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OF THE INTERVAL INTO A FLOW

AUTHOR(S): J. A. Beyer and P. J. Channell

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Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

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**A FUNCTIONAL EQUATION FOR THE EMBEDDING OF
A HOMEOMORPHISM OF THE INTERVAL INTO A FLOW**

W. A. Beyer

and

P. J. Channell

**Los Alamos National Laboratory
Theoretical Division
Los Alamos, NM 87545**

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ABSTRACT

A functional equation $\phi(w(x)) = w'(x) \phi(x)$ for $\phi(x)$ is derived for the problem of finding ϕ so that for a given orientation-preserving everywhere-differentiable homeomorphism $w(x)$ of $[0,1]$ with $w' \neq 0$, there exists a solution $F(x,t)$ to $F_t(x,t) = \phi(F)$ so that $F(x,0) = x$, $F(x,1) = w(x)$. A solution to the functional equation is given for the case where w has a finite number of fixed points a_i with $w'(a_i) \neq 1$. The analogous equation in n -dimensional space is given.

1. INTRODUCTION

Suppose one has a homeomorphism $w(x)$ of $[0,1]$ which is everywhere differentiable and orientation-preserving with $w' \neq 0$ and has a finite number of fixed points $0 = a_0 < a_1 < a_2 < \dots < a_n = 1$. Further, $w'(a_i) \neq 1$. One wishes to embed this mapping into the following problem. Given $w(x)$, find $\phi(y)$ such that the solution $F(x,t)$ to the initial value problem

$$\frac{\partial F(x,t)}{\partial t} = \phi(F(x,t)) \quad , \quad (1.1)$$

$$F(x,0) = x \quad , \quad (1.2)$$

exists for $t \in [0,1]$, $x \in [0,1]$, and

$$F(x,1) = w(x) \quad . \quad (1.3)$$

We impose the conditions on ϕ that

$$(a) \quad \phi(a_i) = 0, \quad (1.4)$$

$$(b) \quad a_i \text{ are the only zeros of } \phi, \quad (1.5)$$

$$(c) \quad \phi(y) \text{ is continuous on } [0,1]. \quad (1.6)$$

This problem is relevant to the study of functional iteration since it permits iteration to be considered in terms of a dynamical system. A recent survey of the problem is given by Utz (1981). Our interest is in the application of a functional equation to construct a ϕ which solves the problem.

Remarks

- a. As Palis (1974) has shown, in a sense he makes precise, the above embedding problem is not "generically" solvable for compact C^∞ manifolds without boundary.
- b. The work of Zdun (1979) on the problem of embedding a continuous map of $[0,1]$ into an iteration semigroup is reviewed by Targonski (1981, 1984). Our hypotheses on ϕ puts ϕ into Zdun's class HE and thus ϕ must be embeddable in a continuous iteration semigroup. See Targonski, 1984, page 32.

In Section 2 we derive a functional equation for $\phi(y)$ and in Section 3 we discuss its solution. In Section 4 we give an example.

2. FUNCTIONAL EQUATION FOR $\phi(x)$

Since we assume that the a_i are the only zeros of ϕ and that ϕ is continuous, any solution $F(x,t)$ for x fixed and $x \neq a_i$ is either strictly increasing or strictly decreasing. See Fig. 1.

Fig. 1. $F(x, t)$ as a function of t for fixed x .

The function ϕ which we construct will satisfy a Lipschitz condition on any subinterval W of (a_i, a_{i+1}) . Thus, if the initial value x is interior to W and if the solution remains in W , the existence and uniqueness theory (see Coddington and Levinson (1955)) of ordinary differential equations applies. For $x = a_i$, we define the solution to be $F(a_i, t) = a_i$ for $t \geq 0$.

Theorem 1. Suppose $\phi(x)$ satisfies (1.4), (1.5) and (1.6) and that $F(x, t)$ defined by (1.1) and (1.2) also fulfills (1.3) with $\omega(x)$ satisfying the conditions given above (1.1). Then $\phi(x)$ satisfies the functional equation

$$\phi(\omega(x)) = \omega'(x)\phi(x) \quad (2.1)$$

Fig. 2. Positions of $F(x, t)$ on x -axis.

Proof. Consider an interval (a_i, a_{i+1}) for which $\omega(x) > x$. (The proof is similar in the contrary case $\omega(x) < x$.) For $t \in (0, 1)$ and $\Delta t > 0$ sufficiently small, consider the four points $F(x, t)$, $F(x, t + \Delta t)$, $\omega(F(x, t))$,

and $w(F(x, t+\Delta t))$. If $x \in (a_i, a_{i+1})$, then all four points are in (a_i, a_{i+1}) . If we assume that $\phi(x) > 0$ in (a_i, a_{i+1}) , then the four points are arranged as shown in Fig. 2. From the mean value theorem, since $F(x, t) \in C^1$, there exists $\theta_1 \in (0, 1)$, $i = 1, 2$ so that

$$F(x, t+\Delta t) - F(x, t) = \frac{\partial}{\partial t} F(x, t+\theta_1 \Delta t) \Delta t = \phi(F(x, t+\theta_1 \Delta t)) \Delta t, \quad (2.2)$$

and

$$w(F(x, t+\Delta t)) - w(F(x, t)) = \frac{\partial}{\partial t} w(F(x, t+\theta_2 \Delta t)) \Delta t = \phi(w(F(x, t+\theta_2 \Delta t))) \Delta t. \quad (2.3)$$

