m}{m*} u \cdot p. \]

For the quasiparticle energy \( \varepsilon'_p \) measured in the moving frame of momentum \( p \) (measured from the rest frame) we have,

\[ \varepsilon'_p = \varepsilon_p + \delta \varepsilon'_p \]

where

\[ \delta \varepsilon'_p = \sum_{k,s} f_{pk} \delta n'_k \]

and \( \delta n'_k \) is the difference of the distribution function measured in the moving and rest frame. This is given by

\[ \delta n'_k \]

\[ = n'_k - n_k \]

\[ = n_{k+\mu} - n_k \approx \mu \cdot \frac{\partial n_k}{\partial k} \]

\[ = \frac{m}{m*} \cdot k \cdot u \cdot \frac{\partial n_k}{\partial k} \]

Here \( n_{k+\mu} \) is the distribution function measured in the rest frame with a center located at the origin of the moving frame. If we plug \( \delta n'_k \) into the equation for \( \varepsilon'_{p-\mu} \) we have,

\[ \varepsilon'_{p-\mu} - \varepsilon_p = \sum_{k,s} f_{pk} \delta n'_k \]

\[ = \frac{m}{m*} \cdot u \cdot p \]

Carrying out the integral we have,

\[ \frac{m}{m*} \left( \chi(\mu) \frac{f_i}{3} + 1 \right) u \cdot p = u \cdot p. \]

From this we get,
\[
\frac{m^*}{m} = 1 + \frac{F^*_1}{3}
\]

or

\[
\frac{m^*}{m} = \frac{1}{1 - \left(\frac{p_{\text{rel}}}{F^*_1}\right) \frac{F^*_1}{3}}.
\]

**The Landau Kinetic Equation**

To describe the transport and low energy, long wavelength response of a Fermi liquid we must investigate the evolution of \( n_{\text{pe}}(r,t) \). This is classically the distribution function for a particle of momentum \( p \) at position \( r \) at time \( t \). In a Fermi liquid, what are the conditions for treating this quasi-classically? Let \( \lambda = h/\varphi \) be the wavelength of the external disturbance, then the quasi-classical condition is,

\[
\lambda \Delta p \gg h.
\]

For a particle localized within distance of the order of an interparticle spacing \( a \) we have that

\[
\Delta p \sim \frac{h}{a} \sim p_F.
\]

Thus, for \( \varphi \sim p_F \) we can use a quasi classical description. Of course, if

\[
\lambda \sim \lambda_F \sim \frac{h v_F}{k_B T}
\]

or

\[
\varphi \sim \frac{k_B T}{v_F},
\]

we also use a classical description but this is too restrictive a range in particular as \( T \to 0 \).

A quasi classical description is also possible if for the energy transfer, \( \omega \), we have that,

\[
\omega \sim v_F \Delta p \sim 1
\]

or

\[
\tau (v_F \Delta p) \sim h
\]

where \( \tau = h/c \) and \( v_F \Delta p \) is a typical quasiparticle energy. If we now set \( \Delta p = h/\varphi = p_F \) we get...
\[ \frac{d \rho_{\sigma}(\mathbf{r}, t)}{dt} = I[\rho_{\sigma}(\mathbf{r}, t)] \]

with \( \rho_{\sigma} = p_{F}^{2} / (2m^{*}) \). The classical limit is the usual limit \( \omega \ll k_{B}T \), however, this is again more restrictive than the quasi-classical constraint.

For \( \omega \ll p_{F} \) and \( q \ll p_{F} \) we have a classical evolution for \( \rho_{\sigma}(\mathbf{r}, t) \):

\[ \frac{d \rho_{\sigma}(\mathbf{r}, t)}{dt} = I[\rho_{\sigma}(\mathbf{r}, t)] \]

with \( \rho_{\sigma}(\mathbf{r}, t) \) the quasiparticle Hamiltonian

\[ \rho_{\sigma}(\mathbf{r}, t) = \rho_{\sigma}^{(0)} + \rho_{\sigma}(\mathbf{r}, t) + \sum_{\eta \neq \sigma} \int_{\mathbf{p}, \eta} \rho_{\eta}(\mathbf{r}, t) \rho_{\eta}(\mathbf{r}, t) \]

(26)

The Poisson bracket is given by

\[ [\rho_{\sigma}(\mathbf{r}, t), \rho_{\sigma}(\mathbf{r}, t)]_{PB} = \nabla_{\rho_{\sigma}} \cdot \nabla_{\rho_{\sigma}} - \nabla_{\rho_{\sigma}} \cdot \nabla_{\rho_{\sigma}} \]

(28)

The applications of the kinetic equation, (K.E.), Eq. (26), will be confined to small deviations from equilibrium where the linearized K.E. (L.K.E.) can be used. In this limit we have

\[ [\rho_{\sigma}(\mathbf{r}, t), \rho_{\sigma}(\mathbf{r}, t)]_{PB} = \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \cdot \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \]

\[ \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \cdot \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \]

(29)

now \( \nabla_{\rho_{\sigma}} (\rho_{\sigma}^{(0)} \text{ or } \rho_{\sigma}^{(0)}) = 0 \). In global equilibrium there is no spatial variations for \( \rho_{\sigma}^{(0)} \) or \( \rho_{\sigma}^{(0)} \).

\[ \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \cdot \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \]

(29)

where \( \nabla_{\rho_{\sigma}} \frac{\partial \rho_{\sigma}}{\partial \rho_{\sigma}} \).

Since \( \frac{\partial}{\partial t} \rho_{\sigma}^{(0)} = 0 \) we get for the L.K.E.,

\[ \frac{\partial}{\partial t} \rho_{\sigma}(\mathbf{r}, t) + \nabla_{\rho_{\sigma}} \left( \rho_{\sigma}^{(0)} + \delta \rho_{\sigma} \right) \left| I[\rho_{\sigma}(\mathbf{r}, t)] \right| \]

(29)
We will return to $I[u_{pe}]$ later. The deviation

$$
\delta \bar{\eta}_{pe}(r,t) = \delta u_{pe}(r,t) - \frac{\partial \eta_{pe}^{(a)}}{\partial \varepsilon_{pe}} \delta \varepsilon_{pe}(r,t)
$$

(30)

is the deviation from the "local equilibrium distribution function" $\bar{\eta}_{pe}(r,t)$, where

$$
\bar{\eta}_{pe}(r,t) = \left\{ 1 + \exp \left[ \frac{\varepsilon_{pe}(r,t) - \mu(r,t)}{T(r,t)} \right] \right\}^{-1}
$$

(31)

and $\delta u_{pe}(r,t)$ is the deviation from global equilibrium. To see this we note that

$$
\delta \bar{\eta}_{pe} = u_{pe}(r,t) - \bar{\eta}_{pe}(r,t)
$$

for $\varepsilon_{pe}(r,t) = \varepsilon_{pe}^{(a)} + \delta \varepsilon_{pe}(r,t)$, then $\bar{\eta}_{pe}(r,t) = \eta_{pe}^{(a)} + \delta \eta_{pe}^{(a)} \frac{\partial \eta_{pe}^{(a)}}{\partial \varepsilon_{pe}}$. From this we recover Eq. (30).

