Change in Dispersion Function from Field Gradient Errors

S. Ohnuma and K. Takayama

Method

We consider changes in the momentum dispersion function induced by field gradient errors of quadrupole magnets located around a ring.

The gradient error in the quadrupole magnet located at $s=s_i$ is represented by a single thin-lens and its strength is specified as

$$k_i = \frac{\Delta B'_i l}{B\rho} \quad (1)$$

where $B\rho$ is the magnetic rigidity. The momentum dispersion function $\eta(s)$ of such a machine should satisfy the equation

$$\ddot{\eta}(s) + [K(s) + \sum_{i=1}^N k_i \delta(s-s_i)] \eta(s) = \Omega(s), \quad (2)$$

where $K(s)$ and $\Omega(s)$ are periodic focussing and bending functions, respectively, and N denotes a total number of quadrupole magnets with a field gradient error; in general

$|K(s_i)| \gg |k_i|$. We express the periodic solution of (2) in the form

$$\eta(s) = \eta^{(0)}(s) + \eta^{(1)}(s) + \eta^{(2)}(s) + \dots, \quad (3)$$

where superscripts stand for the order of expansion. Substituting (3) into (2) and equating terms of the same order, we obtain

$$O(0) \quad \ddot{\eta}^{(0)} + K(s) \eta^{(0)} = \Omega(s), \quad (4a)$$

$$O(1) \quad \ddot{\eta}^{(1)} + K(s) \eta^{(1)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \eta^{(0)}(s), \quad (4b)$$

$$O(2) \quad \ddot{\eta}^{(2)} + K(s) \eta^{(2)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \eta^{(1)}(s), \quad (4c)$$

$$\begin{array}{c} \vdots \\ O(n+1) \quad \ddot{\eta}^{(n+1)} + K(s) \eta^{(n+1)} = \sum_{i=1}^N (-k_i) \delta(s-s_i) \eta^{(n)}(s), \quad (4d) \\ \vdots \end{array}$$

The periodic solution of (4d) is

$$\begin{aligned} \eta^{(n+1)}(s) &= \frac{\sqrt{\beta(s)}}{2s \sin \pi \nu} \int_s^{s+C} \sqrt{\beta(s')} \sum_{i=1}^N (-k_i) \delta(s-s_i) \eta^{(n)}(s') \cos[\nu(\phi(s) - \phi(s') + \pi)] ds' \\ &= \frac{\sqrt{\beta(s)}}{2s \sin \pi \nu} \sum_{i=1}^N (-k_i) \sqrt{\beta(s_i)} \eta^{(n)}(s_i) \cos[\nu(\phi(s) - \phi(s_i) + \pi)], \end{aligned} \quad (5)$$

where $\phi(s)$ is defined by

$$\phi(s) = \int^s \frac{ds'}{\nu \beta(s')},$$

$\beta(s)$ and ν are a betatron function and tune, respectively.

Introducing a $(1 \times N)$ matrix

$$\Delta(s) = \parallel \delta_p \parallel,$$

with

$$\delta_p = \frac{\sqrt{\beta(s)\beta(s_p)}}{2\sin\pi\nu} (-k_p) \cos [\nu(\phi(s) - \phi(s_p) + \pi)], \quad (6)$$

we write (5) in the matrix form

$$\gamma^{(n+1)}(s) = \Delta(s) \cdot X^{(n)}, \quad (7)$$

with

$$X^{(n)} = (\gamma^{(n)}(s_1), \gamma^{(n)}(s_2), \dots, \gamma^{(n)}(s_N))^T.$$

Linearity of (5) with respect to $\gamma(s_i)$ leads to a recursion relation for $\gamma^{(n)}(s_k)$ in the matrix form

$$X^{(n)} = M \cdot X^{(n-1)}, \quad (8)$$

where

$$M = \parallel m_{p,q} \parallel,$$

with

$$m_{p,q} = \frac{\sqrt{\beta(s_p)\beta(s_q)}}{2\sin\pi\nu} (-k_p) \cos [\nu(\phi(s_q) - \phi(s_p) + \pi)]. \quad (9)$$

From (8), we get

$$X^{(n)} = M^n \cdot X^{(0)}. \quad (10)$$

Substituting this expression for $X^{(n)}$ into (7), we obtain

$$\gamma^{(n+1)}(s) = \Delta(s) \cdot M^n \cdot X^{(0)} \quad (11)$$

and, from (3),

$$\gamma(s) = \gamma^{(0)}(s) + \Delta(s) \cdot [M^0 + M^1 + M^2 + \dots] \cdot X^{(0)} \quad (12)$$

If this matrix power series converges, Eq.(12) is equivalent to a simple form

$$\gamma(s) = \gamma^{(0)}(s) + \Delta(s) \cdot (I - M)^{-1} \cdot X^{(0)}, \quad (13)$$

where I is the unit matrix of order N.

