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PLASMAS IN PARTICLE ACCELERATORS:
THE WARM-BEAM EQUILIBRIA

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1. Introduction

Equilibria involving non-neutral plasmas [1] are useful within the context of discussions on particle beams, rotating electron- and ion-layers. Recently, there has also been interest in very cold electron plasmas [2] as well as in nonneutral plasmas studied for the purpose of providing alternate drivers for inertial confinement fusion [3-8]. In most of these problems, the self-consistent field is an important effect.

Within the heavy-ion accelerator context, there is a need to identify and avoid possible space charge driven plasma instabilities so that there is little degradation in the performance of the beam. For modes or local wavelets which involve longitudinal and transverse coupling, the confirmation of this conjecture require studies of the Vlasov stability in six-dimensional phase-space and in time. Even though particle codes exist for such problems, they are costly to run, and their phase-space resolution is strongly limited by cost.

We propose here a set of "warm" beam equilibria as a bridge between the beams with "cold" (i.e. circular transverse orbits) and "hot" transverse temperatures. If any instabilities of cold beams survive through this regime, then special numerical simulations could be devised to pursue their effects under more realistic accelerator beam transport conditions.

Note that the following beam equilibria (Section II) have a complete set of smooth radial density profiles. For the context of heavy-ion drivers, We are actually taking an idealized limit of long

beam bunches in averaged electrostatic focusing fields. In this case, the equilibrium electric field below is to be interpreted as not altogether self-consistent, but including these averaged external focusing forces. As part of this study, it is necessary to specify the particle orbits in these combined electric fields. Since it will be used for future studies of the Vlasov equation, we give a more complete summary of the orbit expansion in the Appendix. To illustrate the use of this equilibrium and orbit expansions, we show, in Section III, that the oscillations of the cold beam [1] can also be obtained from the Vlasov equation using this equilibrium and then taking appropriate limits. This and related procedures can be used to derive a fluid approximation with finite transverse temperature for warm beams.

11. The Inverse Equilibrium Problem

In order to specify the equilibrium of a rotating nonneutral plasma column in the presence of an averaged focusing plus space charge electrostatic force $(-q\phi/r)$ where ϕ is the electrostatic potential, $q = Ze$ is the ion charge, the standard procedure is to start with an equilibrium distribution function $f_0(H_1, p_\theta, p_z)$. Here H_1 is the Hamiltonian for transverse motion and (p_r, p_θ, p_z) are canonical momenta corresponding to the cylindrical coordinates (r, θ, z) aligned along the beam. There is a limited class of equilibria with exactly specified functions $f_0(H_1, p_\theta, p_z)$ which corresponds to simple analytical formulae for the macroscopic particle densities $n(r)$, velocities $\underline{v}(r)$ and temperatures $T(r)$ etc. In view of the fact that these macroscopic quantities

are more directly observable in experiments, it is at least interesting to note the existence of classes of equilibria which include complete sets of macroscopic density profiles and other quantities as well as associated distribution functions. These distribution functions are obtained from given density (and temperature) profiles by solution of the inverse equilibrium problem.

It is easy to see that the conditions for such an inverse problem should probably not include global thermodynamic equilibrium. If space charge effects are important, most density states tend to have differential rotation and are by definition not in global equilibrium. But we will demand that:

(i) These equilibria represent "local thermal equilibrium" at least in the transverse directions so that in these directions the distribution function is locally Maxwellian in structure. A separate Maxwellian or other equilibrium could be specified for the longitudinal motions.

(ii) Consistent with local thermal equilibrium, we will also require that the particle orbits have radial excursions (δr) which are smaller than the typical equilibrium length scales (thus, "warm nonneutral plasma equilibrium"). For convenience of presentation, we will also assume that the azimuthal and axial self-magnetic fields are not important (valid for heavy-ion accelerators) even though we assert that these effects could be readily included.

