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COMPARISON OF TWO SLIDELINE METHODS USING ADINA

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Abstract - Two general methods of treating sliding interface problems were implemented in ADINA. The first method uses constraint equations and is based on the work of Taylor, Hughes, et al. The second method is a penalty function method where finite stiffnesses are used to simulate the contact forces. This is similar to the method used by Hallquist in NIKE2D. These may be used as no-tension sliding interfaces, with or without friction, or as a non-sliding interface to interpolate between different mesh regions. We briefly review the theory behind these methods and describe the implementation in ADINA. Several application problems have been run, including metal forming problems. Based on this experience the methods are compared in terms of ease of use, accuracy, reliability, influence on bandwidth, and solution time. Neither method is ideal. The penalty method is generally more efficient but may be inaccurate, whereas the constraint equations are accurate but less efficient.

Introduction

Slidelines have a long history of use in explicit finite difference wave propagation programs, to model material discontinuities, and as a mesh grading technique. Their use in more conventional structural analysis has been much more limited, basically because reliable algorithms for slidelines have not been implemented in widely used finite element programs. This lack of slidelines in structural programs results from the inherent difficulty of providing this feature, not from its unimportance.

The concept of slidelines, that is, material surfaces that can slide with respect to one another, perhaps with friction, is important to many classes of problems. For example, metal forming processes, impact problems, mechanism design, as well as soil structure and geologic problems. Some problems that are restricted to small relative motions, or in which one surface is rigid, can be solved using simple displacement constraints or truss type elements. Examples of these simpler problems are bolted flange joints in pipes or the squeezing of a body between flat rigid platens. Nevertheless, a general-purpose slideline element simplifies the treatment of these problems and allows for many more general solutions as well.

To study some of the practical problems associated with using slidelines in structural problems we have implemented two different published algorithms in the general purpose finite element program ADINA. The first method is a Lagrange multiplier formulation similar to that of Taylor, Hughes and Sackman [3], while the second method is a penalty formulation as implemented by Hallquist [8]. Some modifications were required to incorporate the algorithms, and additional changes were made in the course of their usage.

We will first review the theory behind the methods, then show several examples of their use. Some examples are contrived, but most are actual applications. Finally, we will review the results and compare the methods from the standpoint of efficiency, reliability, and ease of use.

Lagrange Multiplier Method

The basic idea of the Lagrange Multiplier formulation is the addition of a term of the form

$$\int_c \underline{t} \cdot (\underline{x}^1 - \underline{x}^2) \, dc, \quad (1)$$

to a standard variational formulation for two independent bodies (see [3] for further details). The surface c is the contact region between the deformed bodies, b^1 and b^2 , of the bodies B^1 and B^2 , as shown in Fig. 1. The enforcement of compatibility along the surface c will then be achieved by taking independent variations of $\underline{t}$. ($\underline{x}^a$ are the deformed coordinates of material points in body a with points $\underline{x}^1$ being slave nodes and $\underline{x}^2$ master nodes, and $\underline{t}$ is interpreted as the Cauchy traction vector across the contact surface.)

Assume that B^1 and B^2 are discretized with bilinear elements (See Fig. 2.) and assume $\underline{t}$ consists of Dirac delta functions located at nodal points. Then defining the nodal force components $t_{ijk} = t_{ij} n_j(\underline{x}_k)$, (1) becomes

$$\sum_{j=1}^N [t_{1j} (\underline{x}_j^1 - \underline{x}_j^2) + t_{2j} (\underline{y}_j^1 - \underline{y}_j^2)] \quad (2)$$

where N is the number of slave nodes in B^1 with coordinates $(\underline{x}_j^1, \underline{y}_j^1)$, and $(\underline{x}_j^2, \underline{y}_j^2)$ in B^2 are defined by

$$x_j^2 = N_1^{\ell-1}(a) x_{\ell-1}^2 + N_2^{\ell}(a) x_{\ell}^2$$

$$y_j^2 = N_1^{\ell-1}(a) y_{\ell-1}^2 + N_2^{\ell}(a) y_{\ell}^2$$

where $(x_{\ell-1}^2, y_{\ell-1}^2)$, (x_{ℓ}^2, y_{ℓ}^2) are master node coordinates,

