AN ALTERNATIVE APPROACH TO DISTURBANCES
IN BOUNDARY LAYERS

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ABSTRACT

By modelling the boundary layer on a flat plate as a piece-wise linear velocity profile it is possible to analyze disturbances in the flow in a systematic manner. The approach is that of an initial-value, boundary-value problem but, unlike classical normal modes employed in stability theory, the solutions here can be obtained in closed form and they are non separable. A specific example is treated where the vertical component of the perturbation velocity is prescribed as a localized pulse initially. The amplitude is then depicted as a function of time and the coordinates of the plane of the flat plate at a fixed vertical location. The role of three-dimensionality and the initial transient period of development - heretofore unknown - are both shown to be of significant importance in the dynamics. An argument is given to strongly suggest that the origin of large-scale oscillations known to be prevalent in turbulent shear flows may well be explained in these bases.

1. Introduction

Formally, the means of investigating perturbations in any shear flow is as an initial-value, boundary-value problem, but even the linear system does not lend itself to a tractable framework. The classical method of solution has been by use of a normal mode decomposition. This approach results in a non self-adjoint differential equation that also requires singular perturbation techniques in boundary layers (cf. Lin 1955, Betchov and Criminale 1967, Drazin and Reid 1981). Finally, the eigenvalues that are found are those of a discrete spectrum, thereby hardly sufficient to represent an arbitrary disturbance. There is also a continuous spectrum for this flow and this addition makes the problem complete. The work of Mack (1976) has made perhaps the most notable contribution dealing with this topic for the case of the boundary layer.

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Certainly, using the Laplace integral transform can be done with this problem and both eigen spectra are to be expected. As demonstrated by Gustavsson (1986), the difficulty of the inversion adds to the ones already cited. Essentially, only implications, i.e. the substantiation of the continuous spectrum, can be ascertained by this route.

The boundary layer has been the center of most interest with respect to initial-value problems. This is because experiments have been abundant and, as a prototype flow, this case is of major interest. Three works serve to illustrate what can be achieved using the conventional approach. First, Criminale and Kovasznay (1962) and later, Gaster and Grant (1975), and Gaster (1975) all made attempts to track an initial pulse in the boundary layer (specified in terms of the normal components of the disturbance velocity). Although some allowances were made for change in the Reynolds number, the results obtained were essentially those of one normal mode. Just as normal modes are best suited for asymptotic analysis ($t \rightarrow \infty$) and determination of stability, the ensuing wave packet mirrors what can be determined from such limited data. It is well known that, during the early transient period, the motion can be governed by far different modes, including all the damped normal modes. The existence of a continuous spectrum makes this part of the dynamics even more lacking since it has never been incorporated. In short, the early picture is left unknown.

The more recent work of Criminale and Drazin (1990) has established an alternative means of analysis for initial-value problems. If mean flows are modelled by a piece-wise linear variation, then it is possible to obtain closed form solutions for the respective dependent variables. These solutions contain all of the information of both the discrete and continuous spectra (when such features are possible), making it possible to represent any arbitrary disturbance. It is important to note that the solutions are non separable and hence a single normal mode cannot be a solution of the initial-value problem.

Criminale and Drazin examined several prototypical parallel shear flows under inviscid conditions. In each case, the temporal behaviour was determined for initial conditions where either the velocity (vertical component) or one component of the vorticity was given. This is a sizeable advantage. Moreover, the technique avoids the difficulty of the singular perturbation calculation and even the critical layer so manifest in normal modes. It is clear from the results that structure can play a predominant role in the early period, even driving the perturbations to nonlinearity before any exponentially growing solution will reach critical values. This point was well demonstrated by these authors.

The time dependence of the disturbance variables is akin to what has been found from rapid distortion theory in turbulence (cf. Townsend 1976) but adjusted due to

the fact that boundary conditions had to be met as well. It is known that, even in these circumstances, the turbulence response problem is linear for strong rates of shear (cf. Lee and Hunt 1990). As a result, for large Reynolds number, the laminar and turbulent wall regions for boundary layers have much in common. Large scale oscillations (small wave numbers) and their origins play central roles for both stability theory (these scales are unstable) and in turbulence dynamics. Understanding the origins is at least conceptually possible using this approach, as was suggested by Criminale (1987).