Solving (2.2) for $F(x, t+\Delta t)$ and substituting in (2.3), we obtain

$$w(F(x, t) + \phi(F(x, t+\theta_1 \Delta t)) \Delta t) - w(F(x, t)) = \phi(w(F(x, t+\theta_2 \Delta t))) \Delta t.$$

Hence, there exists θ_3 such that

$$w'(F(x, t) + \theta_3 \phi(F(x, t+\theta_1 \Delta t)) \Delta t) \phi(F(x, t+\theta_1 \Delta t)) \Delta t = \phi(w(F(x, t+\theta_2 \Delta t))) \Delta t.$$

On letting $\Delta t \rightarrow 0$ and replacing F by x , we obtain the functional equation of the theorem.

Remarks. The functional equation (1) can be generalized to n -dimensions as:

$$\phi(w(x)) = Dw(x)\phi(x) \quad (2.4)$$

where $x \in \mathbb{R}^n$, and $w, \phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and Dw is the Jacobian matrix. We will not give the proof here.

b. McKiernan (1963) obtained (2.1) in his work on convergence of a series of iterates. Let $w(z)$ be analytic with the r th iterate $w^{[r]}(z)$ in the domain of w for $r = 1, 2, \dots$. He then shows that the series

$$\phi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} w^{[r]}(z) \quad (2.2)$$

converges in some neighborhood of $z=0$ provided $w(0)=0$ and $w'(0)=\alpha$ for real $\alpha \in (0,1)$ and $\phi(z)$ satisfies $\phi(w(x)) = w'(x)\phi(z)$. It would be of interest to know if (2.2) solves the embedding problem.

c. Jabotinsky also derives an equation which reduces to (2.1). Targonski (1984) calls that equation Jabotinsky's third equation.

d. The functional equation (2.1) is of linear type and is discussed in Kuczma (1968), especially Chapters 2 and 4. If one takes logarithms of both sides of (2.1), one obtains a form of a linear functional equation discussed by Coifman and Kuczma (1969). Our derivation of a solution follows that in Kuczma (1968).

3. SOLUTION OF THE FUNCTIONAL EQUATION FOR ϕ

Again, we suppose x is in the interval (a_1, a_{i+1}) and $w(x) > x$ on the interval. (The proof similar in the contrary case that $w(x) < x$). We wish to find a solution to (2.1) on the interval $[0,1]$. Select $x_0 \in (a_1, a_{i+1})$ so that $w'(x_0) = 1$. This selection is possible by Rolle's theorem. Set

$$\phi(x) = w(x_0) - x_0, \quad (3.1)$$

for $x \in [x_0, w(x_0)]$. For $x \in (w(x_0), w^2(x_0))$, set

$$\phi(x) = w'(w^{-1}(x)) \phi(w^{-1}(x)) = w'(w^{-1}(x))(w(x_0) - x). \quad (3.2)$$

In general, for $x \in (\omega^n(x_0), \omega^{n+1}(x_0))$, $n=1,2,3,\dots$ set

$$\begin{aligned}\phi(x) &= \omega'(\omega^{-1}(x)) \phi(\omega^{-1}(x)) \\ &= \omega'(\omega^{-1}(x)) \omega'(\omega^{-2}(x)) \dots \omega'(\omega^{-n}(x)) (\omega(x_0) - x_0) \quad ,\end{aligned}\quad (3.3)$$

where $\omega^{-k}(x) = \omega^{-1}(\omega^{-(k-1)}(x))$ for $k>1$. For $x \in (\omega^{-1}(x_0), x_0)$, set

$$\phi(x) = \frac{1}{\omega'(x)} \phi(\omega(x)) = \frac{\omega(x_0) - x_0}{\omega'(x)} \quad . \quad (3.4)$$

In general, if $x \in [\omega^{-n}(x_0), \omega^{-n+1}(x_0))$, $n=0, 1, 2, \dots$ set

$$\phi(x) = \frac{\omega'(x_0) - x_0}{\omega'(x) \omega'(\omega(x)) \dots \omega'(\omega^{[n-1]}(x))} \quad . \quad (3.5)$$

Also, define $\phi(a_i) = \phi(a_{i+1}) = 0$. Note that $\lim_{n \rightarrow \infty} \omega^{-n}(x) = a_i$ and

$\lim_{n \rightarrow \infty} \omega^n(x) = a_{i+1}$ for $x \in (a_i, a_{i+1})$.