$N_1^{\ell-1}(a)$ and $N_2^{\ell}(a)$ are the linear shape functions

$$N_1^{\ell-1}(a) = \frac{(1-a)}{2} \quad N_2^{\ell}(a) = \frac{(1+a)}{2} \quad (3)$$

and assuming that (x_c, y_c) is the contact or penetration point of k on $\ell-1, \ell$,

$$a = \frac{2 \cdot [(x_c - x_{\ell-1})^2 + (y_c - y_{\ell-1})^2]^{1/2}}{[(x_{\ell} - x_{\ell-1})^2 + (y_{\ell} - y_{\ell-1})^2]^{1/2}} - 1$$

The t_{ij} are interpreted as nodal contact forces and are unknowns to be added to the list of other nodal variables, i.e., displacements.

For friction, (1) and (2) are cast in normal and tangential forms. Reference [1] presents a simple Coulomb friction derivation for the Lagrange multiplier formulation. In [6], a more sophisticated Coulomb friction is presented for unilateral contact.

Penalty Function Method

The basic idea of this formulation is the addition to the variational equation of a term of the form

$$\epsilon^{-1} \int_C (\underline{v}_n - \underline{g}_0) \underline{v}_n \, dc \quad (4)$$

where ϵ^{-1} is the penalty multiplier, $\underline{v}_n$ is the normal displacement on the boundary C , and $\underline{g}_0$ is the initial gap [10]. Another way of viewing (4) is the insertion of a linear interface spring into the stiffness matrix that couples the penetrating node to two adjacent nodes on the contact surface [7].

Oden [9] takes ϵ^{-1} as the foundation or master surface modulus, whereas Hallquist [7] has found that a contact area weighted bulk modulus is more robust especially when there is plastic deformation of the master surface.

For friction problems, Hallquist [7] uses a tangential spring proportional to the normal springs and the relative tangential motion of the contactor node. In [11], on the other hand, the frictionless problem solution is used as a first approximation of the normal contact pressure, which is subsequently used as data on a second problem with the friction constraint.

Implementation in ADINA

The implementation of the Lagrange multiplier or the penalty function algorithm in ADINA consists primarily of the development of code which finds the point of penetration of a slave node through the master surface and the subsequent computation of a stiffness and an effective load vector. In addition, because contact surfaces will be changing at each iteration, a full Newton-Raphson equilibrium iteration was also implemented in ADINA.

To determine if a slave node has contacted or penetrated the master surface, we first look at the sign of the area of the triangle formed by the

slave node and two master nodes. Let nodes $(l-2, l-1, l, l+1)$ define the master surface and nodes $(k-1, k, k+1)$ be slave nodes. See Fig. 3. (For this algorithm and those in [4] and [8], it is important that the master nodes be defined such that the interior of the master surface be to the right as the nodes come up.) To see if the slave node is interior to the master surface, we compute the area of the triangle $(k, l, l-1)$ (slave, 2nd master node, 1st master node). If the area is positive (e.g., $(k, l, l-1)$ are counterclockwise), then there is a possible penetration. If the area is negative, such as for $(k+1, l-1, l-2)$ (e.g., $(k+1, l-1, l-2)$ are clockwise), then the slave node is exterior to this segment of the master surface.

For slave nodes that pass the area test, two schemes are used to find the contact or penetration point. If the slave node was not in contact for the preceding step or iteration, then the trajectory of the slave node is checked for an intercept with the master surface. If the slave node was in contact or if it failed the intersection test for all master surface segments, then the slave node location is projected onto the nearest master surface segment. If the projected location is outside the range of the master surface then the slave node is released.