The work presented here is a fully viscous development and follows Easthope (1984), but is complimentary to Criminale and Drazin (1990). Lee and Hunt (1990) also considered the boundary layer by rapid distortion theory but were much more interested in effects on turbulent energy spectra than intimate details of the flow variables. It will be seen that large amplitude growth is possible in the early period, three-dimensionality is crucial to the development, and overall bodes well for a signature of streakiness.

2. Problem Statement

The problem to be treated is that of linearized disturbances in the boundary layer in a flat plate. The mean flow is taken in the x direction and depends on y only, the vertical coordinate; z is the transverse axis.

The non-dimensional linearized equations for small disturbances can be written as

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{dU}{dy} v = - \frac{\partial p}{\partial x} + \varepsilon^2 \nabla^2 u \quad (2.1)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = - \frac{\partial p}{\partial y} + \varepsilon^2 \nabla^2 v \quad (2.2)$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = - \frac{\partial p}{\partial z} + \varepsilon^2 \nabla^2 w \quad (2.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.4)$$

where $\varepsilon^2 = 1/Re$, Re being the Reynolds number, ∇^2 is the Laplacian operator, and u, v, w and p are the velocity and pressure perturbations respectively.

In order to attempt an analytic solution to these equations, the mean flow $U(y)$ is approximated by a piece-wise linear profile:

$$U(y) = \begin{cases} \sigma y & y < H \\ 1 & y \geq H \text{ so that } \sigma H = 1, \end{cases}$$

and H can be regarded as the boundary layer thickness.

An initial disturbance is taken as

$$u_i(x, y, z, 0) = Q_i \delta(x - x_0) \delta(y - y_0) \delta(z - z_0),$$

where u_i stands for u, v, w .

The boundary conditions are these of impermeability and non-slip on the plate itself:

$$u = v = w = 0 \quad \text{on} \quad y = 0.$$

All components must also vanish as $y \rightarrow \infty$.

It is now assumed that Fourier transforms may be taken in the x and z directions, which implies that the disturbances are bounded both downstream and transversally.

Therefore define:

$$\hat{u}_i(\alpha; y; \gamma; t) = \iint_{-\infty}^{\infty} e^{i\alpha x} e^{i\gamma z} u_i(x, y, z, t) dx dz$$

as the transformed velocity components.

Taking the transforms of equations (2.1) to (2.4), and combining into a single equation in $\hat{v}$, we have:

$$\begin{aligned} \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right] \frac{\partial \hat{v}}{\partial t} - i\alpha U(y) \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right] \hat{v} \\ + i\alpha \frac{d^2 U}{dy^2} \hat{v} = \varepsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right]^2 \hat{v} \end{aligned} \quad (2.5)$$

$$\left. \begin{aligned} \text{with } \hat{v}(\alpha; y; \gamma, 0) &= q \delta(y - y_0) \\ \hat{v}(\alpha; 0; \gamma, t) &= \frac{\partial \hat{v}}{\partial y}(\alpha, 0, \gamma, t) = 0 \\ \text{and } \hat{v}, \frac{\partial \hat{v}}{\partial y} &\rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \right\} \quad (2.6)$$

with $q = \frac{Q_2}{2\pi} e^{i\alpha x_0} e^{i\gamma z_0}$, and $\tilde{a}^2 = \alpha^2 + \gamma^2$.