The gradient term in Eq. (29) is just the qp. drift term which is the change in $u_{pe}(r,t)$ due to qp. motion without collisions. The flow is governed by the local energy $\varepsilon_{pe}(r,t)$. In the linearized Boltzmann equation $\delta \eta_{pe}$ is replaced by $\delta u_{pe}$.

Fourier transform Eq. (29):

$$
\delta u_{pe}(r,t) = \int \frac{d^3q}{(2\pi)^3} \frac{d\omega}{2\pi} (\omega q \cdot \omega_{pe}) \delta u_{pe}(q,\omega), \text{ etc.}
$$

then

$$
(\omega q \cdot \omega_{pe} \delta u_{pe}(q,\omega) + q \cdot \omega_{pe} \delta \eta_{pe}^{(a)} \frac{\partial \eta_{pe}^{(a)}}{\partial \varepsilon_{pe}} \delta \varepsilon_{pe}(q,\omega)) / |u_{pe}|
$$

(30)

Conservation Laws and Continuity Equations

Before exploring the solutions of Eq. (30) we need to understand some general features of the KE. For example if number is conserved we have,

$$
\frac{\partial u(r,t)}{\partial t} + \nabla \cdot \mathbf{j}(r,t) = 0
$$

(31)

This equation follows from Eq. (26) if the qp. collisions are number conserving. This follows from the conservation of number in collisions, i.e.,

$$
\sum_{i} J_{i} [u_{pe}] = 0
$$

for all $i$. We assume this for now and study the consequences.
For the number density we have that,
\[ n(r, t) = \sum_{p\sigma} n_{p\sigma}(r, t). \]  
(32)

If we integrate Eq. (26) we have,
\[ \frac{\partial}{\partial t} \sum_{p\sigma} n_{p\sigma}(r, t) - \sum_{p\sigma} [\varepsilon_{p\sigma} n_{p\sigma}]_{\rho, \eta} = \sum_{p\sigma} I[n_{p\sigma}] = 0 \]

\[ \sum_{p\sigma} \frac{\partial}{\partial t} n_{p\sigma}(r, t) - \sum_{p\sigma} [\nabla \varepsilon_{p\sigma} \cdot \nabla n_{p\sigma} - \nabla \varepsilon_{p\sigma} \cdot \nabla n_{p\sigma}] \]

\[ - \sum_{p\sigma} \nabla \cdot [(\nabla \varepsilon_{p\sigma} n_{p\sigma})]_{\rho, \eta} - \nabla \cdot [(\nabla \varepsilon_{p\sigma} n_{p\sigma})] \int_{\mu} dS_{\rho} [\nabla \varepsilon_{p\sigma} n_{p\sigma}] \cdot \hat{n} = 0 \]

With the current given by
\[ j(r, t) \sum_{p\sigma} [\nabla \varepsilon_{p\sigma}(r, t)] n_{p\sigma} \]  
(33)

we recover the continuity equation. Here we note that the Landau theory of a F.L. is a
conserving approximation (for number) since it satisfies the continuity equation.

We can express the current in terms of F.L. parameters. Consider small deviations
from equilibrium:
\[ n_{p\sigma} \rightarrow n_{p\sigma}^{(a)} + \delta n_{p\sigma} \]
and \[ \varepsilon_{p\sigma} \rightarrow \varepsilon_{p\sigma} \hat{p}_{p\sigma} \]

then
\[ j(r, t) \sum_{p\sigma} \nu_{p\sigma} n_{p\sigma}^{(a)} n_{p\sigma}^{(a)} \nabla \cdot \varepsilon_{p\sigma} \nabla \cdot \nu_{p\sigma} \nabla \cdot \nu_{p\sigma}^{(a)} \nabla \cdot \hat{p}_{p\sigma}^{(a)} \]

With \[ \nu_{p\sigma} = \nabla \cdot \nu_{p\sigma} \hat{p}_{p\sigma} \], then
\[ \sum_{p\sigma} \nu_{p\sigma}^{(a)} \nabla \cdot \varepsilon_{p\sigma} \nabla \cdot \nu_{p\sigma}^{(a)} \]

\[ \sum_{p\sigma} \nu_{p\sigma} \nabla \cdot \varepsilon_{p\sigma} \nabla \cdot \nu_{p\sigma}^{(a)} \]

since the surface term vanishes.
For \( \mathbf{j}(\mathbf{r}, t) \) we have,

\[
\mathbf{j}(\mathbf{r}, t) = \sum_{\mathbf{p}_\sigma} \mathbf{v}_{\mathbf{p}_\sigma} \left( \delta n_{\mathbf{p}_\sigma} - \frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \delta \mathbf{q}_{\mathbf{p}_\sigma} \right)
\]

\[
= \sum_{\mathbf{p}_\sigma} \mathbf{v}_{\mathbf{p}_\sigma} \delta \tilde{n}_{\mathbf{p}_\sigma}
\]

(33a)

Here we see that the current arises from deviations from "local equilibrium."