Once the contact location of the slave node is known, the incremental equations for the Lagrange multiplier are (see [1], [3], or [13] for more details.):

$$\underline{K}_n \Delta \underline{v}_n^{(i)} = \underline{p}_n - \underline{F}_n \quad (5)$$

where $\Delta \underline{v}_n$ is the incremental displacement vector

$$\Delta u_s^{(i)t} = [\Delta u_k^{(i)}, \Delta u_k^{(i)}, \Delta u_{l-1}^{(i)}, \Delta u_{l-1}^{(i)}, \Delta u_l^{(i)}, \Delta u_l^{(i)}, \Delta t_{1k}^{(i)}, \Delta t_{2k}^{(i)}]^t, \quad (6)$$

$\underline{p}_s$ is the external force vector arising from the internal state of stress in the interface elements, $\underline{K}_s$ is an 8 x 8 stiffness matrix that is assembled into the global stiffness matrix, and $\underline{F}_s$ is a force vector that adds into the out-of-balance loads. Here $\underline{K}_s$ and $\underline{F}_s$ depend on whether the k^{th} slave node is in contact on segment $l-1$, l and the contact constraint (Fig. 3). Appendix A gives the forms that $\underline{K}_s$ and $\underline{F}_s$ take for various conditions.

For the penalty function implementation, the algorithms used in NIKE2D [8] were used with slight modifications for ADINA (see references [8] and [9] for more details on the development, implementation, and restrictions of the algorithm). A contact stiffness and a force vector similar to those in (5) are computed for slave nodes which are found in contact. Although NIKE2D makes the slideline symmetric by considering the master surface nodes as slave nodes to the slave surface automatically, in the ADINA implementation we left it up to the user to define a new slideline with the same result. Because there are no contact forces, the force vector has only 6 elements and the stiffness is a 6 x 6 matrix, i.e.,

$$\Delta u_n^{(i)t} = [\Delta u_k^{(i)}, \Delta u_k^{(i)}, \Delta u_{l-1}^{(i)}, \Delta u_{l-1}^{(i)}, \Delta u_l^{(i)}, \Delta u_l^{(i)}]^t \quad (7)$$

Appendix B gives the form $\underline{K}_n$ and $\underline{F}_n$ take for the penalty function slideline implemented.

EXAMPLE PROBLEMS

Example 1: Ring Sliding on Cylinder.

Figure 4 shows the mesh for this problem, the location of the slidelines between the slider on the cylinder, and the ring on the slider. A radial pressure was applied as shown on the ring and a prescribed displacement was imposed on the slider. Figures 5a and 5b show von Mises contour plots at full radial pressure of 30,000 psi for the Lagrange multiplier and penalty function slidelines, respectively. The slight but noticeable difference in the contours is caused by the different effective stiffnesses of the two methods. Figure 6 is a time history of the radial stress of node 166 for the two methods showing the smoother response of the penalty method, as one would expect from its lower effective stiffness. Figure 7, showing considerable chatter, is a history plot of the radial contact force of the nodes in the vicinity of node 166 for the Lagrange multiplier slidelines. For this problem, the nodes at the top of the slider lift off the cylinder at an earlier time for the Lagrange multiplier.

Example 2: Concentric Spheres With Internal Pressure.

Figures 8a and 8b show the meshes for two concentric spheres separated by a small gap. Internal pressure on the inner sphere causes plastic deformation of the inner shell and subsequent loading of the outer shell. This problem shows very interesting effects of small discrepancies in the initial mesh. Figure 9 shows the motion of the slave (node 5) and master nodes (nodes 277) for mesh 1. They never contact because the slideline is curved in this

problem, and the mesh grading is not uniform on the two surfaces. Figure 9 shows stress concentration points of contact for mesh 1. Figure 10a shows the motion for the same nodes in mesh 2. The implementation of the penalty function inserts a spring when the slave node is "close" to the master surface and thus gives an early contact and late release as seen in Fig. 10b.

Example 3: Superplastic Sheet Metal Forming Problem.

In this problem a uniform pressure is applied over the top surface of a sheet of metal supported above a roughly hemispherical cavity. Fig. 11 shows the arrangement and initial mesh. In the actual problem the metal is heated to a carefully chosen temperature so that the superplastic effect obtained with some metals will allow the sheet to be deformed into the cavity in a reasonable period of time, without instabilities resulting in gross thickness variations or rupturing of the sheet.