We first remind the reader that complete sets of equilibria are easily generated for cold nonneutral plasma columns. For example, we may consider the volume density profiles

$$M^{(k)}(r) = (1 - r^2/a^2)^{2k}, \quad (k = 0, 1, 2, \dots), \quad (1)$$

$$N^{(k)}(r) = (1 + r^2/a^2)^{-k-3/2}, \quad (k = 0, 1, 2, \dots). \quad (2)$$

That these functions are complete in the intervals $(0, a)$ or $(0, \infty)$ follows because

$$\frac{1}{a^2} \int_0^a g(r) M^{(k)}(r) r dr = \int_0^1 g[\xi] \xi^{2k} d\xi, \quad (3)$$

$$\frac{1}{a^2} \int_0^\infty h(r) N^{(k)}(r) r dr = \int_0^1 h[\eta] \eta^{2k} d\eta, \quad (4)$$

where $\xi = (1 - r^2/a^2)$ and $\eta = (1 + r^2/a^2)^{-1/2}$. Polynomials in ξ^2 or η^2 are certainly complete in the interval $(0, 1)$. Since these functions also give simple analytical formulae for the electric fields, we know that we could also easily calculate quantities such as circular rotation frequencies of circular orbits

$$\Omega(r) = \Omega^\pm(r) = \pm \left[-\frac{q}{m} E_r(r) \right]^{\frac{1}{2}}, \quad (5)$$

where the radial electric field

$$E_r(r) = -\frac{q}{r} \left[1 - \frac{1}{r} \int_0^r \omega_p^2(r') r' dr' \right], \quad (6)$$

consists respectively of the averaged focusing field and the space charge components. Here, ν_0 is the coherent oscillation frequency and $\omega_p = [4\pi n q^2/m]^{1/2}$ is the ion plasma frequency, $n(r)$ being the ion number density. The reader may easily generate other examples.

Clearly, there is only need to specify the distribution functions. In order to help isolate equilibria with local thermal equilibrium, we look briefly at particle orbits satisfying condition (ii) given above. If we allow for differential rotation, the betatron vortex gyrations of the particles are no longer circles, but ellipses to the lowest approximation. In an inertial frame the dynamics of the particles with charge $q = Ze$ and mass m is governed by the transverse Hamiltonian (the longitudinal motion simply follows $p_z = \text{const.}$)

$$H_{\perp} = \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} p_{\theta}^2 + q\bar{\Phi}(r), \quad (7)$$

where $\Phi(r)$ is the equilibrium electrostatic potential (assumed independent of θ, z for an axisymmetric plasma column). The equations of motion are as usual

$$p_r = m \dot{r}, \quad p_{\theta} = m \dot{\theta} r^2, \quad p_z = m \dot{z}, \quad (8)$$

$$\dot{p}_r = q E_r(r) + \frac{p_{\theta}^2}{mr^3}, \quad (9)$$

$$\dot{p}_0 = \dot{p}_z = 0 \quad , \quad E_r(r) = - \frac{d\Phi}{dr} \quad , \quad (10)$$

where $\dot{r}$, $\dot{p}_r$, etc. indicate time derivatives of r , p_r .

Let us focus our attention on an orbit characterized by the constants of motion H_1 , p_θ , p_z . Consider as reference the helical orbit of constant radius ρ , which has the same values of p_θ , p_z . This helical orbit has angular frequency $\Omega(\rho)$ given by

$$\Omega^2(\rho) = - \frac{g}{m\rho} E_r(\rho) \quad . \quad (11)$$

its energy (apart from $p_z^2/2m$)

$$H_{\perp\rho} = \frac{1}{2} m \Omega^2(\rho) \rho^2 + \frac{g}{2} \Phi(\rho) \quad , \quad (12)$$

and canonical angular momentum

$$p_\theta = \mathcal{P}(\rho) \equiv m \Omega(\rho) \rho^2 \quad , \quad (13)$$

are both functions of ρ . The properties of the more general $\dot{p}_r \neq 0$ orbit can be obtained from the reference helical orbit, provided $|r-\rho| < \rho$, by means of the expansion ($|\delta r| < \rho$)

$$r = \rho + \delta r, \quad p_r = m \delta \dot{r}, \quad \dot{\rho} = 0, \quad (14)$$

$$\dot{\theta} = \Omega(\rho) + \delta \dot{\theta}, \quad p_\theta = m \Omega(\rho) \rho^2 = \text{constant}. \quad (15)$$