Examples

N=1: From (6) and (9), we have

$$\delta_1 = \frac{1}{2} \sqrt{\beta(s)\beta(s_1)} \cdot \frac{\cos[\nu(\phi(s) - \phi(s_1) + \pi)]}{\sin \pi \nu} (-k_1), \quad (a1)$$

$$m_{11} = \frac{1}{2} \beta(s_1) \cdot \frac{\cos \pi \nu}{\sin \pi \nu} (-k_1), \quad (a2)$$

then,

$$\gamma(s) = \gamma^{(0)}(s) + \frac{\delta_1}{1 - m_{11}} \gamma^{(0)}(s_1). \quad (a3)$$

N=2:

matrix elements

$$\delta_p = \frac{1}{2} \sqrt{\beta(s)\beta(s_p)} \frac{\cos[\nu(\phi(s) - \phi(s_p) + \pi)]}{\sin \pi \nu} (-k_p) \quad (b1)$$

(p=1, 2)

$$\begin{cases} m_{p,p} = \frac{1}{2} \beta(s_p) \cdot \cot \pi \nu \cdot (-k_p), \\ m_{p,q} = \frac{1}{2} \sqrt{\beta(s_1)\beta(s_2)} \frac{\cos[\nu(\phi(s_2) - \phi(s_p) + \pi)]}{\sin \pi \nu} (-k_q), \end{cases} \quad (b2)$$

(p ≠ q)

Then, the matrices of Δ and $(I-M)^{-1}$ become

$$\Delta(s) = \frac{(-1)\sqrt{\beta(s)}}{2\sin\pi\nu} (k_1\sqrt{\beta_1}\cos(\pi\nu + \psi_{s1}), k_2\sqrt{\beta_2}\cos(\pi\nu + \psi_{s2})), \quad (b3)$$

$$(I-M)^{-1} = \frac{1}{D} \begin{pmatrix} 1 + \frac{k_2}{2}\beta_2\cot\pi\nu & -\frac{k_2}{2}\sqrt{\beta_1\beta_2}\frac{\cos(\pi\nu + \psi_{21})}{\sin\pi\nu} \\ -\frac{k_1}{2}\sqrt{\beta_1\beta_2}\frac{\cos(\pi\nu - \psi_{21})}{\sin\pi\nu} & 1 + \frac{k_1}{2}\beta_1\cot\pi\nu \end{pmatrix}, \quad (b4)$$

with

$$\beta_i = \beta(s_i),$$

$$\psi_{ij} = \nu(\phi(s_i) - \phi(s_j)),$$

$$D = 1 + \frac{\cot\pi\nu}{2} (k_1\beta_1 + k_2\beta_2) + \frac{\beta_1\beta_2}{4} k_1k_2 \frac{\sin^2\psi_{21}}{\sin^2\pi\nu}. \quad (b5)$$

From (b3), (b4) and (13),

$$\begin{aligned} \gamma(s) = \gamma^{(0)}(s) + \frac{(-1)\sqrt{\beta(s)}}{2D\sin\pi\nu} \left\{ \gamma^{(0)}(s_1) \left[k_1\sqrt{\beta_1}\cos(\pi\nu + \psi_{s1}) + \frac{k_1k_2\beta_2\sqrt{\beta_1}}{4\sin\pi\nu} P_1(s) \right] \right. \\ \left. + \gamma^{(0)}(s_2) \left[k_2\sqrt{\beta_2}\cos(\pi\nu + \psi_{s2}) + \frac{k_1k_2\beta_1\sqrt{\beta_2}}{4\sin\pi\nu} P_2(s) \right] \right\}, \end{aligned} \quad (b6)$$

with

$$P_1(s) = \cos(2\pi\nu + \psi_{s1}) - \cos(2\pi\nu - \psi_{21} + \psi_{s2}),$$

$$P_2(s) = \cos(2\pi\nu + \psi_{s2}) - \cos(2\pi\nu + \psi_{21} + \psi_{s1}),$$

$$D = 1 + \frac{\cot\pi\nu}{2} (k_1\beta_1 + k_2\beta_2) + \frac{k_1k_2}{4} \beta_1\beta_2 \frac{\sin^2\psi_{21}}{\sin^2\pi\nu}.$$

Discussions

When field gradient errors are caused by power supply ripples, for example, we have

$$k_i = \frac{B'_i l_i}{BP} \times 10^{-\alpha} \quad (14)$$

where B'_i and l_i are the field gradient and length of the i -th quadrupole magnet, respectively, and α represents the magnitude of the power supply ripple. All quadrupoles are assumed to be in series with the same power supply.