To go further we consider the following:

\[
\delta n_{\mathbf{p}_\sigma}^{(a)} = -\frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \sum_{l} \nu_{l} P_{l}(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})
\]

where \( \hat{\mathbf{q}} \) is along the direction of the driving field. For the deviation from local equilibrium we have,

\[
\delta \tilde{n}_{\mathbf{p}_\sigma}^{(a)} = \delta n_{\mathbf{p}_\sigma}^{(a)} \frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \sum_{l} f_{\mathbf{p}_\sigma}^{\mathbf{a}^{(a)}} \delta \mathbf{q}_{\mathbf{p}_\sigma}^{\mathbf{a}^{(a)}}
\]

\[
= \frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \left\{ \sum_{l} \nu_{l} P_{l}(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \right\} \sum_{\mathbf{p}_\sigma'} \sum_{l} f_{\mathbf{p}_\sigma}^{\mathbf{a}^{(a)}} P_{l}(\mathbf{p}_\sigma' \cdot \hat{\mathbf{p}}) \frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \nu_{l} P_{l}(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \left\{ \right\} .
\]

Using the orthogonality of the Legendre polynomials we find that,

\[
\delta n_{\mathbf{p}_\sigma}^{(a)} = \frac{\partial n_{\mathbf{p}_\sigma}^{(a)}}{\partial \mathbf{q}_{\mathbf{p}_\sigma}^{(a)}} \left\{ \sum_{l} \nu_{l} \left( 1 + \frac{F_{l}^{n}}{m_{l}^{n} / m_{l}} \right) P_{l}(\mathbf{p}_\sigma \cdot \hat{\mathbf{p}}) \right\} .
\]

(34)

For \( \mathbf{j}(\mathbf{r}, t) \) we have,

\[
\mathbf{j}(\mathbf{r}, t) = \sum_{\mathbf{p}_\sigma} \mathbf{v}_{\mathbf{p}_\sigma} \delta \tilde{n}_{\mathbf{p}_\sigma}
\]

\[
= \sum_{\mathbf{p}_\sigma} \mathbf{v}_{\mathbf{p}_\sigma} \delta \tilde{n}_{\mathbf{p}_\sigma} \left\{ \sum_{l} \nu_{l} \left( 1 + \frac{F_{l}^{n}}{m_{l}^{n} / m_{l}} \right) P_{l}(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \right\} .
\]

where only the \( l = 1 \) term from Eq (34) survives.
\[ j(r,t) = \frac{1 + F^e/3}{m^*/m} \frac{p_r}{m} \hat{q} m \left[ N(a) \int_{-1}^{1} \frac{d(\cos \theta)}{2} \cos^2 \theta \right] \]
\[ = \left( \frac{1 + F^e/3}{m^*/m} \right) N(a) \frac{p_r}{m} \frac{\nu_1}{3} \hat{q}. \]

(35)

For Galilean invariant systems we have, \( \frac{m^*}{m} = 1 + F^e/3 \), thus

\[ j^0 = \frac{1 + F^e/3}{m^*/m} j_0 \quad \vec{j}_0^0 = \vec{\nu} \]

with \( j^0 = \frac{p_r}{m} \). For a Galilean invariant system we have that,

\[ j_0^0 = j_0^0. \]

**Scattering Amplitudes**

For \( T=0 \) the L.K.E. is given by

\[ (\omega \cdot q \cdot \nu_{ps}) \delta n_{ps}(q,\omega) = \frac{\partial n_{ps}^{[a]}}{\partial \nu_{ps}^{(a)}} \delta n_{ps}(q,\omega) = 0 \]

(36)

where

\[ \delta n_{ps}(q,\omega) = U_{ps}(q,\omega) \sum_{p's'} f_{pp'}^{s's'} \delta n_{p's'}(q,\omega). \]

Consider now,

\[ \frac{\delta n_{ps}(q,\omega)}{\delta n_{p's'}(q,\omega)} f_{pp'}^{s's'} = \sum_{p's'a} f_{pp'a}^{s's'a} \left( \frac{\delta n_{p's'a}(q,\omega)}{\delta n_{p's'}(q,\omega)} \right). \]

(37)

From Eq. (36) we have that

\[ \delta n_{p's'a}(q,\omega) \quad q \cdot \nu_{p's'a} \quad \delta n_{p's'a}(q,\omega) = \delta n_{p's'a}(q,\omega) \]

(38)

We see that from Eq. (38) that

\[ \delta n_{p's'a}(q,\omega) = 0 \]

where we drop the \( q \) and \( \nu \) dependence for now. In the limit Eq. (37) gives.
\[
\lim_{\omega \to 0} \left( \frac{\delta \varepsilon(p)}{\delta n(p')} \right) = f(p')^\sigma.
\] (39)

In the other limit where \( \omega \) goes to zero first then \( q \) goes to zero we have,

\[
\lim_{q \to 0} \left( \frac{\delta n(p''\sigma'')}{\delta n(p'\sigma')} \right) \to \frac{\partial h^{(o)}_{p''\sigma''}}{\partial \varepsilon^{(o)}_{p'\sigma'}} a^{\sigma''\sigma'}_{p''p'},
\]

where \( a^{\sigma''\sigma'}_{p''p'} = \lim_{q \to 0} \left( \frac{\delta \varepsilon(p''\sigma'')}{\delta n(p'\sigma')} \right) \) is the quasiparticle scattering amplitude.

In this limit we obtain a Bethe Salpeter equation for the quasiparticle scattering amplitude given by,

\[
a^{\sigma\sigma'}_{pp'} = f^{\sigma\sigma'}_{pp'} + \sum_{p''\sigma''} f^{\sigma\sigma''}_{pp''} \frac{\partial h^{(o)}_{p''\sigma''}}{\partial \varepsilon^{(o)}_{p'\sigma'}} a^{\sigma''\sigma'}_{p''p'}. \] (40)

For a rotationally invariant system we can write,

\[
a^{\sigma\sigma'}_{pp'} = a^{\sigma\sigma'}_{pp'} + a^{\sigma\sigma'}_{pp'} \sigma \cdot \sigma'
\]

and for \(|p| \approx |p'| \approx p_p\) we have,

\[
a^{\sigma\sigma'}_{pp'} = \sum_{l} a^{\sigma\sigma'}_{l} P_l(\cos \theta).
\]

If we now expand \( f^{\sigma\sigma''} \) and \( a^{\sigma''\sigma'}_{p''p'} \) in Legendre polynomials and plug them into Eq. (40) we find,

\[
a^{\sigma\sigma''}_{l} = \frac{f^{\sigma\sigma''}_{l}}{1 + F^{\sigma\sigma''}_{l}/2l + 1}
\]

or for the dimensionless amplitudes we have,

\[
A^{\sigma\sigma''}_{l} = N_l \frac{f^{\sigma\sigma''}_{l}}{1 + F^{\sigma\sigma''}_{l}/2l + 1} \] (41)

The parameters \( A^{\sigma\sigma''}_{l} \) are the quasiparticle scattering amplitudes. The stability of the Fermi liquid requires all of the amplitudes to be finite.