The form of $U(y)$ suggests that solutions to (2.5) may conveniently be found by splitting the y domain into two zones; above and below $y = H$. Letting the superscripts (0) and (i) represent outer and inner solutions respectively, (2.5) desolves into:

$$\begin{aligned} \left[\frac{\partial}{\partial t} - i\alpha \sigma y \right] \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right] \hat{v}^{(i)} + 2i\alpha \sigma \frac{\partial \hat{v}^{(i)}}{\partial y} \\ = \varepsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right]^2 \hat{v}^{(i)} \end{aligned} \quad (2.7)$$

$$\begin{aligned} \left[\frac{\partial}{\partial t} - i\alpha \right] \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right] \hat{v}^{(0)} \\ = \varepsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{a}^2 \right]^2 \hat{v}^{(0)} \end{aligned} \quad (2.8)$$

The d^2U/dy^2 term has now vanished from these equations, being of the form:

$$\frac{d^2U}{dy^2} = -\sigma\delta(y - H),$$

but will influence the matching conditions at $y = H$.

In fact, $\hat{v}$, $\frac{\partial \hat{v}}{\partial y}$ and $\frac{\partial^2 \hat{v}}{\partial y^2}$ can be shown to be continuous across $y = H$, but

$$\left. \frac{\partial^3 \hat{v}}{\partial y^3} \right|_{H^-}^{H^+} = -\frac{i\alpha\sigma}{\epsilon^2} \hat{v}(H, t) \quad (2.9)$$

The main domain of interest for this paper is the inner layer; the solution for $\hat{v}^{(0)}$ is of lesser importance. It would therefore be useful to avoid solving for both $\hat{v}^{(1)}$ and $\hat{v}^{(0)}$ and carrying out the matching across $y = H$.

Equation (2.7) and (2.8) in fact contain information on the behaviour of $\hat{v}^{(1)}$ toward $y \rightarrow H$, and this can be seen as follows.

Write equation (2.7) in the form:

$$\begin{aligned} & \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \left[\frac{\partial \hat{v}^{(1)}}{\partial t} - i\alpha\sigma y \hat{v}^{(1)} \right] + 2i\alpha\sigma \frac{\partial \hat{v}^{(1)}}{\partial y} \\ & = \epsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right]^2 \hat{v}^{(1)}. \end{aligned}$$

This can be recast as

$$\begin{aligned} & \left[\frac{\partial}{\partial y} - \tilde{\alpha} \right] \left\{ \left[\frac{\partial}{\partial y} + \tilde{\alpha} \right] \left[\frac{\partial \hat{v}^{(1)}}{\partial t} - i\alpha\sigma y \hat{v}^{(1)} - \epsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \hat{v}^{(1)} \right] \right\} \\ & = -2i\alpha\sigma \frac{\partial \hat{v}^{(1)}}{\partial y}. \end{aligned}$$

Regarding, for the moment, the right-hand-side in the nature of a forcing function, this can be integrated once:

$$\begin{aligned} & \left[\frac{\partial}{\partial y} + \tilde{\alpha} \right] \left\{ \frac{\partial \hat{v}^{(1)}}{\partial t} - i\alpha\sigma y \hat{v}^{(1)} - \epsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \hat{v}^{(1)} \right\} \\ & = E(t)e^{\tilde{\alpha}y} + \int_y^H 2i\alpha\sigma e^{\tilde{\alpha}(y-\theta)} \frac{\partial \hat{v}^{(1)}}{\partial \theta}(\theta, t) d\theta \quad (2.10) \end{aligned}$$

and $E(t)$ is an arbitrary function of time, and the fixed limit on the integral can be taken without loss of generality.

If exactly the same operation is now carried out on (2.7), we have:

$$\left[\frac{\partial}{\partial y} + \tilde{\alpha} \right] \left\{ \frac{\partial \hat{v}^{(0)}}{\partial t} - i\alpha \hat{v}^{(0)} - \varepsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \hat{v}^{(0)} \right\} \\ = F(t) e^{\tilde{\alpha} y}, \quad (2.11)$$

and $F(t) = 0$ if the condition on $\hat{v}^{(0)}$ as $y \rightarrow \infty$ is to be satisfied.