If $|m_{pg}| \ll 1$, contributions of higher order terms may be negligible. For such a case, we can write the solution of perturbed equation (2) to the first order, that is,

$$\eta(s) = \eta^{(0)}(s) + \sum_{i=1}^N \delta_i \eta^{(0)}(s_i). \quad (15)$$

Let apply the considerations to the $\bar{p}$ Accumulator Ring¹. Parameters relevant to the present discussion are listed in Table 1. The parameter m_{pg} satisfies the inequality

$$\begin{aligned} |m_{pg}| &< \frac{(\beta_i)_{\max} (B'_i)_{\max} (l_i)_{\max}}{2 \sin \pi \nu \cdot BP} \times 10^{-\alpha} \\ &= \frac{30 \cdot 103 \cdot 1.3}{2 \sin (\pi \cdot 6.61) \cdot 294} \times 10^{-\alpha} \\ &= 7.2756 \times 10^{-\alpha}. \end{aligned} \quad (16)$$

Assuming $\alpha \geq 2$, we see that the first-order solution (15) is accurate enough for our purpose. We can calculate the change in the dispersion function at any point in the ring from a convenient form

$$\eta(s) = \eta^{(0)}(s) + \frac{(-1)^{\sqrt{\beta(s)}}}{2 \sin \pi \nu BP} 10^{-\alpha} \sum_{i=1}^N \sqrt{\beta_i} B'_i l_i \cos[\pi(\nu - 24\nu_i)] \eta_i^{(0)} \quad (17)$$

An amount of change calculated at the particular position ($s=0$) where the dispersion is required to be zero with high precision is compared with the Synch output²;

for $\alpha = 5$

Calculation: $\gamma(0) = \gamma^{(0)}(0) + (-1.89 \times 10^{-4} \text{ m}),$

Synch: $\gamma(0) = \gamma^{(0)}(0) + (-2. \times 10^{-4} \text{ m}).$

Here only gradient errors in large aperture quadrupole magnets are assumed.

References

1. " Design Report Tevatron I Project ", Fermilab, Oct. 1982.
2. D.Johnson, private communications, April(1983).

TABLE 1

NO. OF QMAG	BX (M)	B (KG/M)	L (M)	DNUE*	EX (M)
1	16.2936	103.8087	.6401	.1410	.0918
2	12.1482	-103.8087	1.3117	.1590	.0680
3	27.6176	103.8087	.7010	.1770	.0841
4	15.7332	96.6333	.4572	.2490	.6465
5	2.8710	-97.4126	.8280	.3500	.5935
6	27.8812	96.6333	.7010	.5330	1.7353
7	6.3068	-97.4126	.7010	.5930	.5340
8	21.0945	96.6333	.4572	.7280	-.1799
9	7.7138	-97.4126	.4572	.8170	1.1236
10 (QL)	24.7911	40.8765	.4572	.8970	5.3016
11 (QL)	30.0168	89.3989	.8738	.9350	8.9183
12 (QL)	15.9959	-89.3989	.7722	.9450	7.1831
13 (QL)	12.2270	-89.3989	.7722	.9630	7.3121
14 (QL)	15.8280	89.3989	.6426	.9760	9.1719
15 (QL)	15.8280	89.3989	.6426	1.2280	9.1719
16 (QL)	12.2270	-89.3989	.7722	1.2410	7.3121
17 (QL)	15.9959	-89.3989	.7722	1.2590	7.1831
18 (QL)	30.0168	89.3989	.8738	1.2690	8.9183
19 (QL)	24.7911	40.8765	.4572	1.3070	5.3016
20	7.7138	-97.4126	.4572	1.3870	1.1236
21	21.0945	96.6333	.4572	1.4760	-.1799
22	6.3068	-97.4126	.7010	1.6110	.5340
23	27.8812	96.6333	.7010	1.6710	1.7353
24	2.8710	-97.4126	.8280	1.8540	.5935
25	15.7332	96.6333	.4572	1.9550	.6465
26	27.6176	103.8087	.7010	2.0270	.0841
27	12.1482	-103.8087	1.3117	2.0450	.0680
28	16.2936	103.8087	.6401	2.0630	.0918
29	16.2936	103.8087	.6401	2.3450	.0918
30	12.1482	-103.8087	1.3117	2.3630	.0680
31	27.6176	103.8087	.7010	2.3810	.0841
32	15.7332	96.6333	.4572	2.4530	.6465
33	2.8710	-97.4126	.8280	2.5540	.5935
34	27.8812	96.6333	.7010	2.7370	1.7353
35	6.3068	-97.4126	.7010	2.7970	.5340
36	21.0945	96.6333	.4572	2.9320	-.1799
37	7.7138	-97.4126	.4572	3.0210	1.1236
38 (QL)	24.7911	40.8765	.4572	3.1010	5.3016
39 (QL)	30.0168	89.3989	.8738	3.1390	8.9183
40 (QL)	15.9959	-89.3989	.7722	3.1490	7.1831
41 (QL)	12.2270	-89.3989	.7722	3.1670	7.3121
42 (QL)	15.8280	89.3989	.6426	3.1800	9.1719
43 (QL)	15.8280	89.3989	.6426	3.4320	9.1719
44 (QL)	12.2270	-89.3989	.7722	3.4450	7.3121
45 (QL)	15.9959	-89.3989	.7722	3.4630	7.1831
46 (QL)	30.0168	89.3989	.8738	3.4730	8.9183
47 (QL)	24.7911	40.8765	.4572	3.5110	5.3016