Let us go back and understand these two limits.
i) \( \lim_{q \to 0} \left( \frac{\hbar \varepsilon_{p \sigma}}{\hbar \omega_{p' \sigma'}} \right) = f_{pp'}^{\sigma \sigma'} \).

If we transfer momentum \( q \) to the system it will excite a particle pair of energy \( \varepsilon_{p+q} = \varepsilon_p + q v_F \cos \theta_{pq} \). The maximum energy transfer to the particle-hole pair is \( \omega_{max} \approx q v_F \). Let us now define

\[
\tau_{ph} \sim \frac{\hbar}{\omega_{max}} = \frac{\hbar}{q v_F}
\]

and

\[
\tau \sim \frac{\hbar}{\omega}
\]

where \( \omega \) is the energy transfer. If \( \tau \ll \tau_{ph} (\omega \gg q v_F) \), the system does not have time to respond to the probe, thus no screening.

ii) \( \lim_{q \to 0} \left( \frac{\hbar \varepsilon_{p \sigma}}{\hbar \omega_{p' \sigma'}} \right) = \rho_{pp'}^{\sigma \sigma'} \)

In this limit we have \( \tau \gg \tau_{ph} (\omega \ll q v_F) \), thus, the system, has time to respond, and screening by the other quasiparticles is possible. For the case of finite \( \omega/q v_F = s \) we have,

\[
\rho_{pp'}^{\sigma \sigma'}(s) = \int_{pp'}^{\sigma \sigma'} \sum_{p''p'''} f_{p''p'''}^{\sigma'' \sigma'''} \frac{x^{p''} - x^{p''} + i \delta}{x^{p''} + i \delta} \frac{\partial \eta_{pp''}^{\sigma''}}{\partial \eta_{pp'''}^{\sigma'''}(s)}
\]

(42)

If we carry out the Legendre polynomial expansion of the scattering amplitudes and the quasiparticle interaction we obtain a set of coupled equations for finite \( s \).

**General Properties of the Scattering Amplitude**

Let us consider the most general scattering amplitude with all momenta on the Fermi surface and \( p_1 = 0 \), for \( i = 1, 2, 3 \) and 4. Momentum conservation eliminates one of the momenta since, \( K = p_1 + p_2 + p_3 + p_4 \). For scattering particles and holes it is more convenient to use the following momenta,

\[
K = p_1 + p_2 + p_3 + p_4
\]

\[
q = p_1 + p_2 + p_3 + p_4
\]

\[
p = p_1 + q / 2, p_3 + p - q / 2
\]

\[
p' = p' + q / 2, p_1 + p' - q / 2
\]

where these are 4 momenta with \( q = (q, \omega) \) and \( p = (p, \sigma) \) etc.

The scattering amplitude is now given by,

\[
a_{pp'; pp''}^{\sigma \sigma'} \rho_{pp'}^{\sigma \sigma'}(q)
\]
With all \( |\bar{p}_i| = p_F \) and \( \varepsilon_i = 0 \), for \( i = 1, 2, 3 \) and 4 there are only two relevant scattering angles and the momenta \( q, q' \) and \( K \) are given as follows, (see Fig. 3),

\[
q^2 = |p_1 - p_3| = p_F^2 (1 - \cos \theta)(1 - \cos \phi),
\]
\[
K^2 = 2p_F^2 (1 + \cos \theta),
\]
\[
q'^2 = |p_1 - p_4|^2 = p_F^2 (1 - \cos \theta)(1 + \cos \phi)
\]

When \( \phi \to 0, q \to 0 \) then we recover the scattering amplitudes, \( a_F \), of the Fermi liquid theory. In this limit, \( q \to 0 \) we also have that

\[
q'^2 = 2p_F^2 (1 - \cos \theta) = |p - p'|^2
\]

with \( \theta \) being the expansion angle for the quasiparticle interaction and scattering amplitudes.

Let us now look at the spin structure of the scattering amplitude. The full spin structure can be written as follows,

\[
a_{(\beta, \phi)}^{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = a_{(\theta, \phi)}^{\tau_\gamma \gamma'}.
\]

For spin-rotation invariant interactions we have that,

\[
a^{\alpha, \gamma, \gamma'}(\theta, \phi) = a^\alpha(\theta, \phi) \delta_{\gamma \gamma'} + a^\alpha(\theta, \phi) \sigma_{\alpha \gamma} \cdot \sigma_{\gamma'}
\]  \hspace{1cm} (43)

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Thus, the amplitudes are not independent and they are given as combinations of the spin symmetric and antisymmetric amplitudes,

\[
a^{11}(\theta, \phi) \hspace{1cm} a^{11}(\theta, \phi) \hspace{1cm} a^\alpha(\theta, \phi) + a^\alpha(\theta, \phi).
\]  \hspace{1cm} (43a)

\[
a^{11}(\theta, \phi) \hspace{1cm} a^{12:12}(\theta, \phi) \hspace{1cm} a^\alpha(\theta, \phi) \hspace{1cm} a^\alpha(\theta, \phi).
\]  \hspace{1cm} (43b)

and for the spin flip amplitude we set,

\[
R^{11}(\theta, \phi) \hspace{1cm} a^{12:21}(\theta, \phi) \hspace{1cm} 2 \ a^\alpha(\theta, \phi)
\]  \hspace{1cm} (43c)

The Fermionic nature of the quasiparticles will give rise to definite exchange properties for the scattering amplitudes. For example the triplet amplitude is given by

\[
a^{\alpha, \gamma}(\theta, \phi) = 0
\]

\[
a^{\alpha, 11}(\theta, \phi) = a^{11}(\theta, \phi)
\]

\[
a^{\alpha, 12:12}(\theta, \phi) = a^{12:12}(\theta, \phi)
\]

\[
a^{\alpha, 12:21}(\theta, \phi) = 2a^\alpha(\theta, \phi)
\]