Matching (2.10) and (2.11) across the layer $y = H$, and using the continuity and jump conditions (2.9), it can be seen that:

$$\varepsilon^2 \left[\frac{\partial^3 \hat{v}^{(1)}}{\partial y^3} - \frac{\partial^3 \hat{v}^{(0)}}{\partial y^3} \right] \Big|_{y=H} = - E(t) e^{\tilde{\alpha} H}.$$

$$\text{Or,} \quad \hat{v}^{(1)}(H, t) = \frac{E(t)}{1\alpha\sigma} e^{\tilde{\alpha} H} \quad (2.12)$$

This equation implies that $\hat{v}^{(1)}(y, t)$ should behave like $e^{\tilde{\alpha} y}$ as $y \rightarrow H$ from below.

Solutions to equation (2.7) are now sought, with the initial and boundary conditions (2.6), and the expectation that

$$\hat{v}^{(1)} \sim e^{\tilde{\alpha} y} \text{ as } y \rightarrow H.$$

3. The Inner Solution

Let us, for the moment concentrate on the inner solution. Equation (2.6) clearly supplies two solutions for $\hat{v}^{(1)}$ by inspection, namely,

$$A(t) e^{\tilde{\alpha} y} \text{ and } B(t) e^{-\tilde{\alpha} y},$$

where $A(t)$ and $B(t)$ are arbitrary function of time. The remaining two solutions are given by:

$$\frac{\partial K}{\partial t} - i\alpha\sigma y K = \varepsilon^2 \left[\frac{\partial^2 K}{\partial y^2} - \tilde{\alpha}^2 K \right],$$

$$\text{where} \quad K = \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \hat{v}^{(1)}.$$

If a new dependent variable $\Gamma(y, t)$ is defined as

$$\Gamma(y, t) = e^{\varepsilon^2 \tilde{\alpha}^2 t} e^{-i\alpha\sigma y t} e^{\alpha^2 \varepsilon^2 \sigma^2 t^3 / 3} K(y, t), \quad (3.1)$$

then Γ satisfies the equation:

$$\frac{\partial \Gamma}{\partial t} - 2i\alpha\varepsilon^2\sigma t \frac{\partial \Gamma}{\partial y} = \varepsilon^2 \frac{\partial^2 \Gamma}{\partial y^2}.$$

This can be cast into the familiar diffusion equation by the transformation of coordinates:

$$\begin{aligned}\tau &= t \\ \eta &= y + i\alpha\epsilon^2\sigma^2t^2\end{aligned}\quad (3.2)$$

so that

$$\frac{\partial\Gamma}{\partial\tau} = \epsilon^2 \frac{\partial^2\Gamma}{\partial\eta^2} \quad (3.3)$$

In order to solve equation (3.3), let us introduce an image source δ -function below the plate. That is,

$$\hat{v}^{(i)}(y,0) = q\left\{\delta(y - y_0) + \delta(y + y_0)\right\}.$$

Then,

$$\Gamma(\eta, \tau) = q\left[\frac{\partial^2}{\partial y_0^2} - \tilde{\alpha}^2\right] \left\{ \frac{e^{-\frac{(\eta - y_0)^2}{4\epsilon^2\tau}}}{2\epsilon\sqrt{\pi\tau}} + \frac{e^{-\frac{(\eta + y_0)^2}{4\epsilon^2\tau}}}{2\epsilon\sqrt{\pi\tau}} \right\}.$$

Reverting to the original co-ordinates (3.2), and to K :

$$\begin{aligned}K(y,t) &= qe^{-\epsilon^2\tilde{\alpha}^2t}e^{-i\alpha\sigma y t}e^{\alpha^2\epsilon^2\sigma^2t^3/3} \left[\frac{\partial^2}{\partial y_0^2} - \tilde{\alpha}^2\right] \times \\ &\quad \left[\frac{e^{-\frac{(y - y_0 + i\alpha\epsilon^2\sigma\tau^2)^2}{4\epsilon^2t}}}{2\epsilon\sqrt{\pi t}} + \frac{e^{-\frac{(y + y_0 + i\alpha\epsilon^2\sigma\tau^2)^2}{4\epsilon^2t}}}{2\epsilon\sqrt{\pi t}} \right].\end{aligned}$$

It is now only necessary to solve for $\hat{v}^{(i)}$ from $\left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2\right]\hat{v}^{(i)} = K$, and add the two inviscid $e^{\pm\tilde{\alpha}y}$ solutions.