If higher accuracy is of concern, we will find in general that the guiding center circular motion has also weak dependencies on the vortex energy

$$H_v = H_1 - H_2, \quad (16)$$

for our purposes, it is sufficient to expand equation (9) systematically in δr to find

$$\delta \ddot{r} + \nu^2(\rho) \delta r = 0, \quad (17)$$

where the "generalized vortex frequency" ν (twice the betatron frequency) is the radial oscillation frequency of the orbit,

$$\nu^2(\rho) \equiv \frac{1}{m^2 \rho^3} \frac{d}{d\rho} [p^2(\rho)] = \frac{1}{\rho^3} \frac{d}{d\rho} [\Omega^2(\rho) \rho^4]. \quad (18)$$

(if $d\alpha/d\rho \neq 0$, we find $v^2 = [2\alpha]^2$, a well known result). We now find

$$\delta r = \delta r_0 \cos \chi, \quad p_r/m = \delta \dot{r} = v(\rho) \delta r_0 \sin \chi, \quad (19)$$

where the phase $\chi = -v(\rho)(t-t_0)$, as a function of time t , would include more terms if higher order corrections are included. Also, from

$$\dot{\theta} = \Omega(\rho) - 2\Omega(\rho) \frac{\delta r}{\rho}, \quad (20)$$

we find

$$\rho \delta \theta = 2\Omega(\rho) \frac{\delta r_0}{v} \sin \chi. \quad (21)$$

Therefore, in general, the vortex gyration has elliptical cross-section as we can see by the following χ averages.

$$\begin{aligned} \left[\frac{\langle (r \delta \theta)^2 \rangle_\chi}{\langle (\delta r)^2 \rangle_\chi} \right]^{\frac{1}{2}} &= \left| \frac{2\Omega(\rho)}{v(\rho)} \right| = \left| 1 + \frac{1}{2} \frac{d \ln \Omega}{d \ln \rho} \right|^{-\frac{1}{2}} \\ &= \left[1 + \frac{1}{2\rho^2} \frac{\int_0^\rho \omega_p^2(r) r dr - \omega_p^2(\rho) (\frac{1}{2}\rho^2)}{\rho_0^2 - \frac{1}{\rho^2} \int_0^\rho \omega_p^2(r) r dr} \right]^{-\frac{1}{2}}, \quad (22) \end{aligned}$$

where

$$\omega_p(\rho) = \left[\frac{4\pi n(\rho) q^2}{m} \right]^{\frac{1}{2}}, \quad (23)$$

is the plasma frequency. Thus, in the rotating frame of the reference circular orbit, the betatron vortex oscillation is an ellipse whose orientation is as indicated in Figure 1. The long axis is in the radial direction if $dn/dr < 0$.

These orbit calculations motivate us to consider the integral of motion

$$\begin{aligned} H_1 - H_1(\rho) \left[r(\rho) \right] &\approx \frac{1}{2m} p_r^2 + \frac{1}{2} \frac{\partial H_1}{\partial r^2} \bigg|_{r=\rho} (r-\rho)^2 \\ &\approx \frac{1}{2m} p_r^2 + \frac{1}{2} m v^2(\rho) (r-\rho)^2 \\ &\approx \frac{1}{2} m v_r^2 + \frac{1}{2} m \left[\frac{v(\rho)}{2\Omega(\rho)} \right]^2 v_\theta^2, \end{aligned} \quad (24)$$

where $v_r = p_r/m$, $v_\theta = r\dot{\theta}$, $v_z = p_z/m$ are velocity components with rotation taken out. Thus, if we choose distribution functions of the form

$$f_c(H_1, p_\theta, p_z) = D[\rho(\rho), p_z] \exp \left\{ -\frac{1}{m} \left[H_1 - H_1(\rho) \right] \frac{1}{\Omega(\rho)} \right\} \quad (25a)$$

$$D[\rho(p_0), p_z] = \frac{F[\rho(p_0)]}{\sqrt{2\pi} m c_z^2[\rho(p_0)]} \exp\left\{-\frac{(p_z - p_{zc})^2}{2 m c_z^2[\rho(p_0)]}\right\}, \quad (25b)$$

we will have equilibria which are locally Maxwellian in velocity space. (The longitudinal and transverse temperatures need not be related.)