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Exchange of quasiparticles $p_3$ and $p_1$ in Fig. 3 is equivalent to setting $\phi \to \phi + \pi$, thus

$$a^{11}(\theta, \phi + \pi) = -a^{11}(\theta, \phi) \quad (44)$$

For a spin-singlet the amplitude is even under exchange, thus

$$a^{sing}(\theta, \phi + \pi) = a^{sing}(\theta, \phi) \quad (45)$$

The singlet and triplet amplitudes can be expressed in terms of the $a^a(\theta, \phi)$ and $a^o(\theta, \phi)$. For the singlet we have,

$$a^{sing}(\theta, \phi) = a^a(\theta, \phi) - 3a^o(\theta, \phi)$$

and for the triplet,

$$a^{trip}(\theta, \phi) = a^a(\theta, \phi) + a^o(\theta, \phi)$$

From the Pauli principle we have that the scattering amplitude for Fermions with the same spin and momenta must vanish. For $\theta = 0$, $p_1 = p_2$ and $p_3 = p_1$, thus

$$a^{11}(\theta = 0, \phi) = 0 \quad (46)$$

For $\phi = 0$ we have,

$$N(a) a^{11}(\theta, \phi = 0) = \sum_{\ell = 0}^{\infty} \left( a^a_{\ell} + a^o_{\ell} \right) P\ell(\cos \theta)$$

and from Eq. (46) the forward scattering sum rule follows,

$$N(a) a^{11}(\theta, \phi = 0) \cdot \sum_{\ell = 0}^{\infty} \left( a^a_{\ell} + a^o_{\ell} \right) = 0 \quad (47)$$

Near an instability Eq. (47) has an important effect on the Fermi liquid parameters. Suppose $F^\mu_\nu \to 1$ then,

$$A^\mu_\nu \to \frac{F^\mu_\nu}{1 + F^\mu_\nu} \to \infty$$

From Eq. (47) we have,

$$\sum_{\ell = 1}^{\infty} \left( A^a_{\ell} + A^o_{\ell} \right) + A^a_0 + A^o_0 \to \infty$$

as $F^\mu_\nu \to 1$. The maximum positive value for the $A^\mu_\nu$'s (for a stable F.L.) are
\[
\frac{F_i^{s,a}}{1 + F_i^{s,a}/2\ell + 1} \to 2\ell + 1
\]
as \(F_i^{s,a} \to \infty\). Thus, instability that an infinite number of positive finite amplitudes will be needed to satisfy the sum rule.

**The Collision Integral**

For low T only two body (two qp's) scattering is important:

\[
I[n_{p,\sigma}] = \sum_{p_1, p_2} W(p_1, p_2) \delta(\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}) \delta_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} \delta_{\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}} \delta_{\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}}
\]

(48)

where, the sum is over distinguishable final states and

\[
\delta_{\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}} = \delta_{n_1 + n_2 - n_3 - n_4}
\]

(49)

\(\delta_{n_1 + n_2 - n_3 - n_4}\) = \(n_1 n_2(1 - n_3)(1 - n_4)\) and state 1 is empty which implies that 1 is the increase of state 1.

\(\delta_{n_1 + n_2 - n_3 - n_4}\) = \(n_1 n_2(1 - n_3)(1 - n_4)\), state 1 is full, thus 2 is the depletion of state 1. Here we have that,

\[
W(1, 2; 3, 4) \sim |\gamma(12; 34)|^2
\]

(50)

is the scattering rate and we used the notation that \((i \rightarrow p; \sigma_i, i = 1, 2, 3, 4)\).

**Linearization of the Collision Integral**

At low temperature only two body scattering is possible. In this limit the collision integral is given by,

\[
I[n_1] = \frac{1}{2} \sum_{3, 4} W(12; 34) \delta_{\sigma_1, \sigma_3 + \sigma_4} \delta_{\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}} \delta_{\varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4}}
\]

(51)

where the sum is over all final states and the \(1/2\) takes care of the double counting. Note that the energy conserving delta function contains the exact quasiparticle energies. This is important since it tells us about which energy to linearize. To see this, let,

\[
n_{\mu, i}^{\ell, (\mu, i)} = \left\{ 1 + \exp \left( \frac{\mu - \varepsilon_{i}}{T_i} \right) \right\}^{-1}
\]

then

\[
\nu_i
\]
\[ I[\mu_i^{(r)}] = 0. \]

To see this we note that,

\[ n_i^{(r)} = (1 - n_i^{(r)}) \, e^{-\beta(\varepsilon_i - \mu)}, \]

with \( \beta = 1/T \). Then

\[ n_3^{(r)} \cdot n_4^{(r)} (1 - n_2^{(r)}) \cdot (1 - n_1^{(r)}) - n_1^{(r)} \cdot n_2^{(r)} \cdot (1 - n_3^{(r)}) \cdot (1 - n_4^{(r)}) \]
\[ = n_1^{(r)} \cdot n_2^{(r)} \cdot n_3^{(r)} \cdot n_4^{(r)} \left[ e^{\beta(\varepsilon_1 + \varepsilon_2 - 2\mu)} - e^{\beta(\varepsilon_3 + \varepsilon_4 - 2\mu)} \right] \]

Now since energy is conserved in collisions, i.e., \( \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 \), we have that \( I[\mu_i^{(r)}] = 0 \).

To linearize we now set

\[ n_i = n_i^{(r)} + \delta n_i^{(r)} \]

and define \( \Phi_i \) by

\[ \Phi_i = \frac{1}{T} \, n_1^{(r)} \cdot (1 - n_1^{(r)}) \Phi_i \]

By repeated use of Eqs. (52) and (53) we get for the linearized collision integral,

\[ I[n_i] = \frac{1}{2T} \sum_{2.3.1} W(12;34) \delta n_1^{(r)} \cdot n_2^{(r)} \cdot (1 - n_3^{(r)}) \cdot (1 - n_4^{(r)}) \]
\[ \left[ \Phi_1 \cdot \Phi_2 \cdot \Phi_3 \cdot \Phi_4 \right] \]

The driving term \( \Phi_1 \) will depend on the type of perturbation, e.g., velocity gradients for the viscosity, \( \eta \), field gradients for the spin diffusion, D, etc.