The calculations here are tedious but straightforward; details can be found in Easthope (1984).

Eventually,

$$\begin{aligned}
\hat{v}^{(1)}(y,t) = & q e^{-\epsilon^2 \tilde{\alpha}^2 t} e^{-\alpha^2 \epsilon^2 \sigma^2 t^3/3} \\
& \times \left[e^{i\alpha\sigma y_0 t} \left\{ M(y - y_0, t) + \alpha\sigma t(\alpha\sigma t - 2i\tilde{\alpha})J_1(y - y_0, t) \right. \right. \\
& \quad \left. \left. + \alpha\sigma t(\alpha\sigma t + 2i\tilde{\alpha})J_2(y - y_0, t) \right\} \right. \\
& \left. + e^{-i\alpha\sigma y_0 t} \left\{ M(y - y_0, t) + \alpha\sigma t(\alpha\sigma t - 2i\tilde{\alpha})J_1(y + y_0, t) \right. \right. \\
& \quad \left. \left. + \alpha\sigma t(\alpha\sigma t + 2i\tilde{\alpha})J_2(y + y_0, t) \right\} \right] \\
& + A(t)e^{\tilde{\alpha}y} + B(t)e^{\tilde{\alpha}y}
\end{aligned} \tag{3.4}$$

Here,

$$\begin{aligned}
M(y,t) &= \frac{1}{2\epsilon\sqrt{\pi t}} e^{-\left(y-y_0 - i\alpha\epsilon^2\sigma t^2\right)^2/4\epsilon^2 t} \\
J_1(y,t) &= \frac{e^{\epsilon^2 \tilde{\alpha}^2 t}}{4\tilde{\alpha}} e^{-\tilde{\alpha}y} e^{i\alpha\epsilon^2\sigma t^2} \\
&\quad \operatorname{erfc}\left\{\epsilon\sqrt{t}\left[\tilde{\alpha} - y/2\epsilon^2 t + i\alpha\sigma t/2\right]\right\},
\end{aligned}$$

and $\operatorname{erfc}(z)$ is the complementary error function;

$$J_2(y,t) = \overline{J_1(-y,t)},$$

an overbar indicating complex conjugate.

This solution satisfies the initial condition, and enforcing the two boundary conditions gives $A(t)$ and $B(t)$:

$$\begin{aligned}
A(t) = & -\frac{1}{2}q e^{-\epsilon^2 \tilde{\alpha}^2 t} e^{-\alpha^2 \epsilon^2 \sigma^2 t^3/3} \left[e^{i\alpha\sigma y_0 t} \left\{ M(-y_0, t) \right. \right. \\
& \times \left[1 + \frac{y_0}{2\epsilon^2 \tilde{\alpha} t} - \frac{3i\alpha\sigma t}{2\tilde{\alpha}} \right] \\
& \left. \left. + 2\alpha\sigma t(\alpha\sigma t + 2i\tilde{\alpha})J_2(-y_0, t) \right\} \right. \\
& \left. + e^{-i\alpha\sigma y_0 t} \left\{ M(y_0, t) \left[1 - \frac{y_0}{2\epsilon^2 \tilde{\alpha} t} - \frac{3i\alpha\sigma t}{2\tilde{\alpha}} \right] \right. \right. \\
& \left. \left. + 2\alpha\sigma t(\alpha\sigma t + 2i\tilde{\alpha})J_2(y_0, t) \right\} \right],
\end{aligned}$$