If $\delta r \ll \rho$ for the typical orbits of the equilibrium, then we find that $F[\rho(p_0)]$ and $\rho(p_0)$ are related very simply to the number density $n[\rho(p_0)]$ and the radial "temperature" $T_r[\rho(p_0)]$. To see this we calculate [using Eqs. (8) and (24)]

$$\begin{aligned} n(r) &= \iiint f_c d^3v = \iiint f_c \frac{1}{m^3 r} dp_r dp_\theta dp_z \\ &\approx \sum_{\text{sgn} \Omega} \int_0^\infty \frac{2\pi c_r^2(\rho)}{m r} F[\rho(p_0)] \exp\left\{-\frac{v^2(\rho)(\rho-r)^2}{2 c_r^2(\rho)}\right\} \left|\frac{dk_\theta}{d\rho}\right| d\rho \end{aligned} \quad (26)$$

where p_θ and hence c_r, v are functions of ρ through Eq. (13). The fact that the typical orbit has $|\rho - r| \ll \rho$ implies that c_r is much smaller than $(v\rho)$. Therefore, the remaining integral in Eq. (26) can be evaluated asymptotically by Laplace's method, we find, between $F[\rho, \text{sgn}\Omega]$ and $n[\rho, \text{sgn}\Omega]$ corresponding to beam particles with corresponding signs of $\text{sgn}(\Omega)$, the relation

$$F[\rho, \text{sgn}(\Omega)] = \frac{n[\rho, \text{sgn} \Omega] |\Omega(\rho)|}{\pi m^2 v(\rho) c_r^2(\rho)}, \quad (27)$$

where $\rho(p_\theta)$ is defined in Eq. (13). This equation together with Eq. (25) gives the distribution function.

Note that if we have solid body rotation, $\Omega = \text{constant}$, then Eq. (25) implies indeed a form $f_0 \sim \exp [-(H - \Omega p_\theta)/m c_r^2]$ because

$$\begin{aligned} H_{\perp} p &= \frac{1}{2} m \Omega^2 r^2 + \int \bar{\Phi}(r) \\ &= m \Omega^2 r^2 + \text{constant} \\ &= \Omega p_\theta + \text{constant} , \end{aligned} \quad (28)$$

according to Eqs. (11) - (13).

Defining $\langle \dots \rangle_r$ to be average over velocity part of distribution function we find that

$$\langle v_r \rangle_r = 0 , \quad \langle v_r^2 \rangle_r \approx \frac{1}{m} \Theta [\rho(r)] , \quad (29)$$

$$\langle v_\theta \rangle_r \approx \Omega(r) r , \quad \langle [v_\theta - \langle v_\theta \rangle_r]^2 \rangle_r = \frac{v_\theta^2(r) \Theta [\rho(r)]}{m [\frac{1}{2} \Omega(r)]^2} , \quad (30)$$

where it is helpful to note that $v_\theta r = u(\rho) \rho^2$. Thus, we have

$$\frac{\langle [\dot{\vartheta} - \langle \dot{\vartheta} \rangle_r]^2 \rangle_r}{\langle \dot{\vartheta}^2 \rangle_r} = \frac{v^2(r)}{[2 \Omega(r)]^2} \quad (31)$$

Note that this ratio is not in contradiction to equation (23) which is an average over the orbit of one particle with fixed guiding center radius (or over particles with same guiding center radius). The distribution function average $\langle \dots \rangle_r$ is actually taken over particles with different guiding center radii ρ but whose particle positions will arrive at radius r at time $t=0$. To perform this average from orbit theory, we note, using Eqs. (19) - (20) that these orbits have guiding center radii $\rho = r - \delta r_0 \cos[v(r)t_0]$, and at $t = 0$

$$\dot{r} \approx v(r) \delta r_0 \sin[v(r) t_0] \quad (32)$$

$$\begin{aligned} \dot{\vartheta} &\approx \Omega(r) + \frac{d\Omega}{dr}(\rho - r) - 2\Omega(r) \delta r_0 \omega_s(v t_0) \\ &\approx \Omega(r) - \frac{v^2(r)}{2\Omega(r)} \frac{\delta r_0}{r} \cos[v(r) t_0] \end{aligned} \quad (33)$$