NO. OF QMAG	BX*(M)	B'(KG/M)	L(M)	DNUE*	EX*(M)
48	7.7138	-97.4126	.4572	3.5910	1.1236
49	21.0945	96.6333	.4572	3.6800	-.1799
50	6.3068	-97.4126	.7010	3.8150	.5340
51	27.8812	96.6333	.7010	3.8750	1.7353
52	2.8710	-97.4126	.8280	4.0580	.5935
53	15.7332	96.6333	.4572	4.1590	.6465
54	27.6176	103.8087	.7010	4.2310	.0841
55	12.1482	-103.8087	1.3117	4.2490	.0680
56	16.2936	103.8087	.6401	4.2670	.0918
57	16.2936	103.8087	.6401	4.5490	.0918
58	12.1482	-103.8087	1.3117	4.5670	.0680
59	27.6176	103.8087	.7010	4.5850	.0841
60	15.7332	96.6333	.4572	4.6570	.6465
61	2.8710	-97.4126	.8280	4.7580	.5935
62	27.8812	96.6333	.7010	4.9410	1.7353
63	6.3068	-97.4126	.7010	5.0010	.5340
64	21.0945	96.6333	.4572	5.1360	-.1799
65	7.7138	-97.4126	.4572	5.2250	1.1236
66 (QL)	24.7911	40.8765	.4572	5.3050	5.3016
67 (QL)	30.0168	89.3989	.8738	5.3430	8.9183
68 (QL)	15.9959	-89.3989	.7722	5.3530	7.1831
69 (QL)	12.2270	-89.3989	.7722	5.3710	7.3121
70 (QL)	15.8280	89.3989	.6426	5.3840	9.1719
71 (QL)	15.8280	89.3989	.6426	5.6360	9.1719
72 (QL)	12.2270	-89.3989	.7722	5.6490	7.3121
73 (QL)	15.9959	-89.3989	.7722	5.6670	7.1831
74 (QL)	30.0168	89.3989	.8738	5.6770	8.9183
75 (QL)	24.7911	40.8765	.4572	5.7150	5.3016
76	7.7138	-97.4126	.4572	5.7950	1.1236
77	21.0945	96.6333	.4572	5.8840	-.1799
78	6.3068	-97.4126	.7010	6.0190	.5340
79	27.8812	96.6333	.7010	6.0790	1.7353
80	2.8710	-97.4126	.8280	6.2620	.5935
81	15.7332	96.6333	.4572	6.3630	.6465
82	27.6176	103.8087	.7010	6.4350	.0841
83	12.1482	-103.8087	1.3117	6.4530	.0680
84	16.2936	103.8087	.6401	6.4710	.0918

BP =294.0756 (KG-m)
 NUEx =6.61177
 * :VALUES AT THE CENTER OF EACH MAGNET
 QL :LARGE APERTURE QUADRUPOLE MAGNET