Solving this problem with ADINA required not only the slideline modifications but also large deformation pressure and large deformation creep modifications. To date, we have solved this problem with assumed properties for the sheet metal simply to demonstrate the ability of the program to solve this type of problem. Although this might not be realistic for any particular metal it does make an excellent test of the slideline routines.

In our initial runs on this problem we had considerable trouble with spurious penetration of nodes. This resulted in an apparent necking when the slideline algorithm noticed the penetrated node and pushed it back out to the surface. This penetration was traced to the value used for the repenetration tolerance in the slideline. Originally we assumed this might have been caused

by a material instability. Only through a very detailed examination of the deformation process did we finally identify the actual error.

Given appropriate constants for the slideline parameters and the material properties for the sheet, the problem is handled easily, as shown in Fig 12.

Example 4: Rolling Diaphragm.

This problem is related to creating an environmental seal between a plunger and a vessel. This is done by using a thin metal membrane to connect the two parts, thus requiring the membrane to roll back on itself. Figures 13 and 14 show the initial configuration and a series of deformed shapes as the plunger is moved downward by a prescribed displacement. The slidelines are required to confine the membrane between the tube and the hole in the vessel. In plane geometry this problem would be fairly simple, but in axisymmetric geometry the selection of the roll radii and thickness of the membrane is difficult because of the geometric constraints.

The combination of material instabilities and slideline behavior has made this a most difficult problem for us to solve.

SUMMARY AND CONCLUSIONS

In comparing the two slideline methods, we found that both methods had advantages and disadvantages which, depending on the particular problem considered, could make either one look good or bad. For example, the Lagrange multiplier method satisfies the contact constraint more precisely and there is no need to determine or input a "spring constant." On the other hand, it

adds extra degrees of freedom requiring additional storage and computation. In addition, the Lagrange multiplier method is not symmetric, because a slave node cannot contact two master surfaces at the same time; neither can a slave node simultaneously be a master node. As a result, when using the Lagrange multiplier method, intersecting slidelines are not always possible.

The penalty function slideline has the advantage that no extra degrees of freedom are introduced, and thus no extra storage requirement and fewer computations. Slidelines can intersect, and for a given problem a solution is assured with sufficiently soft springs. On the other hand, the analyst may have to solve the problem several times to get a good solution, because of required adjustments to the spring constants to minimize unacceptable penetrations of slave nodes. In addition, the solution may converge very slowly or not at all if the stiffness is as high as desired.

In our implementation of the two methods we found that two areas were very significant in the results of the methods. They were the check for contact and/or penetration and the check for release of the node. We found that slight modification of tolerances for these checks could mean good results for one problem and bad for another. For example, a positive tension cutoff for the Lagrange multiplier on the slider problem reduced slave node chattering on the cylinder. However, a similar tension cutoff on the spheres problem meant a delayed release. It is important to provide some hysteresis in release tests in order to obtain convergence in many problems. We have also observed a wide variation in the details of these algorithms by various authors, and at times we have used several different variations.

Finally, we found that no one method could work all of the problems to our satisfaction. Even though the penalty function method was faster on a per run basis, iterating on the problem several times to get the proper

stiffness negated its efficiency. Interestingly enough, we also found that the two methods implemented in ADINA could work well together. In most of the cases where either method had difficulties, we found that the algorithm implementation was the cause of the problem.

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APPENDIX A

Lagrange Multiplier Contact Stiffness and Out-of-Balance Force Contribution Matrices

In the subsequent equations we will consider only slave node k and master surface segments $\ell-1$, ℓ and ℓ , $\ell+1$ and the "(i)" iteration superscript will be omitted for the sake of clarity. That is, all nodal coordinates have been updated to the current configuration. In addition, we let $N_1 = N_1^{\ell-1}(a(i))$ and $N_2 = N_2(a(i))$ (see equation (4)).