### Collective Excitations In a Normal Fermi Liquid

The Fourier transform of the K.E. is given by Eq. (30).

\[ (\omega - q \cdot \mathbf{v}_p) \cdot \delta n_p^0 + q \cdot \mathbf{v}_p \cdot \frac{\partial n_p^0}{\partial \mathbf{r}_p} \cdot \delta \mathbf{r}_p \cdot I[n_p^0] \]

To determine the linear response we solve this equation with

\[ \delta n_p^0 + \Gamma_\sigma \cdot \sum_{p \sigma} \int_{p \sigma} \cdot \delta n_{p \sigma}^0 \]

\[ \Phi_0 \]
where for the density response

\[ U_\sigma = U \quad (U^* \to 0) \]

and spin-density response

\[ U'_\sigma = \sigma U^* \quad (U^* \to 0) \]

To determine the free oscillations of the system we set \( U = 0 \), then we have

\[ (s - x)\dot{\eta}_{\mu \sigma} + x \frac{\partial \eta_{\mu \sigma}^{(0)}}{\partial \mu_{\nu \sigma}} \sum_{\nu' \sigma'} f_{\nu \nu'} \dot{\eta}_{\nu' \sigma'} = \frac{i}{q v_F} I[\eta_{\nu \sigma}] \tag{56} \]

To study the propagation of sound in a Fermi liquid we study the oscillations of the density.

\[ \dot{\eta}(q, \omega) \sum_{\mu \sigma} \dot{\eta}_{\mu \sigma}(q, \omega) \tag{57} \]

If we write

\[ \dot{\eta}_{\mu \sigma} = \frac{\partial \eta_{\mu \sigma}^{(0)}}{\partial \mu_{\nu \sigma}} \sum_{\nu} V_\nu^\sigma(q, \omega) P_{\nu}(\mu \cdot \hat{q}) \]

then,

\[ \dot{\eta} = N(\sigma) e_\nu^\sigma \quad (e_\nu^\sigma \quad \mu \downarrow + \mu \uparrow) \]

where we drop the dependence on \( q \) and \( \omega \) for now. The density fluctuations are proportional to \( e_\nu^\sigma \) since we have applied a uniform compression to the system.

To obtain an equation for the evolution of \( e_\nu^\sigma \) we project out the \( \ell = 0 \) component of Eq. (56). This can be done by summing over the momentum (which is confined to be close to \( p_F \) in magnitude) and summing on \( \sigma \).

\[ \sum_{\mu \sigma} (s - \tau) \dot{\eta}_{\mu \sigma} + \sum_{\mu \sigma} x \frac{\partial \eta_{\mu \sigma}^{(0)}}{\partial \mu_{\nu \sigma}} \sum_{\nu' \sigma'} f_{\nu \nu'} \dot{\eta}_{\nu' \sigma'} = \frac{i}{q v_F} \sum_{\mu \sigma} I[\eta_{\mu \sigma}] \quad 0 \]

The vanishing of the sum over the collision integral is a consequence of charge conservation in quasiparticle collisions. This gives,
\[ s \, N(a) \, v_0^a - N(a) \, \frac{\mu_1^a}{3} = N(a) \, \frac{f_1^a}{3} \, \frac{\mu_1}{3} \, N(a) = 0 \]

\[ s \, v_0^a - (1 + \frac{F_1^a}{3}) \, \frac{\mu_1}{3} = 0 \]  

(58)

which is just the Fourier transform of the continuity equation.

To obtain equation for \( v_1^a \) we consider the current along the \( \hat{q} \) direction.

\[ \sum_{\sigma} p \cdot \hat{q} \sum_{\sigma} (p \cdot \hat{q}) \frac{\partial n_{p\sigma}}{\partial x} \sum_{p'\sigma'} f_{p\sigma}^{p'\sigma'} \delta n_{p'\sigma'} = \frac{i}{q \nu_F} \sum_{\sigma} \hat{q} \cdot p \, I[n_{p\sigma}] = 0 \]

where momentum conservation leads to the vanishing of the term involving \( I[n_{p\sigma}] \).

From the term \( (p \cdot \hat{q}) x \cdot p \cdot x^2 \cdot \frac{3}{2} p \cdot (P_2(x) + \frac{1}{2} P_0) \) we see that \( v_1^a \) will couple to both \( v_0^a \) and \( v_2^a \), thus

\[ s \, v_1^a \left( 1 + \frac{F_0^a}{3} \right) v_0^a = 2 \left( 1 + \frac{F_2^a}{5} \right) \frac{\mu_0^a}{5} = 0 \]  

(59)

Generally from Eq. (56) we see that every moment couples to a higher moment. We will assume in what follows that \( v_{1+l}^a \propto v_1^a \) for \( l \propto 2 \), (this will be justified below). With this assumption the \( v_2^a \) equation is given by,

\[ s \, v_2^a = 2 \left( 1 + \frac{F_1^a}{3} \right) \frac{\mu_1^a}{3} \approx \frac{1 + \frac{F_2^a}{5}}{iq \nu_F \nu q} \, v_2^a \]  

(60)

where we defined \( \nu_q \) by

\[ \nu_q \frac{I[n_{p\sigma}]}{q \nu_F} \left( 1 + \frac{F_2^a}{5} \right) (\delta n_{p\sigma})_\gamma \approx \frac{1}{iq \nu_F \nu q} \]  

(61)

and \( (\delta n_{p\sigma})_\gamma = N(0) v_0^a \). The form of Eq. (61) is just is a relaxation time approximation for the collision integral that conserves charge and momentum.

If we now combine Eqs. (58) (60) we have,

\[ s^2 \, v_1^2 q^2 \, \frac{2^2}{45} (q v_1)^2 \left( 1 + \frac{F_2^a}{5} \right) \frac{v_2^2}{v_0^a} \]  

(62)

where,

\[ v_1^2 = \left( 1 + \frac{F_1^a}{3} \right) \frac{m^a}{m} \, v_0^a \]

\[ v_2^2 = \left( 1 + \frac{F_2^a}{5} \right) \frac{m^a}{m} \, v_0^a \]
The phase space limitations on quasiparticle scattering leads to a viscous relaxation time, \( \tau_\eta \propto T^{-2} \). The limit \( T \to 0 \) is the collisionless (zero sound) regime where \( \omega \tau_\eta \to \infty \) and \( \frac{\nu^2_1}{\nu^2_0} \to 2 \). If we now set \( \omega^2 = \nu^2_0 q^2 \), then we have

\[
\nu^2_1 - \nu^2_0 \approx \frac{4}{15} \frac{1 + F_2^s/5}{1 + F_2^s/5} \nu^2_1 \mu^2_0 \tag{63}
\]

It should be noted that when zero sound propagates \( c_0 > c_1 \). In the hydrodynamic limit we have \( \omega \tau_\eta \ll 1 \), thus

\[
\frac{\nu^2_1}{\nu^2_0} \to \frac{-2i\omega \tau_\eta}{(1 + F_2^s/5)}
\]

Setting \( \nu^2_1 = \frac{\omega}{c_1} + i\alpha \), we get

\[
\omega \approx c_1 \eta
\]

and

\[
\alpha \approx \frac{2}{15} \left( 1 + F_2^s \right) \frac{\nu^2_1}{\nu^2_0} \omega^2 \tau_\eta,
\]

\[
= \frac{2}{3} \frac{\omega^2 \eta}{\mu^2_0 c_1^2}
\]

with \( \eta = \frac{2}{3} n \cdot E \cdot \tau_\eta \) and \( \mu = mn \).