Therefore, in agreement with Eq. (31), we find

$$\frac{\langle [\dot{\vartheta} - \Omega(r)]^2 \rangle_r}{\langle \dot{\vartheta}^2 \rangle_r} \approx \frac{v^2(r)}{[2 \Omega(r)]^2} \quad (34)$$

In summary, the posing of the inverse equilibrium problem has advantages. For example, suppose for the ion beam we can measure the beam and rotation velocities U_z , $U_\theta(r) = [a(r) r]$, then from Eq. (5) we obtain the electric field. From Poisson's equation (6) we deduce the charge density (nq). After that distribution functions are obtained from Eqs. (25), (27), and (29) if the radial temperatures $T_r(r)$ are known. But the greatest advantage in this equilibrium is that it allows more freedom when solving the Vlasov equation for warm beams. It does not require singular distributions in phase space nor discontinuous ones in configuration space [9, 10].

III. An Example: wave Perturbations In The Cold-Beam Limit

In order to illustrate the use of these equilibria as compared to the more frequently applied distributions (e.g., K-V) in accelerator physics, we show the transition to the "Cold-beam" limit for waves with infinite longitudinal wavenumber. Here the perturbed electrostatic potential in polar coordinates (r, θ, z) takes the form

$$\phi_1(r, \theta, t) = \phi(r) \exp \left\{ -i [\omega t - \ell \theta] \right\}, \quad (35)$$

where ω and ℓ are the frequency and azimuthal wavenumber respectively. We assume growing modes so that $\text{Im}(\omega) > 0$. The Vlasov equation for the perturbed distribution function $f_1(p_r, p_\theta, r, \theta, t)$ gives

$$f_1 e^{i[\omega t - \ell \theta]} = Ze \frac{\partial f_0}{\partial H_1} [\phi(r) + \omega \hat{I}] + \ell Ze \frac{\partial f_0}{\partial k} \hat{I}, \quad (36)$$

where $\hat{I}$ is the usual integral over the equilibrium orbit,
 $r'(t'), \theta'(t')$,

$$\hat{I} \equiv \int_{-\infty}^t \phi(r') \exp\{-i[\omega(t'-t) - \ell(\theta' - \theta)]\} dt'. \quad (37)$$

If we take into account the fact that the equilibrium orbit has a radial excursion which is exactly periodic with period $(2\tau_v)$ during which time the corresponding angular excursion is $(2\theta_v)$, we find that

$$\hat{I} = - \frac{\tau_v}{2\pi \sin(\omega_0 \pi)} \int_{-\pi}^{\pi} \phi[r'(s)] \exp\{-i[\omega s - \ell \theta(s)]\} ds, \quad (38)$$

where

$$\omega = \omega_0 \frac{\tau_v}{\pi} - \ell \frac{\theta_v}{\pi}, \quad (39)$$

$$s = \frac{\pi}{\tau_v} (t' - t + \tau_v), \quad (40)$$

and $\phi[r'(s)]$, $\theta(s)$ are periodic in s with period 2π .

In considering the transition from "warm-beams" to the "cold beam" limit, we define the parameter

$$\epsilon_0 = O\left(\frac{\delta r}{r}\right) = O\left(\frac{c_r}{v r}\right), \quad (41)$$

assuming that the radial excursions in the orbits are becoming so small that

$$\left| \frac{d \ln f}{d \ln r} \right| \ll \frac{1}{\epsilon_0}. \quad (42)$$

We could easily confirm the orderings

$$\iint \left\{ \begin{array}{c} f_0 \\ \frac{\partial f_0}{\partial H_1} \\ \frac{\partial f_0}{\partial p_\theta} \end{array} \right\} \frac{dr dp_\theta}{r} = O \left\{ \begin{array}{c} 1 \\ \epsilon_0^{-2} \\ \epsilon_0^{-2} \end{array} \right\}, \quad (43)$$