1. No Contact Condition

When there is no contact, then $\underline{F}_a$ and $\underline{K}_a$ are

$$\underline{F}_a^T = [0, 0, 0, 0, 0, 0, 0, 0], \text{ and}$$

$$\underline{K}_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 1 & 0 \\ \text{symmetric} & & & & & & & 1 \end{bmatrix} \quad (A.1)$$

2. Stick Condition

For a stick or no slip condition,

$$\underline{F}_a^T = [t_{1k}, t_{2k}, -N_1 t_{1k}, -N_2 t_{1k}, -N_2 t_{2k},$$

$$x_k - (N_1 x_{\ell-1} + N_2 x_\ell), y_k - (N_1 y_{\ell-1} + N_2 y_\ell)]$$

$$\frac{K}{s} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 & 0 & N_1 & 0 \\ & & & 0 & 0 & 0 & 0 & N_1 \\ & & & & 0 & 0 & N_2 & 0 \\ \text{symmetric} & & & & & 0 & 0 & N_2 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} \quad (A.2)$$

3. Friction

For the friction or slip constraint let θ denote the angle between segment $l-1, l$ and the x -axis, measured counterclockwise from x (Fig. 3), then

$$\theta = \arctan \frac{(y_l - y_{l-1})}{(x_l - x_{l-1})} \quad (A.3)$$

Let $c = \cos \theta$, $s = \sin \theta$, and then the tangential force is

$$q_k = ct_{1k} + st_{2k}$$

and the normal force is

$$r_k = -sat_{1k} + ct_{2k} \quad (A.4)$$

The normal force r_k is subsequently checked at each iteration for tension or a tension cut-off, i.e.,

$$r_k = T_n \quad \text{where } T_n \geq 0.$$

The tension cut-off was implemented because it helped the stability of some problems and in others allowed specification of "bonding" strength.

For a slip or sliding condition, the tangential force q_k is set to zero. For friction, it

$$|q_k| > \mu_s |r_k|, \quad (A.6)$$

where μ_s is a static coefficient of friction, then q_k is recomputed as

$$q_k^* = \mu_d \operatorname{sign}(q_k) \cdot |r_k|$$

where μ_d is the dynamic coefficient of friction. If (A.6) is not true, then the stick condition (A.2) is imposed.

If the slave node is sliding with friction (e.g., use q_k^* for tangential force) or without (set $q_k^* = 0$) then the force vector and stiffness matrix are

$$\underline{F}_n^T = [f_1, f_2, -N_1 f_1, -N_1 f_2, -N_2 f_1, -N_2 f_2, -sg_n, cg_n]$$

where

$$f_1 = cq_k + sr_k, \quad f_2 = -sq_k + cr_k \quad \text{and}$$

$$sg_n = [-sx_k + cy_k - N_1 (-sx_{k-1} + cy_{k-1}) - N_2 (-sx_k + cy_k)] \quad (A.7)$$

$$K_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & s^2 & -sc \\ & 0 & 0 & 0 & 0 & 0 & -sc & c^2 \\ & & 0 & 0 & 0 & 0 & -n^2 N_1 & nc N_1 \\ & & & 0 & 0 & 0 & nc N_1 & c^2 N_1 \\ & & & & 0 & 0 & -n^2 N_2 & nc N_2 \\ & & & & & 0 & nc N_2 & -c^2 N_2 \\ & & \text{Symmetric} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

where $n^2 = \sin^2 \theta$, $c^2 = \cos^2 \theta$, $nc = \sin \theta \cos \theta$.

APPENDIX B

Penalty Function Contact Stiffness and Out-of-Balance Force Contribution Matrices.