The result for \( c_0 \) given by Eq. (63) depended on \( \nu^2_{l+1} \sim \nu^2_g \) for \( l \geq 2 \). Under what conditions is this valid? To determine these conditions we return to Eq. (56) and project out the \( l \)th moment of \( \delta n_{pq} \).

\[
\delta n_{pq} = \left[ a_l^q \left| a_l^1 + a_l^2 + a_l^3 a_{l+1}^1 a_{l+1}^2 \right| a_l^p \mu q r_{l+1} \right] r_l^p
\]

where \( a_l^q = 1 + F_{l+1}^q \). Assume that \( \eta \) is the same for all \( \nu^2_g \).

\[
\left( a_l^q \right) \left( \mu q r_{l+1} r_n \right) r_l^p = \left( a_l^1 + a_l^2 + a_l^3 a_{l+1}^1 a_{l+1}^2 \right) \left( \mu q r_{l+1} r_n \right) r_l^p
\]
If \( \nu_{v+1}^\sigma < \nu_{v}^\sigma \) we have

\[
\nu_{v}^\sigma \approx \frac{\nu_{v+1}^\sigma}{s \left( 1 - \frac{ag}{\omega \tau_D} \right)}
\]

thus, we see that \( \nu_{v}^\sigma \ll \nu_{v+1}^\sigma \) if \( s \gg 1 \) or \( \omega \tau_D \ll 1 \). In the hydrodynamic regime collisions damp out all higher harmonics \( (\ell \geq 2) \) of the Fermi surface. In the zero sound regime if \( s \gg 1 \) then we have \( c_n \gg v_F \), thus the restoring forces, in particular \( F_\sigma^\sigma \), suppress the higher harmonics, \( \nu_{v}^\sigma \).

If \( F_\sigma^\sigma \) is small but positive, \( s \approx 1 \), with \( c_n/v_F \approx 1 + 2 - 2 \left( 1 + \frac{1}{\ell} \right) \), with \( F_\sigma^\sigma \ll 0 \) for \( \ell \geq 1 \). Here the expansion does not truncate and we must return to Eq. (56). If we assume that all \( \nu_{v} \)'s contribute and the \( F_\sigma \)'s truncate we can obtain the above solution. If \( -1 < F_\sigma^\sigma < 0 \) zero sound does not propagate. Transverse zero sound propagates if \( F_\sigma^\sigma > 0 \). The propagation of a sound wave and as well transverse shear waves in a liquid at \( T = 0 \) is a clear a non-classical result. These waves would not propagate in a classical fluid.

**Transverse Spin Waves**

To study transverse spin waves we want to study the evolution of the magnetization for arbitrary directions. In general we have that

\[
(\mu_p)_{\alpha,t} = \mu_p \delta_{\alpha,t} + m_p \cdot \sigma_{\alpha,t}
\]  

and

\[
(\mu_p)_{\alpha,t} = \mu_p \delta_{\alpha,t} + h_p \cdot \sigma_{\alpha,t}
\]

The kinetic equation (K.E.) can be used to describe the evolution of \( \mu_p \) (as above) and another one can be derived to study the evolution of \( m_p \). For small changes in \( m_p \) these equations decouple. The form of the K.E. for \( m_p \) is,

\[
\frac{\partial}{\partial \tau} \lambda m_p(\tau,t) + v_p \cdot \nabla \left[ \lambda m_p - \frac{\partial m_p}{\partial \tau} + \hat{h}_p \right] \quad J|m_p|
\]

where

\[
\hat{h}_p = \mu_B \sum_{\ell \alpha} f_{\ell \alpha} \lambda m_p
\]

and

\[
J|m_p| = \left( \frac{\tau m_p}{\partial \tau} \right)_{\text{precess.}} + \hat{h}_p \quad |m_p|
\]
\[ \approx 2 \left( m_p^2 \times \Delta h_p + \Delta m_p \times h_p^2 \right) \]  \hspace{1cm} (67)

where

\[ h_p^2, B_0(r) + 2 \sum_{p'} f_{pp'}^a m_p^2 \]

\[ = B_0 + f_{pp'}^a m \]

and

\[ m'' = 2 \sum_p m_p^2 \times B_0 \cdot \]

For the internal field, $\mathbf{h}$, we have,

\[ \mathbf{h} = B_0 \left( 1 + \frac{E_{\gamma}}{1 + E_{\gamma}} \right) \]

\[ \frac{B_0(r)}{1 + E_{\gamma}} \]

Thus, the quasiparticles precess about the internal field $\mathbf{h}$. However, spin conservation requires the uniform mode to precess at the Larmor frequency $\omega_{L} = 2 \mu B$ and not $2 \mu (h/|h|)$.

To understand this we go back to the lineared Eq. (67) for precession. For the noninteracting case we have (after introducing $\mathbf{H} = \mathbf{H}_0 + \Delta \mathbf{H}$ with $\Delta \mathbf{H} = H, r + H_p$ and defining $\Delta H^1$ as $\Delta H^1 = H_r + r H_p, m_p^1 = m_p^0 + r m_p^0$, etc.)

\[ \frac{\partial m^1}{\partial \mathbf{h}} = \omega_{\mu} H_0 \cdot m^1 \]

where

\[ m^1, m'' r, \omega_{\mu}^1 \text{ and } \omega_{\mu}^1 = \omega_{\mu} H_0 \cdot \]

For the interacting case we have,

\[ \frac{\partial}{\partial \mathbf{h}} \Delta m_p^1 = m_p^{(0)} \Delta H^1 + \left[ B_0^2 \sum_{p'} f_{pp'}^{(a)} m_p^{(a)} \right] \Delta m_p^1 \]

\[ + 1 \left( \sum_{p'} f_{pp'}^{(a)} \Delta m_p^{(a)} \right) m_p^{(a)} \]

\hspace{1cm} (68)
There are 3 terms: 1) External tipping field, 2) The internal field, and 3) Change in the internal field due to interactions with the other quasiparticles.