if we note that

$$\frac{\partial f_0}{\partial H_1} = - \frac{1}{m \zeta_r^2(r)} f_0, \quad (44)$$

$$\frac{\partial f_0}{\partial p_\theta} = \frac{2 R(r) f_0}{m^2 r v'(r)} \left\{ m \frac{\partial \ln D(\cdot; H_2)}{\partial \rho} + \frac{2}{\zeta_r^3(r)} [H_2 - H_1 \rho] \frac{d c_r}{d \rho} + \frac{1}{\zeta_r^2} \frac{d H_1 \rho}{d \rho} \right\}. \quad (45)$$

Thus unless there are cancellations within the integrals $\iint f_1 dp_r dp_\theta$, the perturbed density $n_1 \sim \epsilon_0^{-2}$ which could force rapid spatial

oscillations $\nabla^2 \phi_1 \sim \epsilon_0^{-2}$ in contradiction with assumption (42) needed for transition to the cold beam limit. But in fact there is such a cancellation of $O(\epsilon_0^{-2})$ and $O(\epsilon_0^{-1})$ terms in n_1 . If we expand

$$\phi(r') = \phi(r) + \frac{d\phi}{dr}(r'-r) + \frac{1}{2} \frac{d^2\phi}{dr^2}(r'-r)^2 + \dots, \quad (46)$$

in the integral $\hat{I}$ of equation (38), then we find in n_1 that the coefficients of $\phi(r)$ of $O(\epsilon_0^{-2})$ and $O(\epsilon_0^{-1})$ both vanish while the leading term of $O(\epsilon_0^{-1})$ in the coefficient of $d\phi/dr$ also vanishes. Correct to $O(1)$ then, we have a second order ordinary differential equation in $\phi(r)$ which is the cold-beam approximation.

The detailed calculation is straightforward but lengthy. We have to calculate the quantity ($j = 0, 1, 2$)

$$\hat{I}^{(j)} = - \frac{\tau_v}{2\pi \sin(\omega\pi)} \int_{-\pi}^{\pi} (r'-r)^j \exp \left\{ -i \left[\hat{\omega} z - \ell \hat{\theta}(z) \right] \right\} dz, \quad (47)$$

to the relevant orders in ϵ_0 using the orbit expansions summarized in the appendix. More directly, in order to calculate the perturbed density $n_1(r, \theta, t)$, we need, to the relevant orders in ϵ_0 the coefficients $N^{(j)}$ of the expansion

$$4\pi \int e r^2 n_1(r, \theta, t) \exp [i(\omega t - \ell \theta)] = \sum_{j=0}^{\infty} \frac{d^j \phi(r)}{dr^j} N^{(j)}. \quad (48)$$

According to equations (25), (27), (36), (44) - (46), we have

$$(j!)N^{(j)} = \delta_{j0} \mathcal{L}[1] + \mathcal{L}\left\{ \left[\omega - \ell m c_r^2(\rho) \frac{\partial f_0}{\partial H_0} \right] \hat{I}^{(n)}(\xi, \eta, \rho) \right\}_{(49)}$$

where δ_{j0} is the Kronecker-delta,

$$\begin{aligned} \mathcal{L}[h] &= 4\pi(Ze)^2 \iint h \frac{\partial f_0}{\partial H_2} d\rho dH_2 \\ &= - \sum_{\text{sgn} \Omega} \int_{-\infty}^{\infty} h \frac{\omega_p^2(\rho)}{2\pi^2} \left[\frac{c_r^2(\rho)}{c_r^2(\rho)} \right]^2 \exp[-\beta(\rho)(\xi^2 + \eta^2)] d\xi d\eta, \end{aligned} \quad (50)$$

$$\mathcal{L}[\rho] = \frac{1}{2} \left[\frac{\rho v(\rho)}{c_r(\rho)} \right]^2, \quad \omega_p^2 = \frac{4\pi(Ze)^2 n[\rho, \text{sgn} \Omega]}{m}. \quad (51)$$

Here ξ , η are velocity related variables defined by

$$\xi = E \sin S_0, \quad \eta = E \cos S_0, \quad (52)$$

with e given by equation (A3); and in terms of the constant t_0 of equation (A7) and the radial orbit period ($2\tau_v$) of equation (A10) we have used

$$S_0 = \frac{\pi}{\tau_v} (t_0 - t + \tau_v). \quad (53)$$

We may also note that according to the orbit theory of the appendix,

$$\rho = r - r\eta + r[A_2(r)\eta' - 2B_2(r)\xi'] + \dots, \quad (54)$$

varies with ξ and η (nb., $\xi \sim \eta \sim \epsilon_0$).