Using the notation in [8] and letting

$$a = \frac{[(x_c - x_{l-1})^2 + (y_c - y_{l-1})^2]^{1/2}}{[(x_l - x_{l-1})^2 + (y_l - y_{l-1})^2]^{1/2}} \quad (B.1)$$

where (x_c, y_c) is the contact point c between segment $l-1, l$ (see Fig. 3), the stiffness matrix is

$$K_B = \begin{bmatrix} k s^2 & -k s c & -(1-a)k s^2 & (1-a)k s c & -k a s^2 & k a s c \\ & k c^2 & (1-a)k s c & -(1-a)k c^2 & k a s c & -k a c^2 \\ & & (1-a)^2 k s^2 & -(1-a)^2 k s c & (1-a)k a s^2 & -(1-a)k a s c \\ & & & (1-a)^2 k c^2 & -(1-a)k a s c & (1-a)k a c^2 \\ & & & & k a^2 c^2 & -k a^2 s c \\ & & & & & k a^2 c^2 \end{bmatrix} \quad (B.2)$$

where k is a spring constant based on the contact area and material properties of the master surface, a is given by (B.1), and $c = \cos\theta$ and $s = \sin\theta$ where θ is defined by (A.3). In addition, the capability to scale k for each slave node independently was also provided in the ADINA implementation. The force vector F_B is

$$\underline{F}_g = \kappa \delta \begin{bmatrix} -s \\ c \\ (1-a)s \\ -(1-a)c \\ as \\ -ac \end{bmatrix} \quad (B.3)$$

where δ measures the penetration of node k through segment $l-1, l$ and is assumed to be negative. Let $s_l = .001 \cdot (\text{segment length})$, then if a slave node is within s_l of the master segment $l-1, l$, then K_g is inserted in the global stiffness with $\underline{F}_g$ equal to zero.

The types of constraints available are: sliding only, tied, sliding with voids, and friction sliding with voids. For sliding only, initial gaps in the slideline are closed and at each iteration the normal penetration of the slave node k and segment $l, l+1$ is used for δ . For the tied condition two passes are made. The first pass uses the relative vertical displacement difference of the slave node and the contact point for δ with $c=1$ and $a=0$. The second pass uses the horizontal displacement difference for δ , and $c=0$ and $a=1$. For the sliding with voids, if the normal displacement of the slave node with respect to the master surface is positive or greater than a given tolerance, the node is released, i.e., no stiffness or force vectors are added to the global matrices. If there is penetration, $\delta < 0$, then (B.2) and (B.3) are used. For slave nodes in contact with friction, a pass is first made with δ being the normal penetration and followed by one, where $c = \sin \theta_i$, $a = \cos \theta_i$, and $K = .01 \cdot k$ and the tangential force K_g depends on the current tangential displacement δ_t , a friction constant and the

slave node's previous tangential force.

Finally, the convectivity array for each slave node must be updated at each iteration for both types of slidelines.

Fig. 1. Contacting Bodies. The original bodies B^i deform into b^i with a part of their boundaries forming a contact line c .

Fig. 2. Typical configuration for slave node x_k^1 and target segment reference point x^1 .

Fig. 3. Slave surface penetrating the master surface at point c.

Fig. 4. Finite element mesh for ring sliding on a cylinder.

(a) Lagrange Multiplier slideline

(b) Penalty function slideline

Fig. 5. Von Mises contour plot at full radial pressure

Fig. 6. Comparison of radial stress at node 166 for penalty function and Lagrange multiplier formulation shows smoother behavior of the penalty method.

Fig. 7. Time history of force at nodes adjacent to 166 shows considerable chatter.

(a) Mesh a

(b) Mesh b

Fig. 8. Two concentric spheres separated by a small gap

(a) Lagrange Multiplier slideline

(b) Penalty function slideline

Fig. 9. Nodal histories of slave node 5 and master node 277 for mesh 1. The lack of contact is caused by the curved interface and lack of alignment at the nodes across the interface.

(a) Lagrange Multiplier slideline

(b) Penalty function slideline

Fig. 10. Nodal histories of slave node 5 and master node 7 for mesh 2, with the nodes aligned opposite one another.

Fig. 11. Finite element mesh for Example Problem A.

(a)

(b)

(c)

(d)

Fig. 12. Deformed plots of superplastic sheet metal forming problem.

Fig. 13. Initial configuration and mesh for rolling diaphragm problem. The inner and outer guide tubes are elastic elements, and are very stiff compared to the diaphragm.

(a)

(b)

(c)

(d)

Fig. 14. Sequence of deformed mesh plots shows formation of initial roll and contact with both inner and outer guide tubes.