$$\begin{aligned}
B(t) = & \frac{1}{2} q e^{-\frac{1}{2} \tilde{\alpha}^2 t} e^{-\frac{1}{2} \epsilon^2 \sigma^2 t^3 / 3} \left[e^{i \alpha \sigma y_0 t} \left\{ M(-y_0, t) \right. \right. \\
& \times \left[1 + \frac{y_0}{2 \epsilon^2 \tilde{\alpha} t} - \frac{3 i \alpha \sigma t}{2 \tilde{\alpha}} \right] + 2 \alpha \sigma t (\alpha \sigma t - 2 i \tilde{\alpha}) J_1(-y_0, t) \left. \right\} \\
& + e^{-i \alpha \sigma y_0 t} \left\{ M(y_0, t) \left[1 + \frac{y_0}{2 \epsilon^2 \tilde{\alpha} t} - \frac{3 i \alpha \sigma t}{2 \tilde{\alpha}} \right] \right. \\
& \left. \left. + 2 \alpha \sigma t (\alpha \sigma t - 2 i \tilde{\alpha}) J_1(y_0, t) \right\} \right].
\end{aligned}$$

It remains to examine the form of (3.4) as $y \rightarrow \Pi$. If $\epsilon^2 t \ll 1$ and $\gamma^2 \gg \epsilon^2 t$, $M(y \pm y_0, t)$ will be exponentially small.

Hence,

$$\begin{aligned}
\hat{v}^{(1)} \sim & q e^{-\frac{1}{2} \tilde{\alpha}^2 t} e^{-\frac{1}{2} \epsilon^2 \sigma^2 t^3 / 3} \frac{\alpha \sigma t}{2 \tilde{\alpha}} (\alpha \sigma t - 2 i \tilde{\alpha}) e^{i \tilde{\alpha} \alpha \epsilon^2 \sigma t^2} \\
& \times \left\{ e^{i \alpha \sigma y_0 t} e^{-\tilde{\alpha}(y-y_0)} + e^{-i \alpha \sigma y_0 t} e^{\tilde{\alpha}(y-y_0)} \right\} \\
& + A(t) e^{\tilde{\alpha} y} + B(t) e^{-\tilde{\alpha} y} \quad \text{as } y \rightarrow \Pi.
\end{aligned}$$

That is, toward the outer edge of the inner layer, the solution is dominated by inviscid solutions of the form $e^{\pm \tilde{\alpha} y}$, as might have been anticipated.

This is not quite of the correct form, since the $e^{-\tilde{\alpha} y}$ term will not match the required solution at $y = \Pi$, but (3.4) is taken to be an adequate approximation for investigative purposes, particularly in the near wall region.

4. The Other Velocity Components, Pressure

Having obtained an expressed for $\hat{v}^{(1)}$, it is possible to determine $\hat{u}, \hat{w}, \hat{p}$ and the vorticity components.

In particular from (2.4)

$$i \alpha \hat{u} + i \gamma \hat{w} = - \frac{\partial \hat{v}}{\partial y} \quad (3.5)$$

The pressure can be obtained from (2.1), (2.3) and (3.5):

$$\begin{aligned}
i \tilde{\alpha}^2 \hat{p} = & - \frac{\partial^2 \hat{v}}{\partial y \partial t} + i \alpha \sigma y \frac{\partial \hat{v}}{\partial y} + \alpha \sigma \hat{v} \\
& + \epsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \frac{\partial \hat{v}}{\partial y}. \quad (3.6)
\end{aligned}$$

Similarly, the vertical component of vorticity, $\hat{\Omega}_y$, is given by

$$\hat{\Omega}_y = -i\gamma\hat{u} + i\alpha\hat{w}, \quad (3.7)$$

and satisfies the equation

$$\frac{\partial \hat{\Omega}_y}{\partial t} - \frac{i\alpha\sigma\gamma}{\tilde{\alpha}} \hat{\Omega}_y - \varepsilon^2 \left[\frac{\partial^2}{\partial y^2} - \tilde{\alpha}^2 \right] \hat{\Omega}_y = i\gamma\sigma\hat{v}. \quad (3.8)$$