In order to calculate $N^{(j)}$ to several orders in ϵ_0 , we need to use equation (54) to expand out all occurrences of ρ in equations (49) - (50). Also, we must evaluate the relevant orders in ϵ_0 , the quantities $\partial \ln f_0 / \partial p_0$, $\hat{\omega}^{(j)}$, τ_y , $\hat{\omega}$, $\theta(s)$, and $(r' - r)^j$. To illustrate, we calculate the coefficient $N^{(2)}$. Since we only need the dominant order of this quantity, we may note that

$$\left[\omega - \ell m c_r^2 \frac{\partial \ln f_0}{\partial p_0} \right] \approx \omega - \ell \Omega(r). \quad (55)$$

Also $\tau_y = \pi/v(r)$ and

$$\hat{\omega} \approx \hat{\omega}_0(r) \equiv \frac{\omega - \ell \Omega(r)}{v(r)}, \quad (56)$$

according to equations (A10) - (A11); $\theta(s)$ can be neglected; and

$$r' - r = r \left[\xi \sin s + \eta (1 + \cos s) \right]. \quad (57)$$

Taking these into account, we find that

$$N^{(2)} = \sum_{sgn \Omega} \frac{\omega_p^2 [r, sgn(\Omega)] r^2}{\nu^2(r) - [\omega - \ell \Omega(r)]^2}, \quad (58)$$

where the distinction between ρ and r does not matter to this order. Similarly we find that

$$N^{(1)} = \sum_{sgn \Omega} r \frac{d}{dr} \left\{ \frac{r \omega_p^2(r)}{\nu^2(r) - [\omega - \ell \Omega(r)]^2} \right\}, \quad (59)$$

$$N^{(0)} = \sum_{sgn \Omega} \left\{ \frac{\ell^2 \omega_p^2}{\nu^2(\hat{\omega}_0^2 - 1)} + \frac{\ell r}{\nu \hat{\omega}_0} \frac{\partial}{\partial r} \left[\frac{2 \Omega \omega_p^2}{\nu^2(\hat{\omega}_0^2 - 1)} \right] \right\}_0. \quad (60)$$

Using equations (48) and (58) - (60) in Poisson's equation gives the cold-beam equation for $\phi(r)$,

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left\{ r \left[1 - \sum_{sgn \Omega} \frac{\omega_p^2}{\nu^2(\hat{\omega}_0^2 - 1)} \right] \frac{d\phi}{dr} \right\} - \frac{\ell^2}{r^2} \left[1 - \sum_{sgn \Omega} \frac{\omega_p^2}{\nu^2(\hat{\omega}_0^2 - 1)} \right] \phi \\ + \frac{\ell \phi}{r} \sum_{sgn \Omega} \frac{1}{\nu \hat{\omega}_0} \frac{d}{dr} \left[\frac{2 \Omega \omega_p^2}{\nu^2(\hat{\omega}_0^2 - 1)} \right] = 0. \end{aligned}$$

except for the fact that we have two counter-rotating streams of the same charge species, we find this to be the same as equation (2.6.20) of Ref. [1].

If we introduce in equation (36) the additional perturbation $\exp(ik_z z)$, with $k_z \neq 0$, then we find that the transition to the cold-beam limit appears to be non-trivial. It is likely that $n_1 \sim \epsilon_0^{-2}$ even as $c_r \rightarrow 0$ except in special circumstances. Since the $k_z \neq 0$ case is indeed the main focus of introducing the warm-beam equilibria, the study of this $k_z \neq 0$ case is under active consideration.