Lemma 1: ϕ is continuous in $(0,1)$.

The proof is a matter of checking the formulae at the interval end points.

Lemma 2: For $x \in (a_i, a_{i+1})$, $\int_x^{\omega(x)} \frac{dy}{\phi(y)} = 1$.

Proof. Case 1. Suppose $x \in (\omega^n(x_0), \omega^{n+1}(x_0)]$. ($n = 0, 1, 2, 3, \dots$)

Then

$$\int_x^{\omega(x)} \frac{dy}{\phi(y)} = \int_x^{\omega^{n+1}(x_0)} \frac{dy}{\phi(y)} + \int_{\omega^{n+1}(x_0)}^{\omega(x)} \frac{dy}{\phi(y)}$$

$$\begin{aligned}
&= \int_x^{w^{n+1}(x_0)} \frac{dy}{\phi(y)} + \int_{w^n(x_0)}^x \frac{w'(u)du}{\phi(w(u))} \\
&= \int_x^{w^{n+1}(x_0)} \frac{dy}{\phi(y)} + \int_{w^n(x_0)}^x \frac{du}{\phi(u)} \\
&= \int_{w^n(x_0)}^{w^{n+1}(x_0)} \frac{dy}{\phi(y)} = \dots = \int_{x_0}^{w(x_0)} \frac{dy}{\phi(y)} = 1.
\end{aligned}$$

Case 2. Suppose $x \in (w^{-(n+1)}(x_0), w^{-n}(x_0)]$. $n = 0, 1, 2, 3, \dots$

Then

$$\begin{aligned}
\int_x^{w(x)} \frac{dy}{\phi(y)} &= \int_x^{w^{-(n)}(x_0)} \frac{dy}{\phi(y)} + \int_{w^{-(n)}(x_0)}^{w(x)} \frac{dy}{\phi(y)} \\
&= \int_x^{w^{-(n)}(x_0)} \frac{dy}{\phi(y)} + \int_{w^{-(n+1)}(x_0)}^x \frac{w'(u)du}{\phi(w(u))} \\
&= \int_x^{w^{-(n)}(x_0)} \frac{dy}{\phi(y)} + \int_{w^{-(n+1)}(x_0)}^x \frac{du}{\phi(u)} = \int_{w^{-(n+1)}(x_0)}^{w^{-(n)}(x_0)} \frac{dy}{\phi(y)} \\
&= \int_{w^{-(n)}(x_0)}^{w^{-(n-1)}(x_0)} \frac{w'(u)du}{\phi(w(u))} = \int_{w^{-(n)}(x_0)}^{w^{-(n-1)}(x_0)} \frac{du}{\phi(u)}
\end{aligned}$$

$$= \dots = \int_{x_0}^{\omega(x_0)} \frac{du}{\phi(u)} = 1 \quad ,$$

which completes the proof of the lemma.

Theorem 2. If $\phi(x)$ is defined as above, then for $x \in (a_i, a_{i+1})$, the initial value problem (1) and (2) has the unique solution $F(x, t)$ which satisfies $F(x, 1) = \omega(x)$.

Proof. It follows from (1) and (2) that

$$\int_0^t \frac{1}{\phi(y(\tau))} \frac{dy(\tau)}{d\tau} d\tau = t \quad .$$

Now

$$\int_0^t \frac{1}{\phi(y(t))} \frac{dy(t)}{dt} dt = \int_x^{F(x, t)} \frac{dy}{\phi(y)} \quad .$$

Put $t = 1$. Then

$$1 = \int_x^{F(x, 1)} \frac{dy}{\phi(y)} \quad . \tag{3.6}$$

From Lemma 2,

$$1 = \int_x^{\omega(x)} \frac{dy}{\phi(y)} \quad , \tag{3.7}$$

so that, since $\phi(y) > 0$, comparing (3.6) and (3.7), we have

$$F(x, 1) = \omega(x) \quad .$$

4. Example with $w(x) = \sqrt{x}$.

It is useful to present an example illustrating the solution to the functional equation (2.1). Select $w(x) = \sqrt{x}$. Then $a_0 = 0$ and $a_1 = 1$. Eq. (2.1) becomes

$$\phi(\sqrt{x}) = \frac{1}{2\sqrt{x}} \phi(x) .$$

The value x_0 is determined by

$$w'(x_0) = 1 ,$$

so that $x_0 = \frac{1}{4}$. Also $w(x_0) = \frac{1}{2}$. Hence, by (3.1)

$$\phi(x) = \frac{1}{2} \quad \frac{1}{4} \leq x < \frac{1}{2} .$$

In general, for

$$x \in (w^{[n]}(x_0), w^{[n+1]}(x_0)) ,$$

we have

$$\phi(x) = \frac{1}{2^{n+2}} \frac{1}{x^{2^{n-1}}} , \quad -\infty < n < \infty ,$$

by (3.3) and (3.5).

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