Once the behaviour of $\hat{v}$ has been determined, (3.6) permits a solution for the pressure and (3.8) determines $\hat{\Omega}_y$. Besides the forced solution for $\hat{\Omega}_y$ there is the homogeneous solution where $\hat{v} = 0$; this solution has been termed the Squire mode by some authors. Finally, (3.5) and (3.7) can be inverted to give

$$\hat{u} = \frac{1}{\tilde{\alpha}^2} \left[i\alpha \frac{\partial \hat{v}}{\partial y} + i\gamma \hat{\Omega}_y \right] \quad (3.9)$$

and

$$\hat{w} = \frac{1}{\tilde{\alpha}^2} \left[i\gamma \frac{\partial \hat{v}}{\partial y} + i\alpha \hat{\Omega}_y \right].$$

5. Computational Results

Equation (3.4) summarises the characteristics of the vertical velocity disturbance in y and t , in Fourier space for x and z . The more detailed behaviour of the flow can only be revealed by numerical inversion of the Fourier transforms in α, γ .

This requirement for computer solutions necessitated the replacement of the initial-value δ -function by more computationally tractable Gaussians. The form of equation (3.4) remained essentially unaltered.

A number of calculations were carried out with the aim of investigating the behaviour of the disturbance in both space and time; the results are illustrated in Figures 1 to 8. The Reynolds number is here taken to be 800; x and z mark the downstream and transverse directions respectively.

Figures 1 to 6 illustrate the disturbance at about one-tenth of the boundary layer thickness ($y = 0.106$ in dimensionless units), for a sequence of dimensionless times, 0.01, 0.1, 0.25, 0.5, 1.0, and 10.0.

The non-dimensionalisation parameters are the free-stream velocity and the boundary-layer thickness. This implies that the time scale is in fact extremely short, and we are therefore displaying only the very early history of the disturbance. Fortunately, the more interesting behavioural aspects are manifest within this span; as computational constraints set limits on the feasible values of t .

The perspective plots show a gradual evolution from a nearly symmetric initial disturbance, to a longitudinal extension with laterally symmetric peaks, to a more uniform trough-like disturbance. The twin peaks are reminiscent of the phenomena observed by Gaster and Grant (1975), while the overall longitudinal stretching is as predicted by Criminale and Kovaszny (1962).

This spreading of the wave-packet can be more easily appreciated from Figure 7, which shows clearly the laterally symmetric expansion, approximately as $\sqrt{t}$. Initially, the wave-packet accelerates more rapidly in the downstream direction, but by about $t \sim 10$, is again symmetric.

Figure 8 shows the structure of the wave-packet at about one-third of the boundary-layer thickness ($y = 0.35$), at $t = 1$. This shows more structure than was apparent nearer the plate. Clearly, the physical mechanism is changing as well where vorticity is giving way to pressure by a dominant balance. It should also be recalled that the solution is only approximate in this outer region as well.

Some of the Gaster and Grant experimental findings have here been duplicated, most notably the double transverse maxima. Other features are, however, seemingly different. For example, the oxbow shape of the wave-packet observed by Gaster and Grant is here replaced by a longitudinal ellipse form. But there is little or none of the normal modes involved in these pictures as would be expected for longer time.

It is relevant to ask what happens if a more 2-D initial disturbance were used, rather than a point pulse. This can be viewed as an attempt to mimic the vibrating ribbon used by Klebanoff, Tidstrom and Sargent (1962). The numerical results obtained showed a wave-packet with considerable lateral spread, but the transverse maxima were absent. It is clear that the form of the initial disturbance is critical in determining the shape of the resulting disturbance.

The numerical results also indicated a very rapid growth in the amplitude of the wave-packet at early times. This can in fact be demonstrated analytically, by taking an approximation to (3.4) as $t \rightarrow 0^+$.