APPENDIX A: Orbit Expansions to Higher Orders

from Equations (14), (19), and (24), we conclude that

$$H_1 - H_{1\rho} - \frac{1}{2m} p_r^2 \approx \frac{1}{2} m v^2(\rho) (5r_0)^2 \cos^2 \chi, \quad (A1)$$

where $(5r_0/\rho)^2$ is proportional to $(H_1 - H_{1\rho})$.

Let us also note that $(H_1 - H_{1\rho} - p_r^2/2m)$ is dependent only on r (fixed ρ) and so is exactly periodic with $r(t)$. Therefore, we may define the exact phase λ by

$$H_1 - H_{1\rho} - \frac{1}{2m} p_r^2 \equiv \frac{1}{2} m v^2(\rho) \epsilon^2 \cos^2 \lambda, \quad (A2)$$

where

$$\epsilon^2 \equiv \frac{2(H_1 - H_{1\rho})}{m v^2(\rho) \rho^2}, \quad (A3)$$

solving for r in terms of ρ and $(\epsilon \cos \lambda)$ gives

$$\frac{r}{r_0} = 1 + \epsilon \cos \lambda + A_2(\rho) \epsilon^2 \cos^2 \lambda + A_3(\rho) \epsilon^3 \cos^3 \lambda + \dots, \quad (A4)$$

with

$$A_2(\rho) = \frac{1}{3} \left[1 + \frac{2}{3} \frac{d^2 \ln v(\rho)}{d \ln \rho} \right], \quad (A5)$$

$$A_2(\rho) = \frac{1}{3} + \frac{25}{4} \left[\frac{\Omega(\rho)}{\nu(\rho)} \right]^2 + \frac{4}{3} \frac{d\ell_n \nu}{d\ell_n \rho} - \frac{2}{9} \left[\frac{d\ell_n \nu}{d\ell_n \rho} \right]^2 - \frac{1}{2} \frac{\rho^2}{\nu} \frac{d^2 \nu}{d\rho^2} \quad (A6)$$

If we also expand

$$t = t_c + \frac{\tau_n \lambda}{\pi} - \frac{1}{\nu(\rho)} \left[2 B_2(\rho) \varepsilon \sin \lambda + 3 B_3(\rho) \varepsilon^2 \sin(2\lambda) + \dots \right], \quad (A7)$$

then using equations (8), (A2) and (A4), we get

$$\varphi = \varphi_c + \frac{\ell_n \nu}{\pi} \lambda + \frac{\Omega(\rho)}{\nu(\rho)} \left[2 A_2(\rho) \varepsilon \sin \lambda + 3 A_3(\rho) \varepsilon^2 \sin(2\lambda) + \dots \right], \quad (A8)$$

where

$$B_2 = 1 - A_2, \quad B_3 = -A_3 + 2A_2 - \dots \quad (A9)$$

and $(2\pi_\nu)$, $(2\theta_\nu)$ are respectively the exact period Δt and angular excursion $\Delta\theta$ during one radial oscillation $\Delta\lambda = 2\pi$.

We find that

$$\tau_n(\rho, \varepsilon) = \frac{\pi}{\nu(\rho)} \left[1 - \frac{3}{2} B_3(\rho) \varepsilon^2 + \dots \right], \quad (A10)$$

$$\theta_\nu(\rho, \varepsilon) = \frac{\pi}{\nu(\rho)} \left[1 + \frac{3}{2} A_3(\rho) \varepsilon^2 + \dots \right]. \quad (A11)$$

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Figure 1. Transverse rosette orbit of a beam particle in anharmonic space charge force is indicated by solid and dashed curves. One radial oscillation period $\Delta t = 2\tau_p$ of the orbit as indicated by the solid part of curve; in this case the period begins and ends when the orbit radial position is r . During this radial oscillation period, the particle traverses the azimuthal angle $2\theta_v$.

Figure 2. Schematic illustration of the analyses of the transverse orbits (such as that of Fig. 1) into a rotation and betatron-vortex gyration of the ions in the cases where ion density n is monotone decreasing or increasing in magnitude. The smaller betatron oscillations are here viewed in the rotating frame of the reference orbit.