Consider for now arbitrary distortions $\delta m_p$, then

$$\delta m_p = \frac{\partial n_{p}^{(o)}}{\partial \delta m_p} \sum_{l m} \gamma_{l m} Y_{l m}(\hat{\phi})$$

(69)

and

$$m_{p}^{(o)} = \frac{\partial n_{p}^{(o)}}{\partial \gamma_{o}} \gamma_{o}.$$

Inserting Eq. (69) into (68) we find,

$$i \frac{\partial}{\partial t} \gamma_{l m} + 2 \gamma_{o}^{m} \delta B \gamma_{l m} + 2 \left( \hbar + \frac{f_{n}^{m}}{2t + 1} \right) \gamma_{l m},$$

where $m_{p}^{o} = |m_{p}^{o}|$. The free oscillations occur for $\delta B = 0$, with $\gamma_{l m} = m_{l m} + i \omega_{l m}$ we have

$$\omega_{l m} = 2 \left( \hbar + \frac{f_{n}^{m}}{2t + 1} \right),$$

$$= \omega_{0} \left[ 1 + \frac{B_{o}}{2t + 1} \right]$$

(70)

where we used $m_{p}^{o} = \frac{B_{o}}{1 + f_{n}^{m}}$ $R_o$ and $h = \frac{B_{o}}{1 + f_{n}^{m}}$. These are just the Silin modes.

Note that $\omega_{l m}$ is the Larmor frequency.

Let us return to Eq. (66) to study the attenuation and dispersion of these modes. If we introduce a gradient in the magnetic field we have,

$$\frac{\partial}{\partial t} \gamma_{p} + \nabla \cdot \left[ \gamma_{p} \nabla \gamma_{p} \right]$$

$$+ \frac{\partial n_{p}^{(o)}}{\partial \gamma_{o}} \gamma_{o}$$

$$= J_{\mu p}$$

(71)

For the $J_{\mu p}$ we use a relaxation time approximation that conserves spin,

$$J_{\mu p} \left( \gamma_{p} \gamma_{p}^{(o)} \right)$$

(72)

with $\gamma_{p}$ the spin diffusion lifetime, where the spin diffusion $D_{s}$ is given by.

$$3^{o}$$
\[
D = \frac{\nu_F}{3} (1 + F_n^a) \tau_D
\]  

(72a)

This will generate as usual a set of coupled moment equations. It can be shown that

\[
\nu_{l+1,m} \ll \nu_{l,m} \quad \text{for } l \geq 1
\]

for all \( T \) with \( q r_F \ll |F_n^a| \omega_l \) thus for our purpose we need only consider the \( l = 0, 1 \) terms.  
For these we have,

\[
\Omega_n \nu_{l,m} = \frac{Q_l}{\sqrt{4\pi}} \sum_{m=-1}^{1} \nu_{l,-m}^* Y_{l,m}^* (\hat{p}) (1)^m
\]  

(73a)

\[
\Omega_l \nu_{1,m} = \frac{Q_l \nu_{1,m}^*}{\sqrt{4\pi}} Y_{1,m} (\hat{p}) \quad i J_{1,m}
\]  

(73b)

where

\[
\Omega_l = \omega_l + i \frac{2m^a}{N(a)} \left[ F_n^a - \frac{F_n^a}{2l+1} \right]
\]

(74a)

\[
Q_l = \frac{4}{3} \pi q r_F \left( 1 + \frac{F_n^a}{2l+1} \right)
\]

(74b)

and

\[
J_{1,m} = \frac{(1 + F_n^a/3)}{4} \nu_{1,m}^* \tau_D
\]

(74c)

The dispersion and attenuation of the \( l = 0, 1 \) Silin modes is given by,

\[
(\omega - \omega^a) \left[ \omega - \omega^a \right] = \left[ 1 + \frac{F_n^a/3}{\tau_D} \right] \left( 1 + F_n^a \right) \left( 1 + \frac{F_n^a}{3} \right) (q r_F)^2
\]

(75)

For \( T = 0 \) we have,

\[
\omega^a_{1,}(q) = \omega_l + \frac{1 + F_n^a (q r_F)^2}{3 \lambda} \omega_l
\]

(76a)

\[
\omega^a_{1,}(q) = \omega_l + \frac{1 + F_n^a (q r_F)^2}{3 \lambda} \omega_l
\]

(76b)

with
\[
\lambda = \left( \frac{1}{1 + F_o^a} - \frac{1}{1 + F_i^a / 3} \right)
\]

\[
\omega_1^+ \approx \omega_I \left( \frac{1 + F_i^a / 3}{1 + F_o^a} \right)
\]

For higher \( T \) the mode is diffusive,

\[
\omega - \omega_o^1 \approx -i \tau_D (1 + F_o^a) \frac{q^2 v_F^2}{3}
\]

\[
\approx -i D q^2
\]

(77)

At low \( T \) the attenuation of the spin wave is given by

\[
\text{Im} (\omega - \omega_o^1) \sim \frac{(q v_F)^2}{\lambda^2 \omega_L^2 \tau_D}
\]

and this vanishes as \( T \to 0 \).

An interesting new regime arises when \( \lambda \to 0 \). In this limit for \( T \to 0 \) we find from Eq. (75) that,

\[
\omega = \omega_L \left( 1 + F_o^a \right) \frac{q v_F}{\sqrt{3}}
\]

(78)

with the imaginary part given by,

\[
\text{Im} (\omega - \omega_L) \approx \frac{1}{2} \left[ \frac{1 + F_o^a}{\tau_D} \right]
\]

(79)

Note the difference between these modes and the \( \lambda \neq 0 \) results. This rather dramatic change in the dispersion and attenuation of these modes arises when the two modes \( \omega_o^1 \) and \( \omega_1^+ \) cross.
FIGURE CAPTIONS

Fig. 1. The momentum distribution function for a free Fermi gas at $T = 0$.

Fig. 2. The rest frame for the Fermi liquid is given by the solid lines and the frame moving with velocity $u$ is given by the dashed lines. It should be noted that adding a particle to the Fermi liquid is equivalent to adding a single quasiparticle.

Fig. 3. The scattering geometry for quasiparticle scattering on the Fermi surface. With $|p_i| = p_F$ for $i = 1, 2, 3$, and 4 only two angles are needed to characterize the scattering.
Fig. 2