Using Gaussian initial sources instead of δ -functions, we have (in physical space):

$$v(x - y, z, 0^+) \approx \frac{Q_2}{4\pi\beta} e^{-r^2/4\beta} \left[\frac{e^{-(y-y_0)^2/4\lambda}}{2\sqrt{\pi\lambda}} + \frac{e^{-(y+y_0)^2/4\lambda}}{2\sqrt{\pi\lambda}} \right] \\ - \frac{Q_2}{4\pi^2} \frac{e^{-y_0^2/4\lambda}}{\sqrt{\pi\lambda}} \iint_{-\infty}^{\infty} \left\{ e^{-1\alpha(x-x_0)} e^{-1\beta(z-z_0)} \right. \\ \left. \times e^{-\alpha^2\beta} \cosh(\tilde{\alpha}y) d\alpha d\beta \right\},$$

$$\text{where} \quad r^2 = (x - x_0)^2 + (z - z_0)^2, \quad (3.8)$$

and β and λ control the sharpness of the initial Gaussian peaks in (x,z) and y respectively.

Equation (3.8) demonstrates firstly the symmetric nature of the disturbance in x,z - as also illustrated in Fig. 1 - but also that the main contributor to the initial growth is the $e^{\tilde{\alpha}y}$ term in $\cosh(\tilde{\alpha}y)$. In fact, if $\beta = \lambda$, the main contributor to v in (3.8) can be shown to be of the form

$$e^{(y^2 - y_0^2 - r^2)/4\beta},$$

and clearly has a powerful effect for locations within a hyperboloid of revolution extending vertically above the source.

6. Conclusions

The shear wall dynamics within a modelled boundary layer have been solved explicitly by an alternative means of linear analysis. In effect, this work is an extension of the inviscid theory presented by Criminale and Drazin (1990) for initial-value problems to encompass viscous boundary conditions. And, because the bases of the technique is like that of rapid distortion theory, the concepts are valid for either a laminar or a turbulent wall layer.

In terms of laminar boundary layers it has been shown that acute three-dimensionality (structure) can give rise to rapid growth of the perturbations which could lead to nonlinearity before the normal means of development of unstable Tollmien-Schlichting waves. In addition, the wave-packet characteristics shown double maxima and an elongated streaky solution in the downstream direction. This early transient period is unlike the predictions that have been made from the asymptotics of limited normal mode decomposition.

Interestingly enough, there is yet another important result. Even though the normal component of the velocity, v , will decay after rapid rise, the other components of the velocity can increase algebraically as a power of t . This can be seen at once if v is

considered with no x -dependence. Clearly, $\hat{\Omega}_y$ will increase at least linearly with t and consequently u, w as well. Such a finding has already been reported by Landahl (1980) using a completely different approach.

The analysis shows that it is the large scales (small wave numbers) that are most energetic in this study. Layer values of the shear rate mean layer values of the Reynolds number and thus the problem is turbulent like. Can streaky structure or large scale oscillations have their origins in this manner? It is highly suggestive.

Finally, an interesting use of this strategy is in progress. If each of the solutions for the perturbations velocity, namely u, v, w , are replaced by $\dot{x}, \dot{y}, \dot{z}$ and taken as particle paths, a nonlinear non-autonomous system results. Solutions of the coupled three equations bear all of the aspects that are now common place in dynamical systems. We believe the solutions of this system will go a long way to understanding what is seen by various visualization methods such as dye, hydrogen bubbles or smoke and give greater insight into the problem of mixing. This work is in progress.

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Figure 1 Pulse at $y = 0.106$, $t = 0.01$

Figure 2 Pulse at $y = 0.106$, $t = 0.1$

Figure 3 Pulse at $y = 0.106$, $t = 0.25$

Figure 4 Pulse at $y = 0.106$, $t = 0.5$

Figure 5 Pulse at $y = 0.106$, $t = 1.0$

Figure 6 Pulse at $y = 0.106$, $t = 10.0$

Figure 7 Spread of wave packet at $y = 0.106$

Figure 8 Pulse at $y = 0.35$, $t = 1$

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